MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

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Shreds and Slices

A Note on Convexity

A function f is **convex on** I (I an interval) if

$$f(\alpha x+(1-\alpha)y)\leq \alpha f(x)+(1-\alpha)f(y) \qquad \forall\, x,y\in I, \quad \alpha\in[0,1],$$

and **J–convex on** I(see [1]) if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \qquad \forall\, x,y\in I.$$

Similarly, f is **concave on** I if

$$f\left(\alpha x+(1-\alpha)y\right)\geq \alpha f(x)+(1-\alpha)f(y) \qquad \forall\, x,y\in I,\ \alpha\in[0,1],$$

and **J–concave** is defined similarly. Several sources claim that J–convexity is sufficient for convexity (see [2]), but we intend to make this more precise. First, f convex clearly implies that f is J–convex. We will prove that:

(i) If f is convex on an open interval I, then f is continuous on I, and

(ii) If f is J–convex and continuous on an interval I, then f is convex on I.

Proof. (i) Let $a\in I$, $\epsilon>0$. We will show that on some interval around a, $f(x)<f(a)+\epsilon$ and $f(x)>f(a)-\epsilon$.

Choose any $b\in I$. Assume $b>a$. Then the graph of $f(x)$ on $[a,b]$ lies under the chord joining $(a,f(a))$ and $(b,f(b))$ (see Figure 1), which in turn lies under the line $y=f(a)+\epsilon$ on some interval to the right of a, including a. Applying a similar argument when $b<a$, we find an interval around a on which $f(x)<f(a)+\epsilon$.

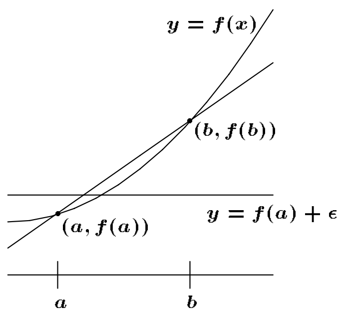


Figure 1.

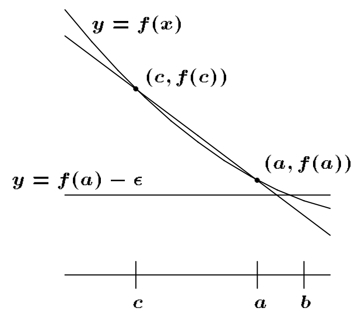


Figure 2.

Now, if $f(x) \geq f(a)$ for all $x \in I$, then we are done, so assume $f(b) < f(a)$ for some $b \in I$. Assume $b > a$. Then $f(x) > f(a)$ for all $x < a$ by convexity, which certainly implies $f(x) > f(a) - \epsilon$. Take some $c < a$, and consider the line joining $(c, f(c))$ and $(a, f(a))$. Then the graph of $f(x)$ to the right of a lies above this line, which in turn lies above the line $y = f(a) - \epsilon$ on some interval to the right of a, including a. The case $b < a$ is similar.

(ii) Since f is J–convex, it is clear that

$$f(\alpha x + (1 - \alpha)y) \ \leq \ \alpha x + (1 - \alpha)f(y)$$

for $\alpha = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$, and by an induction argument, for any dyadic rational between 0 and 1; that is, a fraction of the form $\frac{m}{2^n}$. Since the dyadic rationals are dense in $[0, 1]$, we can find a sequence which converges to any given real α in $[0, 1]$. By taking a limit along this sequence, the identity is shown to be true for all $\alpha \in [0, 1]$.

Remark. In (i), I must be open, as seen in the example

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 0 \text{ or } x = 1 \end{cases}.$$

Then f is convex, but not continuous.

Hence, (ii) allows for an easy way to check convexity, which is useful for setting up Jensen's inequality, without resorting to the second derivative test if calculus does not appeal to you. Also, the corresponding results hold for f concave.

Example 1. Show that $\sin x$ is concave on $[0, \pi]$.

Proof.

$$\sin x + \sin y \ = \ 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \ \leq \ 2 \sin\left(\frac{x + y}{2}\right),$$

so that

$$\sin\left(\frac{x+y}{2}\right) \geq \frac{\sin x + \sin y}{2}.$$

Example 2. Show that $\dfrac{a}{x+b}$ is convex on $(-b, \infty)$, where $a > 0$.

Proof. By two applications of AM-GM,

$$\frac{a}{\frac{x+y}{2}+b} \leq \frac{a}{\sqrt{(x+b)(y+b)}} \leq \frac{1}{2}\left(\frac{a}{x+b} + \frac{a}{y+b}\right).$$

Example 3. Let k be a positive integer. Show that x^k is concave on $[0, \infty)$.

Proof. We show by induction that

$$\left(\frac{x+y}{2}\right)^k \leq \frac{x^k + y^k}{2}$$

for all x, $y \geq 0$. The result is certainly true for $k = 1$. Assume it holds for some k. Assume without loss of generality that $x \geq y$. Then

$$(x^k - y^k)(x - y) = x^{k+1} - x^k y - xy^k + y^{k+1} \geq 0,$$

so that $x^k y + xy^k \leq x^{k+1} + y^{k+1}$. Then,

$$\begin{aligned} \left(\frac{x+y}{2}\right)^{k+1} &= \left(\frac{x+y}{2}\right)^k \left(\frac{x+y}{2}\right) \\ &\leq \left(\frac{x^k+y^k}{2}\right)\left(\frac{x+y}{2}\right) \\ &= \frac{x^{k+1} + x^k y + xy^k + y^{k+1}}{4} \\ &\leq \frac{2(x^{k+1} + y^{k+1})}{4} = \frac{x^{k+1} + y^{k+1}}{2}. \end{aligned}$$

References

1. E. Lozansky and C. Rousseau, *Winning Solutions*, Springer-Verlag, New York, 1996.

2. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

The Equation of the Tangent to the n^{th} Circle

Krishna Srinivasan

Let n circles of radius r be tangent to each other in a row, such that the centre of each lies on the x-axis, and the first circle passes through the origin. Let l be the tangent of the n^{th} circle passing through the origin, as shown in the diagram. What is the equation of this tangent l?

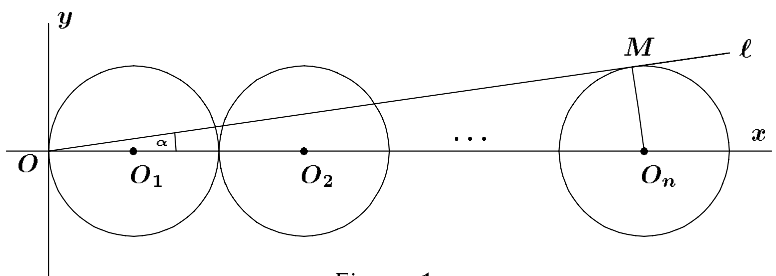


Figure 1.

Let O_n be the centre of the n^{th} circle, M the point of tangency, and α the angle formed between ℓ and the x-axis. Then the slope of ℓ is $\tan\alpha$. Also, $\angle OMO_n = 90°$ (since the radius is perpendicular to the tangent). Therefore, the value of α is $\sin^{-1}(\frac{MO_n}{OO_n})$. Since the radius is r, $MO_n = r$ and $OO_n = 2nr - r$. Consequently, the slope of the tangent is $\tan(\sin^{-1}(\frac{1}{2n-1}))$.

Let us derive $\tan(\sin^{-1} x)$, for $0 \leq x < 1$. Let $\theta = \sin^{-1} x$. Then

$$\tan(\sin^{-1} x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{\sqrt{1-x^2}}.$$

Substituting $x = \frac{1}{2n-1}$, we find the value of the slope is

$$\frac{1}{\sqrt{(2n-1)^2-1}} = \frac{1}{2\sqrt{n^2-n}}.$$

And finally, since the tangent passes through the origin, the equation of ℓ is

$$y = \frac{x}{2\sqrt{n^2-n}}. \tag{1}$$

It is interesting to note that the radius r does not appear in (1). This shows that the line $y = x/2\sqrt{2}$ (substituting $n = 2$ into (1)), for example, is always tangent to the second circle, regardless of the radius. [Ed: This makes sense geometrically. Why?]

Coordinatizing the plane, as we have done, can make a problem simpler, as the following problems show:

Problem: Circles P and Q are tangent; each has radius 1. PQ extended meets the circles at A and B, and AC and BTC are tangents to circle P, as shown in the diagram. Compute AC.

(1993 ARML, Team Questions)

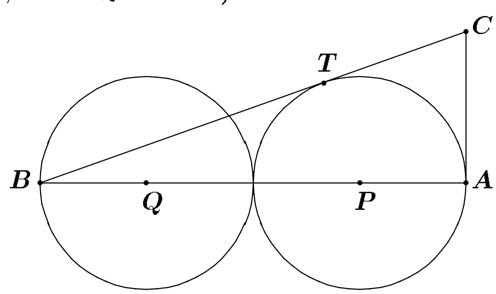


Figure 2.

Solution: The slope of BC is $1/2\sqrt{2}$, by (1). Hence, $AC = AB/2\sqrt{2} = \sqrt{2}$.

Problem: Three equal circles are tangent as shown. The line AD is drawn from point A on the left circle, and tangent to the circle at the right at point D. How long is the chord BC of the circle in the middle?

(1996-1997 Scarborough Mathematics League)

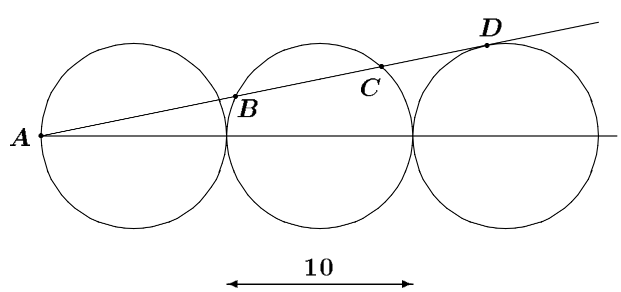


Figure 3.

Solution: Establish a coordinate system, as in Figure 1. As found above, the equation of the tangent is $y = x/2\sqrt{6}$ and the equation of the second circle is $(x - 15)^2 + y^2 = 25$. Solving for these two equations gives the points of intersections, which are $B(\frac{72-8\sqrt{6}}{5}, \frac{6\sqrt{6}-4}{5})$ and $C(\frac{72+8\sqrt{6}}{5}, \frac{6\sqrt{6}+4}{5})$. With this information, the length of the chord BC can be calculated to 8 units.

[Ed: Knowing the slope of AD leads to a nice Euclidean solution. By secant theorem, $AB \cdot AC = 10 \cdot 20 = 200$ and by similar triangles, $AB + BC/2 = 6\sqrt{6}$. Now find BC.]

Combinatorial Games

Adrian Chan

Introduction

Definition: One type of *combinatorial game* is a two-person game such that:

(i) Players alternate removing counters from a finite collection according to a set of rules, and

(ii) The last player to remove a counter wins.

The classic combinatorial game of this sort is *Bouton's Nim* or *Nim*. The game consists of some number of piles with some number of counters in each. A player, on his turn, may remove any number of counters from any *one* pile.

Example: Say there are three piles of 1, 3, and 5 counters. The game could proceed as follows:

$$(1,3,5) \xrightarrow{1} (1,3,2) \xrightarrow{2} (1,1,2) \xrightarrow{1} (1,1,0) \xrightarrow{2} (1,0,0) \xrightarrow{1} (0,0,0),$$

and player 1 wins!

A Simple Game: One Pile Nim with Restriction

Let us look at a simple one pile game of Nim, with the restriction that a player may only remove one or two counters at a time. Consider the game where there are 7 counters to play with, denoted $\text{Nim}(7;1,2)$. First, try playing with a friend. You'll probably see a strategy for one of the players. The strategy can be developed by looking at the game in a different way.

Construct a *directed graph*, where the vertices represent the number of counters, and the directed edges represent possible moves. Note that the graph has no cycles since positions cannot repeat. Hence, a game is represented by a path from the initial vertex (7 counters) to the terminal vertex (0 counters).

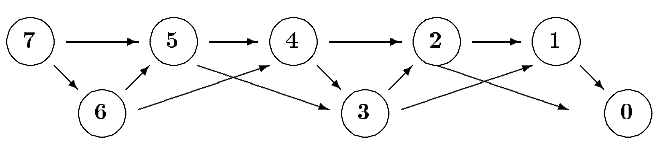


Figure 1.

Safe or Unsafe?

A vertex is *safe* if a player moving to that vertex has a winning strategy, and *unsafe* otherwise. We can label each vertex of our directed graph as safe or unsafe according to the following instructions:

(i) The terminal position is safe,

(ii) If all moves from a vertex lead to an unsafe vertex, then that vertex is safe, and

(iii) If there is a move from a vertex to a safe vertex, then that vertex is unsafe.

We can relabel our directed graph of Nim$(7; 1, 2)$ to get:

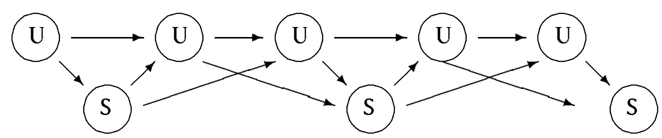


Figure 2.

The Winning Strategy

1. A player starts on an unsafe vertex, and can move to a safe vertex (iii).
2. The other player is forced to move to an unsafe vertex (ii).
3. The original player can again move to a safe vertex (iii). In fact, the original player can always move to a safe vertex, and after some number of moves, he will land on the terminal vertex.

Since the terminal position is a safe vertex, the original player will win!

Exercise: Who should win a similar Nim game with a pile of 11 counters? With 12 counters?

Grundy-Values for Games

For any combinatorial game, we can label each vertex of the directed graph with a non-negative integer instead of the labels safe and unsafe. This number, the *Grundy-Value*, is found by the following process:

(i) The terminal vertex is labelled 0.

(ii) For every other vertex, consider the set of the labels of the vertices it points to. The label of the vertex is then the smallest non-negative integer not appearing in this set.

Example: Our game of Nim$(7; 1, 2)$ with Grundy-Values appears as

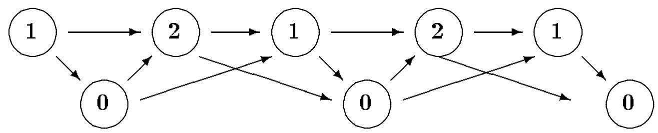


Figure 3.

Is it a coincidence that the safe positions coincide with those vertices assigned a Grundy-Value of 0? Let us investigate. Denote the sets of vertices with safe positions, unsafe positions, Grundy-Values of 0, and positive Grundy-Values as S, U, Z, and N respectively.

Proposition. $S = Z$ and $U = N$.
Proof by Induction. The terminal position is in both S and Z by definition. Assume that there exists some subset of vertices P in which the proposition is true; that is, for any vertex in P, it is in S if and only if it is in Z, and it is in U if and only if it is in N.

Let us look at vertices that are not members of P. There exists a vertex not in P such that all of its emanating edges lead to a vertex in P (to the reader: why?). Let this vertex be A, and consider all vertices that A leads to.

If A leads to a vertex in P that is in S (and so in Z), then A is unsafe so A is in U, and A leads to a vertex with Grundy-Value 0, so A itself must have positive Grundy-Value, and A is in N.

Otherwise, A leads to vertices in P that are only in U (and so in N), so A is safe and A is in S, and every vertex A leads to has a positive Grundy-Value, so A itself has Grundy-Value 0 and A is in Z.

Hence, the statement is true for P and A, a larger set of vertices. We can repeat the argument until we have included all vertices.

The Original Game of Nim

First consider a one-pile game of Nim, with no restrictions on how many counters are taken. Obviously, the first person can take all the counters and win. If we compute the Grundy-Values for such a game, the Grundy-Value of a pile of n counters is n.

Now consider a game of Nim with two piles of counters. Say there are 6 in one and 7 in the other, denoted Nim$(6, 7)$. We can look at the game as a grid instead of a directed graph: You start with a marker at $(6, 7)$ and a move is a translation going left or down, but not both. Whoever lands on $(0, 0)$ wins. (See Figure 4.)

Exercise: Fill the grid with Grundy-Values. Which positions are safe (winning)? Who will win, the first player or the second player?

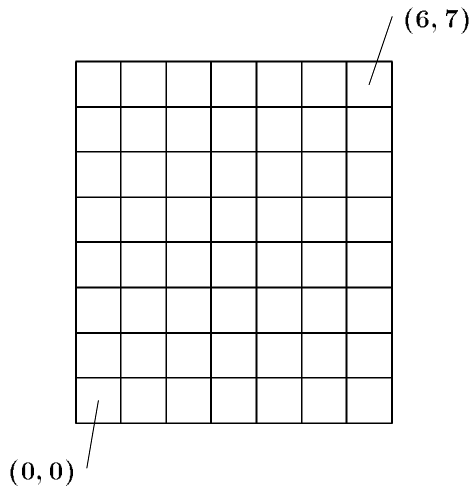


Figure 4.

Finding the Strategy: Nim Sum

Definition: The *Nim Sum* of a combinatorial game is the binary addition of the Grundy-Values of each independent game. However, there is the additional condition that there is no carrying. For example, the nim sum of 1_2 and 1_2 is 0_2, and the nim sum of 1110_2 and 101_2 is 1011_2.

Example: Let us look at the game Nim$(4, 5, 6, 7)$. Notice that the game is the same as the simultaneous games of Nim(4), Nim(5), Nim(6), and Nim(7), where the player can choose which game to play in and make a move in.

	Grundy-Value	Binary Representation
Nim(4)	4	1 0 0
Nim(5)	5	1 0 1
Nim(6)	6	1 1 0
Nim(7)	7	1 1 1
Nim Sum		0 0 0

If the Nim Sum is 0, then the position is *balanced*. Otherwise, it is *unbalanced*.

Problem: Prove that all balanced positions are safe positions. Hint: Look at the definition of safe and unsafe positions.

Exercise: Analyze Nim$(n; 1, 2, 3)$ and Nim$(n; 1, 3, 4)$. What are the Grundy-Values? Does the first or second player win?

With this concept of the Nim Sum, you will be able to find the strategy in any combinatorical game and determine if you will win or lose! The concept of Nim Sum is a very effective tool not only in these games, but also other areas of mathematics.

Mayhem Book Reviews

Donny Cheung

The Art of Problem Solving: A Resource for the Mathematics Teacher, Edited by Alfred S. Posamentier, published by Corwin Press Inc., 2455 Teller Road, Thousand Oaks CA, 91320-2218, 1996, ISBN 0-8039-6362-9, softcover, 465 pages.

The subtitle does not quite do this book justice. This wonderful collection of twenty independent contributions has brought together a diverse spectrum of professionals to cover a wide range of topics. The result is a problem solving resource which is both entertaining and informative.

Having no singular theme other than problem solving in general, this book is ideal as an exposition of the many different aspects of problem solving. Unlike other books on problem solving, this book is not singularly interested in documenting individual problem solving techniques, teaching methods, or historical anecdotes. Rather, a good mixture has been obtained, and this is the strength of the book.

The book begins with a chapter titled "Strategies for Problem Exploration", which individually explores 13 powerful problem solving strategies. But also contained within the book are a chapter promoting the use of problem solving as a teaching paradigm, a chapter investigating the reasons that many students make the errors that they do, a chapter on cooperative learning, a chapter dedicated to the pigeonhole principle, and a chapter exploring mathematically gifted students from a psychological viewpoint.

For the most part, this book is written in a conversational tone, and is very readable. There are problems scattered throughout the book, and there is a handy system of boxed and circled numerals which make it easy to find specific problems from a certain topic of mathematics, or specific problems which highlight a certain problem solving technique.

Perhaps this is not the ideal book for those looking for hundreds of problems to solve, but for anyone who is interested in problem solving in itself, and things related to problem solving, I highly recommend this book.

Students! Get Ready for the Mathematics for SAT I, by Alfred S. Posamentier, published by Corwin Press Inc., 1996, ISBN 0-8039-6415-3, softcover, 206 pages.

Teachers! Prepare Your Students for the Mathematics for SAT I, by Alfred S. Posamentier and Stephen Krulik, published by Corwin Press Inc., 1996, ISBN 0-8039-6416-1, softcover, 116 pages.

Each year, college-bound students around the world write the Scholastic Assessment Test, whose scores are used in college admissions, mostly in the United States. So for these students, preparing for this test is usually considered a very important task both in the classroom and in homes.

To this end, there have been a great number of books which are aimed at helping students prepare for the SAT. The two books reviewed here are designed to help a student prepare for the Mathematics portion of the SAT, and to help a teacher prepare his or her students.

Without an effort on the part of the student, no book can help prepare anyone for the SAT. However, a well-designed book can certainly be helpful.

This set of books is organized and well laid out. The book for students includes a very useful summary of basic mathematical facts, and includes both sample problems with in-depth solutions and timed sample tests for practice. Throughout the sample problems are also good tips for answering different types of questions that are asked on the SAT.

The book for teachers details ten important problem solving strategies, with excellent illustrative problems and in-depth analyses of the problems. It also discusses some various more non-routine types of problems, and some more technical points about the test, such as efficient calculator use. And, it includes, as an appendix, the summary of mathematical facts that appears in the student's book.

On the whole, I think that these books are worth looking into for anyone who is going to write the SAT, or has students who are going to.

J.I.R. McKnight Problems Contest 1980

1. Sum to k terms the series whose n^{th} term is

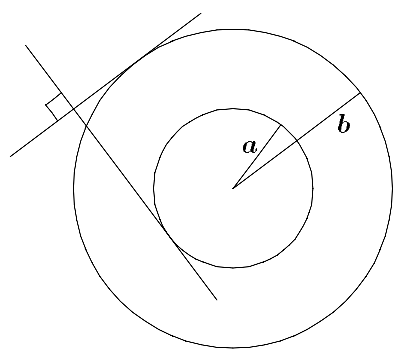
$$\frac{n^4 + 3n^3 + 2n^2 - 1}{n^2 + n}\,.$$

2. Find all the functions of the form $f(x) = \frac{a+bx}{b+x}$, where a and b are constants, $f(2) = 2f(5)$, and $f(0) + 3f(-2) = 0$.

3. Solve for x and y:

$$\begin{aligned} 2^x \cdot 3^y &= 6 \\ \frac{2^{x+1}}{3^{y-1}} &= 5 \end{aligned}$$

4. Perpendicular tangents are drawn to the circles represented by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Find the equation of the locus of the point of intersection of these tangents, and name the locus.



5. Prove that

$$\frac{\sin x + \sin y + \sin z - \sin(x+y+z)}{\cos x + \cos y + \cos z + \cos(x+y+z)} = \tan\left(\frac{x+y}{2}\right)\tan\left(\frac{y+z}{2}\right)\tan\left(\frac{z+x}{2}\right)\,.$$

6. A square sheet of tin, a inches on a side, is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner for the box to have as large a volume as possible?

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 February 1998, for publication in the issue 5 months ahead; that is, issue 4 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Solutions

H214. Show that $3(a+b+c) - 8 \leq abc + c$ for all positive integers a, b, $c \geq 2$.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

Let $a = x + 2$, $b = y + 2$, and $c = z + 2$, so x, y, $z \geq 0$. Then

$$3(6+x+y+z) - 8 \leq (2+x)(2+y)(2+z) + 2 + z$$

implies that

$$18 + 3(x+y+z) - 8 \leq 8 + 4(x+y+z) + 2(xy+yz+xz) + xyz + 2 + z\,,$$

which implies that

$$10 + 3(x+y+z) \leq 10 + 4(x+y+z) + 2(xy+yz+xz) + xyz + z\,,$$

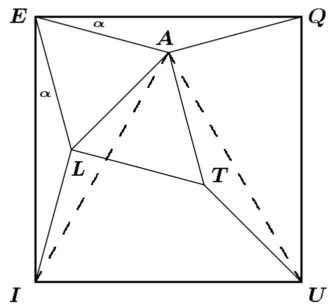
which, in turn, imples that

$$0 \leq x + y + 2z + 2(xy+yz+xz) + xyz\,,$$

and the last inequality is trivially true.

H215. Consider a square $EQUI$. Let L, A, and T be points on the interior such that EQA and IEL are isoceles triangles with bases EQ and IE respectively. Show that $UT = IL$ if EAL and ALT are equilateral triangles.

Solution.



We have the following facts: $EA = AQ$ and $EL = LI$, since EAQ and ELI are isosceles triangles. Also, $AL = LI$ and $AT = TL = AL$, since EAL and ALT are equilateral triangles.

To show that $UT = IL$, we will show that $\triangle ALI$ is congruent to $\triangle ATU$.

Let $\alpha = \angle AEQ$. Then $\angle LEI = \alpha$, since $\triangle AEQ \cong \triangle LEI$. Hence, $90° = \angle IEQ = 2\alpha + \angle LEA = 2\alpha + 60°$, so $\alpha = 15°$.

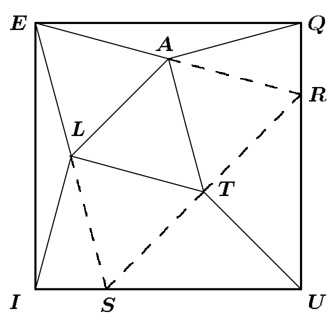
Then $\angle ILA = 360° - \angle ELA - \angle ELI = 360° - 150° - 60° = 150°$. Therefore, $\triangle ALI \cong \triangle ELI$, and $AI = EI = IU$ so $\angle AIU = \angle EIU - \angle EIA = 90° - 30° = 60°$. Therefore, $\triangle AIU$ is equilateral, so $AI = AU$.

Finally, $\triangle ALI \cong \triangle ATU$ since $AL = AT$, $AI = AU$, and $\angle LAI = 60° - \angle IAT = \angle TAU$. Hence, $UT = IL$ as required.

H216. Triangulate a square such that all edges of the triangles on the interior of the square are equal in length and this length is smaller than the length of the side of the square. (Hint: See the previous problem **H215**.)

Solution.

Consider the figure in the solution of problem **H215**. Extend EA through A to meet QU at R and extend EL through L to meet UI at S as shown.



Now ER is the diameter of the circumcircle of $\triangle ERQ$ since $\angle EQR = 90°$. Further, A must be the centre of the circle since $EA = AQ$ and A is on the diameter ER. Thus, $AR = RE$. Likewise, $LS = LE$ in $\triangle ESI$.

Note that $\triangle ERS$ is equilateral and $SR = 2AL$. Therefore, $ST = TR$ and this value is equal to all the other lengths in the interior shown in the solution of problem **H215**.

Rider. Are there other possible triangulations of the square with all the lengths in the interior equal?

Advanced Solutions

A191. Taken over all **ordered** partitions of n, show that

$$\sum_{k_1+k_2+\cdots+k_m=n} k_1k_2\cdots k_m = \binom{m+n-1}{2m-1}.$$

Additional Solution by Waldemar Pompe.

Consider n soccer players. Let $A(n,m)$ be the number of ways they can be divided into m teams (blocks) such that in each team there is a goal-keeper indicated. A partition which has k_1, k_2, ... , k_m players in the blocks gives $k_1k_2\cdots k_m$ ways of choosing goal-keepers. Adding these products over all partitions gives $A(n,m)$.

On the other hand, $A(n,m)$ can be found as follows: Place $m+n-1$ players in a row and choose $2m-1$ of them. Starting with the first of these $2m-1$ players, we make every alternate player a goal-keeper (resulting in m of them), and the remaining $m-1$ are discarded, creating $m-1$ gaps in our row of players. There are then m blocks, with a goal-keeper in each. It can be verified that there is a one to one correspondence between such formations and the formations above. Hence, the equality in question holds.

Challenge Board Solutions

We begin with a couple of corrections. J. Chris Fisher of the University of Regina has kindly pointed out that the statement of problem **C70** (and hence the solution last issue) is wrong. The problem was as follows:

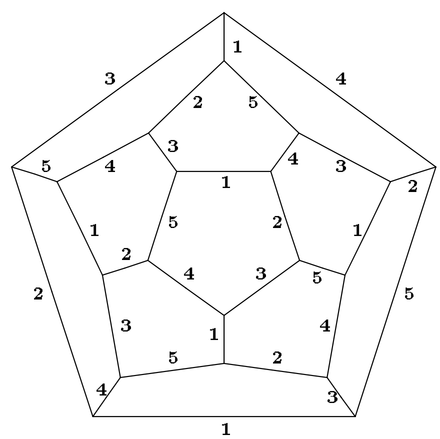
C70. Prove that the group of automorphisms of the dodecahedron is S_5, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is A_5.

In fact, the group of automorphisms of the dodecahedron is $A_5 \times C_2$, where C_2 is the two-element group. The published proof showed that the rotation group was A_5, and then incorrectly described the rest of the problem as similar.

In fact, the isomorphism between the automorphism group of the dodecahedron and $A_5 \times C_2$ can be described explicitly: given a dodecahedron with edges numbered as described in the problem, any automorphism induces a permutation of the numbers which can be checked to be even (and hence an element of A_5). Also, the map to C_2 sends an automorphism to the identity if it is a rotation, and to the other element if it is not (that is, if it is a reflection). The reverse map is similar: there is one rotation that permutes the 5 types of edge-labels in any even way, and one reflection.

Chris suggests Coxeter's *Introduction to Geometry* (especially p. 273) as a reference. One of the warning flags which tipped Chris off was that the erroneous proof implied that S_5 is an isometry group of Euclidean three-space. Coxeter lists the possible finite isometry groups, and S_5 does not appear (and indeed S_5 doesn't turn up as an isometry group until Euclidean four-space).

(Also, the figure accompanying the solution contained a typo: the "3" on the central line of symmetry should be a "1".)



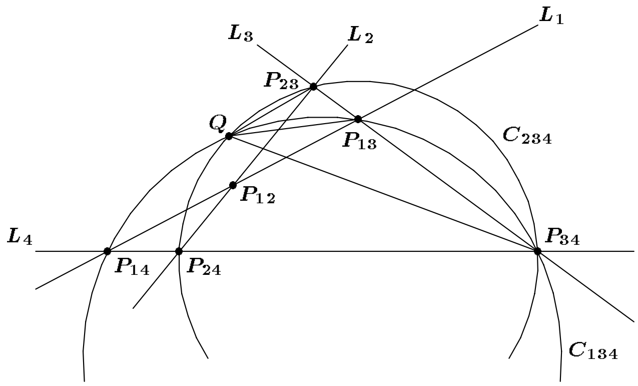
Thank you, Chris, for pointing out the error and explaining how to rectify it! And if any other reader is suspicious of anything written in these pages, please let us know.

C71. Let L_1, L_2, L_3, L_4 be four general lines in the plane. Let p_{ij} be the intersection of lines L_i and L_j. Prove that the circumcircles of the four triangles $p_{12}p_{23}p_{31}$, $p_{23}p_{34}p_{42}$, $p_{34}p_{41}p_{13}$, $p_{41}p_{12}p_{24}$ are concurrent.

Solution.

For convenience, let C_{123}, C_{234}, C_{341}, and C_{412} be the circles. By symmetry, it suffices to show that the first three are concurrent.

This proof is slightly diagram-dependent, but it is short and sweet and geometric. Label the three angles as in the figure. By the exterior angle theorem, $c = a + b$. Let Q be the intersection of C_{234} with C_{341}, other than p_{34}. Then



$$\begin{aligned} \angle p_{13}Qp_{23} &= \angle p_{23}Qp_{34} - \angle p_{13}Qp_{34} \\ &= c - b \\ &= a \\ &= \angle p_{13}p_{12}p_{23}. \end{aligned}$$

Thus Q is on the circle C_{123} as well.

Very rough sketch of another solution.

This solution is intended to intrigue and entice the reader, rather than rigorously solve the problem. All statements made here can be made rigorous.

This problem originally came to our attention because of a connection to a sophisticated result:

Lemma. Let C be a degree d curve in the plane. Let A and B be degree e curves that each meet C at de distinct points (say $a_1, \ldots, a_{de}$ and $b_1, \ldots, b_{de}$ respectively). Suppose that $a_i = b_i$ for $i = 1, \ldots, de - 1$. Then, if $d \geq 3$, $a_{de} = b_{de}$; that is, the last points are also equal.

Amazingly, this result fails if d is 1 or 2. The proof relies on the fact that there is no rational parametrization of curves of degree greater than 2, while a line such as $x = 0$ has the parametrization $(0, t)$ and a conic such as the circle $x^2 + y^2 = 1$ has the parametrization

$$\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

(An even more sophisticated result: these deep facts are related to the fact that the diophantine equations $x + y = z$ and $x^2 + y^2 = z^2$ have lots of solutions in integers, while if $n > 2$, $x^n + y^n = z^n$ has only a finite number of solutions even if x, y, and z are allowed to be quite general, for example of the form $r + s\sqrt{2}$ where r and s are integers. This result, stated appropriately, is a consequence of Siegel's Theorem, and relates in an obvious way to Fermat's Last Theorem.)

Even the application of the lemma to this problem has some subtleties. As before, it suffices to show that C_{123}, C_{234}, and C_{341} are concurrent. Let Q be the intersection of C_{234} and C_{341} (other than p_{34}) as before, and let P be the intersection of C_{123} and C_{234} (other than p_{23}). Let C be the cubic curve that is the union of C_{123} and L_4, A the cubic that is the union of C_{234} and L_1, and B the cubic that is the union of C_{341} and L_2. Then C and A intersect at the six points $\{p_{12}, \ldots, p_{34}\}$, and at P. They also intersect at two points at infinity $(\frac{1}{0}, \frac{i}{0})$ and $(\frac{1}{0}, \frac{-i}{0})$ (whatever that means!). The cubics C and B intersect at the same nine points (including the two strange ones!), except the P is replaced by Q. By the lemma, $P = Q$, and we are done.

Of course, a lot of further explanation is required to even make sense of all this, but such explanation is beyond the scope of ***Mayhem***.

Comments.

1. This problem is a first of an infinite sequence of theorems, called the *Clifford Theorems*. Call the intersection of two lines in general position their *Clifford point*. For three lines in general position, there are three Clifford points of pairs of lines; call the circle through the three points the *Clifford circle* of the three lines. Then, according to this problem, given four lines in general position, the four Clifford circles (of the four triples of lines) are concurrent; call this point the *Clifford point* of the four lines. In general, if n is odd, given n general lines, then the n Clifford points of all $(n-1)$-subsets of the n general lines are concyclic, and the resulting circle is called the *Clifford circle* of the n general lines. Similarly, if n is even, given n general lines, then the n Clifford circles of all $(n-1)$-subsets of the n general lines are concurrent, and the resulting point is called the *Clifford point* of the n general lines. The theorems implicit in these definitions are the Clifford theorems.

   This theorem is discussed in Liang-shin Hahn's book *Complex Numbers and Geometry* (published by the Mathematical Association of America). This is a wonderful, beautiful book, and possibly the best place to learn about how complex numbers can be used to make Euclidean plane geometry simple. Hahn proves all the Clifford theorems (in Section 2.3) using a simple lemma, which he proves using complex numbers, but which can also be proved with ordinary Euclidean geometry:

   Lemma. Suppose there are four circles C_1, C_2, C_3, and C_4 in a plane. Let C_1 and C_2 intersect at z_1 and w_1, C_2 and C_3 intersect at z_2 and w_2, C_3 and C_4 intersect at z_3 and w_3, C_4 and C_1 intersect at z_4 and w_4. Then the points z_1, z_2, z_3, z_4 are concyclic if and only if w_1, w_2, w_3, w_4 are concyclic.

   If anyone has a slick proof of the Clifford theorems (or even the next case), we would be interested in seeing it.

2. What needs to be done to make the first (geometric) proof rigorous?

3. Can you make sense of the weird points at infinity $(\frac{1}{0}, \frac{i}{0})$ and $(\frac{1}{0}, \frac{-i}{0})$ described in the sketch of the second solution? In your way of making sense of them, can you see why they lie on every circle?

4. Without understanding the intricacies of the second sketch, can you use the sketch's lemma, with a little hand waving, to prove Hahn's lemma above? Can you use it to prove any other well-known results in Euclidean geometry? (Pappus' theorem seems like a good possibility.) If so, we would love to hear from you!