Dissecting Rectangular Strips Into Dominoes

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The First Problem

A domino is defined to be a 1×2 or 2×1 rectangle. The first one is said to be horizontal and the second vertical. In our Mathematics Club, we learned to count the number g_n of different ways of dissecting a $3 \times 2n$ strip into dominoes. The sequence $\{g_n\}$ satisfies the recurrence relation

$$g_n = 4g_{n-1} - g_{n-2}, \tag{1}$$

for all $n \geq 2$, with initial conditions $g_0 = 1$ and, as shown in Figure 1, $g_1 = 3$.

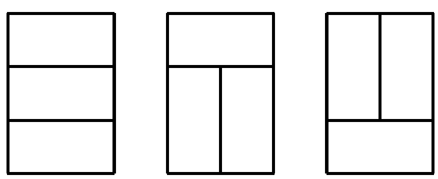

Figure 1

The generating function $G(x)$ for the sequence is defined to be the formal power series

$$g_0 + g_1x + g_2x^2 + \cdots .$$

It is easy to deduce from (1) that

$$G(x) = \frac{1-x}{1-4x+x^2}. \tag{2}$$

In examining the dissections of the $3 \times 2n$ strip, we observed that they fall into two kinds. A dissection of the first kind can be divided by a vertical line into two substrips without splitting any dominoes. Such a line is called a *fault line*. A dissection of the second kind, called a *fault-free dissection*, has no fault lines. Figure 2 shows an example of each, using 3×4 strips.

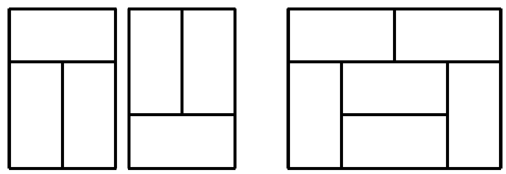

Figure 2

Let f_n be the number of fault-free dissections of the $3 \times 2n$ strip. From Figure 1, $f_1 = 3$. For all $n \geq 2$, a fault-free dissection cannot start with three horizontal dominoes. It must start off as shown in Figure 3, and continue by adding horizontal dominoes except for a final vertical one.

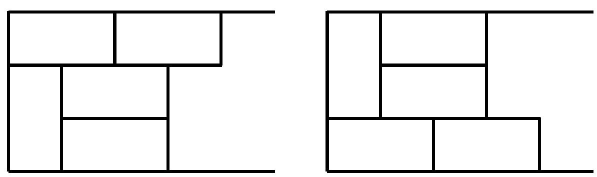

Figure 3

It follows that $f_n = 2$ for all $n \geq 2$, and the sequence satisfies a trivial recurrence relation

$$f_n = f_{n-1}$$

for all $n \geq 3$, with initial conditions $f_0 = 1, f_1 = 3$ and $f_2 = 2$. Let $F(x) = f_0 + f_1x + f_2x^2 + \cdots$ be the generating function for the sequence. Then $F(x) = -1 + x + 2(1 + x + x^2 + \cdots) = -1 + x + \frac{2}{1-x}$. This simplifies to

$$F(x) = \frac{1 + 2x - x^2}{1 - x}. \tag{3}$$

Having solved the simpler problem of counting fault-free dissections of the $3 \times 2n$ strip, we make use of our result to find an alternative solution to the general problem of finding all dissections of this strip. They can be classified according to where the first fault line is. This is taken to be the right end of the strip if the dissection is fault-free. Then the strip is divided into a $3 \times 2k$ substrip on the left and a $3 \times 2(n-k)$ substrip on the right, where $1 \leq k \leq n$.

Since the first substrip is dissected without any fault lines, it can be done in f_k ways. The second substrip can be dissected in g_{n-k} ways as we do not care whether there are any more fault lines. Hence

$$g_n = f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0. \tag{4}$$

From the values of f_n, $g_n = 3g_{n-1} + 2g_{n-2} + 2g_{n-3} + \cdots + 2g_0$. If we subtract from this expression $g_{n-1} = 3g_{n-2} + 2g_{n-3} + \cdots + 2g_0$, we have $g_n - g_{n-1} = 3g_{n-1} - g_{n-2}$, which is equivalent to (1).

We now derive (2) in another way. It follows from (4) that for all $n \geq 1$,

$$2g_n = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0. \tag{5}$$

Multiplying $F(x)$ and $G(x)$ yields

$$\begin{aligned} F(x)G(x) &= (f_0 + f_1 x + f_2 x^2 + \cdots)(g_0 + g_1 x + g_2 x^2 + \cdots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0)x + (f_0 g_2 + f_1 g_1 + f_2 g_0)x^2 + \cdots . \end{aligned}$$

In view of (5), this becomes $F(x)G(x) = g_0 + 2g_1 x + 2g_2 x^2 + \cdots = 2G(x) - 1$ so that

$$G(x) = \frac{1}{2 - F(x)}. \tag{6}$$

Substituting (3) into (6) yields (2).

The Second Problem

Let g_n be the number of ways of dissecting a $4 \times n$ strip into dominoes. Then $g_0 = 1, g_1 = 1$ and, as shown in Figure 4, $g_2 = 5$. It is not hard to verify that $g_3 = 11$. We wish to determine the infinite sequence $\{g_n\}$ via recurrence relations and generating functions.

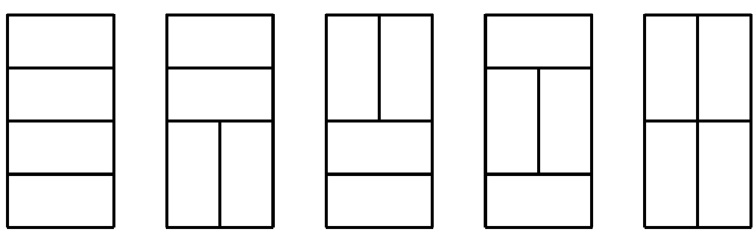

Figure 4

Let f_n be the number of fault-free dissections of the $4 \times n$ strip. We have $f_0 = 0, f_1 = 1$ and from Figure 4, $f_2 = 4$. For odd $n \geq 3$, the only fault-free dissections are the extensions of the second and third ones in Figure 4, with horizontal dominoes except for a final vertical one. Hence $f_n = 2$. For even $n \geq 4$, $f_n = 3$ since we can also include similar extensions of the fourth dissection in Figure 4.

As in the solution of the First Problem, we have

$$g_n = f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0.$$

This leads to

$$g_n = g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4} \tag{7}$$

for all $n \geq 4$, with initial conditions $g_0 = 1, g_1 = 1, g_2 = 5$ and $g_3 = 11$. Also,

$$\begin{aligned} F(x) &= 1 + x + 4x^2 + 2x^3 + 3x^4 + 2x^5 + 3x^6 + \cdots \\ &= -2 - x + 2x^2 \\ &\quad +3(1 + x^2 + x^4 + x^6 + \cdots) + 2x(1 + x^2 + x^4 + \cdots) \\ &= -2 - x + x^2 + \frac{3 + 2x}{1 - x^2} \\ &= \frac{1 + x + 3x^2 + x^3 - x^4}{1 - x^2}. \end{aligned} \tag{8}$$

Substituting (8) into (6), which is still valid here, we have

$$G(x) = \frac{1 - x^2}{1 - x - 5x^2 - x^3 + x^4}. \tag{9}$$

We now give an alternative solution to the Second Problem, along the line of the solution to the First Problem we learned at the Mathematics Club. We classify the dissections of the $4 \times n$ strip into five types according to how they start. These correspond to those in Figure 4 if we ignore the vertical dominoes in the second column. Call these Types A, B, C, D and E, and let their numbers be a_n, b_n, c_n, d_n and e_n, respectively.

By symmetry, we have $b_n = c_n$ so that

$$g_n = a_n + 2b_n + d_n + e_n \tag{10}$$

for all $n \geq 1$. In a Type A dissection, we are left with a $4 \times (n-2)$ strip, which can be dissected in g_{n-2} ways. Hence

$$a_n = g_{n-2} \tag{11}$$

for all $n \geq 3$, with $a_1 = 0$ and $a_2 = 1$.

In a Type B dissection, if we complete the second column with a vertical domino, the remaining $4 \times (n-2)$ strip can be dissected in g_{n-2} ways. The only alternative is to fill the second column with two horizontal dominoes. The remaining part can be dissected in b_{n-1} ways, so that

$$b_n = g_{n-2} + b_{n-1} \tag{12}$$

for all $n \geq 3$, with $b_1 = 0$ and $b_2 = 1$.

In a Type D dissection, the situation is similar except that if we fill the second column with two horizontal dominoes, we must then also fill the third column with two more horizontal dominoes. The remaining part can be dissected in d_{n-2} ways, so that

$$d_n = g_{n-2} + d_{n-2} \tag{13}$$

for all $n \geq 3$, with $d_1 = 0$ and $d_2 = 1$.

Finally, in a Type E dissection, after filling the first column with two vertical dominoes, we are left with a $4 \times (n-1)$ strip which can be dissected in g_{n-1} ways. Hence

$$e_n = g_{n-1} \tag{14}$$

for all $n \geq 2$, with $e_1 = 1$. Now (7) follows from (10), (11), (12), (13) and (14) since

$$\begin{aligned} g_n &= a_n + 2b_n + d_n + e_n \\ &= g_{n-2} + (2g_{n-2} + 2b_{n-1}) + g_{n-2} + d_{n-2} + g_{n-1} \\ &= g_{n-1} + 4g_{n-2} + 2(g_{n-3} + b_{n-2}) + d_{n-2} \\ &= g_{n-1} + 4g_{n-2} + 2g_{n-3} + 2b_{n-2} + d_{n-2} \\ &= g_{n-1} + 4g_{n-2} + 2g_{n-3} + g_{n-2} - a_{n-2} - e_{n-2} \\ &= g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4}. \end{aligned}$$

Using (7), $(1 - x - 5x^2 - x^3 + x^4)G(x)$ simplifies to $1 - x^2$. Hence (9) also follows.

Supplementary Problem

Let f_n denote the number of fault-free dissections of the $4 \times n$ strip. Find a recurrence relation for the sequence $\{f_n\}$ with initial conditions.

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