PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 June 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Where to send your solutions and proposals

There has been an increase in the number of solutions and proposals sent to the Canadian Mathematical Society's Head Office in Ottawa, Ontario. Please note the instructions above and send them directly to the Editor-in-Chief.

Solutions submitted by FAX

There has been an increase in the number of solutions sent in by FAX, either to the Editor-in-Chief's departmental FAX machine in St. John's, Newfoundland, or to the Canadian Mathematical Society's FAX machine in Ottawa, Ontario. While we understand the reasons for solvers wishing to use this method, we have found many problems with it. The major one is that hand-written material is frequently transmitted very badly, and at times is almost impossible to read clearly. We have therefore adopted the policy that we will no longer accept submissions sent by FAX. We will, however, continue to accept submissions sent by email or regular mail. We do encourage email. Thank you for your cooperation.

2287. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let G denote the point of intersection of the medians, and I denote the point of intersection of the internal angle bisectors of a triangle. Using only an unmarked straightedge, construct H, the point of intersection of the altitudes.

2288. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are a circle (without centre) and five points A, B, C, D, E, on it such that $\text{arc } AB = \text{arc } BC$ and $\text{arc } CD = \text{arc } DE$. Using only an unmarked straightedge, construct the mid-point of $\text{arc } AE$.

2289★. *Proposed by Clark Kimberling, Evansville, IN, USA.*

Use any sequence, $\{c_k\}$, of 0's and 1's to define a *repetition-resistant sequence* $s = \{s_k\}$ inductively as follows:

1. $s_1 = c_1$, $s_2 = 1 - s_1$;
2. for $n \geq 2$, let

$$\begin{aligned} L &= \max\{i \geq 1 \;:\; (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \ldots, s_n, 0) \quad \text{for some } m < n\}, \\ L' &= \max\{i \geq 1 \;:\; (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \ldots, s_n, 1) \quad \text{for some } m < n\}. \end{aligned}$$

(so that L is the maximal length of the tail-sequence of $(s_1, s_2, \ldots, s_n, 0)$ that already occurs in $(s_1, s_2, \ldots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if} \quad L < L', \\ 1 & \text{if} \quad L > L', \\ c_n & \text{if} \quad L = L'. \end{cases}$$

$\Big($For example, if $c_i = 0$ for all i, then

$$s = (0,1,0,0,0,1,1,0,1,0,1,1,1,0,0,1,0,0,$$
$$1,1,1,1,0,1,1,0,0,0,0,0,1,0,1,0,\ldots)\Big)$$

Prove or disprove that s contains every binary word.

2290. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

For $x, y, z \geq 0$, prove that

$$\big((x+y)(y+z)(z+x)\big)^2 \geq xyz(2x+y+z)(2y+z+x)(2z+x+y).$$

2291. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let a, b, c denote the side lengths of a Pythagorean triangle. Suppose that each side length is the sum of two positive integer squares. Prove that $360|abc$.

2292. *Proposed by K.R.S. Sastry, Dodballapur, India.*

A convex quadrilateral Q has integer values for its angles, measured in degrees, and the size of one angle is equal to the product of the sizes of the other three.

Show that Q is either a parallelogram or an isosceles trapezium.

2293. *Proposed by Claus Mazanti Sorensen, student, Aarhus University, Aarhus, Denmark.*

A sequence, $\{x_n\}$, of positive integers has the properties:

1. for all $n > 1$, we have $x_{n-1} < nx_n$;
2. for arbitrarily large n, we have $x_1x_2 \ldots x_{n-1} < nx_n$;
3. there are only finitely many n dividing $x_1x_2 \ldots x_{n-1}$.

Prove that $\sum_{k=1}^{\infty} \frac{(-1)^k}{x_k k!}$ is irrational.

2294. *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For the annual Sino-Japanese "Go" tournament, each country sends a team of seven players, C_i's and J_i's, respectively. All players of each country are of different ranks (strengths), so that

$$C_1 < C_2 < \ldots < C_7 \qquad \text{and} \qquad J_1 < J_2 < \ldots < J_7.$$

Each match is determined by one game only, with no tie. The winner then takes on the next higher ranked player of the opponent country. The tournament continues until all the seven players of one country are eliminated, and the other country is then declared the winner. (For those who are not familiar with the ancient Chinese "Chess" game of "Go", a better and perhaps more descriptive translation would be "the surrounding chess".)

(a) What is the total number of possible sequences of outcomes if each country sends in n players?

(b)* What is the answer to the question in part (a) if there are three countries participating with n players each, and the rule of the tournament is modified as follows:

> The first match is between the weakest players of two countries (determined by lot), and the winner of each match then plays the weakest player of the third country who has not been eliminated (if there are any left). The tournament continues until all the players of two countries are eliminated.

2295. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find three positive integers a, b, c, in arithmetic progression (with positive common difference), such that $a+b$, $b+c$, $c+a$, are all perfect squares.

2296. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Show that $\sin^2 \frac{\pi x}{2} > \frac{2x^2}{1+x^2}$ for $0 < x < 1$.

Hence or otherwise, deduce that $\pi < \frac{\sin \pi x}{x(1-x)} < 4$ for $0 < x < 1$.

2297. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Given is a circle of radius 1, centred at the origin. Starting from the point $P_0 = (-1,-1)$, draw an infinite polygonal path $P_0P_1P_2P_3\ldots$ going counterclockwise around the circle, where each P_iP_{i+1} is a line segment tangent to the circle at a point Q_i, such that $|P_iQ_i| = 2|Q_iP_{i+1}|$. Does this path intersect the line $y = x$ other than at the point $(-1, 1)$?

2298. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

The "Tickle Me" Feather Company ships its feathers in boxes which cannot contain more than 1 kg of feathers each. The company has on hand a number of assorted feathers, each of which weighs at most one gram, and whose total weight is 1000001/1001 kg.

Show that the company can ship all the feathers using only 1000 boxes.

2299. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, $z > 0$ be real numbers such that $x + y + z = 1$. Show that

$$\prod_{\text{cyclic}} \left(\frac{(1-y)(1-z)}{x}\right)^{(1-y)(1-z)/x} \geq \frac{256}{81}.$$

Determine the cases of equality.

2300. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle with circumradius R. The circle passing through A and touching BC at its mid-point has radius R_1. Define R_2 and R_3 similarly. Prove that

$$R_1^2 + R_2^2 + R_3^2 \geq \frac{27}{16}R^2.$$