THE OLYMPIAD CORNER

No. 186

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This number we give the 24 problems proposed to the jury, but not selected for the 37th International Mathematical Olympiad in July 1996 at Mumbai, India. My thanks go to Ravi Vakil, Canadian Team Leader to the 37th IMO for collecting this and other contest material and forwarding it to me.

PROBLEMS PROPOSED TO THE JURY
BUT NOT USED AT THE
37th INTERNATIONAL MATHEMATICAL OLYMPIAD
July 1996 — Mumbai, India

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca} \leq 1.$$

When does equality hold?

2. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \cdots + a_n^k \geq 0.$$

Let $p = \max\{|a_1|, \ldots , |a_n|\}$. Prove that $p = a_1$ and that

$$(x-a_1)(x-a_2)\cdots(x-a_n) \leq x^n - a_1^n$$

for all $x > a_1$.

3. Let $a > 2$ be given, and define recursively:

$$a_0 = 1, \quad a_1 = a, \quad a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) a_n.$$

Show that for all integers $k > 0$, we have

$$\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k} < \frac{1}{2}(2+a-\sqrt{a^2-4}).$$

4. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers, not all zero.

(a) Prove that $x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root.

(b) Let $A = \sum_{j=1}^{n} a_j$, and $B = \sum_{j=1}^{n} ja_j$ and let R be the positive real root of the equation in (a). Prove that $A^A \leq R^B$.

5. Let $P(x)$ be the real polynomial, $P(x) = ax^3 + bx^3 + cx + d$. Prove that if $|P(x)| \leq 1$ for all x such that $|x| \leq 1$, then

$$|a| + |b| + |c| + |d| \leq 7.$$

6. Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x)(x+1)^n + g(x)(x^n + 1)$$

for some polynomials $f(x)$, $g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n.

7. Let f be a function from the set of real numbers $\mathbb{R}$ into itself such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a non-zero real number c, such that $f(x+c) = f(x)$ for all $x \in \mathbb{R}$).

8. Let the sequence $a(n)$, $n = 1, 2, 3, \ldots$, be generated as follows: $a(1) = 0$, and for $n > 1$,

$$a(n) = a(\lfloor n/2 \rfloor) + (-1)^{n(n+1)/2}.$$

(Here $\lfloor t \rfloor$ is the greatest integer less than or equal to t.)

(a) Determine the maximum and minimum value of $a(n)$ over $n \leq 1996$ and find all $n \leq 1996$ for which these extreme values are attained.

(b) How many terms $a(n)$, $n \leq 1996$, are equal to 0?

9. Let triangle ABC have orthocentre H, and let P be a point on its circumcircle, distinct from A, B, C. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X. Prove that EX is parallel to AP.

10. Let ABC be an acute-angled triangle with $|BC| > |CA|$, and let O be the circumcentre, H its orthocentre, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.

11. Let ABC be equilateral, and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \;\geq\; A_1B \cdot B_1C \cdot C_1A.$$

12. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ respectively, with $a < c \leq d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$(b^2 - a^2)^2 \;\leq\; (bc - ad)^2 + (bd - ac)^2.$$

13. Let ABC be an acute-angled triangle with circumcentre O and circumradius R. Let AO meet the circle BOC again in A', let BO meet the circle COA again in B' and let CO meet the circle AOB again in C'. Prove that

$$OA' \cdot OB' \cdot OC' \;\geq\; 8R^3.$$

When does equality hold?

14. Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, R_D denote the circumradii of the triangles DAB, ABC, BCD, CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

15. On the plane are given a point O and a polygon $\mathcal{F}$ (not necessarily convex). Let P denote the perimeter of $\mathcal{F}$, D the sum of the distances from O to the vertices of $\mathcal{F}$, and H the sum of the distances from O to the lines containing the sides of $\mathcal{F}$. Prove that $D^2 - H^2 \geq P^2/4$.

16. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and the next number on the circle, moving in a clockwise direction; that is, the numbers
a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$. Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes?

17. A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called *quadratic* if for each i in the set $\{1, 2, \ldots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

(a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

18. Find all positive integers a and b for which

$$\left\lfloor \frac{a^2}{b} \right\rfloor + \left\lfloor \frac{b^2}{a} \right\rfloor = \left\lfloor \frac{a^2 + b^2}{ab} \right\rfloor + ab,$$

where, as usual, $\lfloor t \rfloor$ refers to the greatest integer which is less than or equal to t.

19. Let N_0 refer to the set of non-negative integers. Find a bijective function f from N_0 into N_0 such that for all $m, n \in N_0$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

20. A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemes are regarded as different if at least one vertex is coloured differently in the two schemes.)

21. Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, 3 \ldots, k-1, k\}$ such that no n distinct elements of S add up to m.

22. Determine whether or not there exist two disjoint infinite sets $\mathcal{A}$ and $\mathcal{B}$ of points in the plane satisfying the following conditions:

(a) No three points in $\mathcal{A} \cup \mathcal{B}$ are collinear, and the distance between any two points in $\mathcal{A} \cup \mathcal{B}$ is at least 1.

(b) There is a point of $\mathcal{A}$ in any triangle whose vertices are in $\mathcal{B}$, and there is a point of $\mathcal{B}$ in any triangle whose vertices are in $\mathcal{A}$.

23. A finite number of beans are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one bean is chosen. Two beans are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one bean on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

24. Let U be a finite set and f, g be bijective functions from U onto itself. Let

$$S = \{w \in U : f(f(w)) = g(g(w))\}$$

and

$$T = \{w \in U : f(g(w)) = g(f(w))\},$$

and suppose that $U = S \cup T$. Prove that, for $m \in U$, $f(w) \in S$ if and only if $g(w) \in S$.

As always we welcome your nice original solutions which differ from the official solutions provided by the proposers and the selection committee.

As an example of an Olympiad which may not be as widely circulated, and for which you may not have already seen solutions, we give the four problems of the 4th Class for the Croatian National Mathematical Competition of May 13, 1994 and the three problems of the Croatian Mathematical Olympiad of May 14, 1994.

My thanks go to Richard Nowakowski, Canadian Team Leader at the 35th IMO in Istanbul for collecting these problems.

CROATIAN NATIONAL MATHEMATICAL COMPETITION

Fourth Class

May 13, 1994

1. One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.

2. For a complex number z let $w = f(z) = \dfrac{2}{3-z}$.

(a) Determine the set $\{w : z = 2 + iy, \quad y \in R\}$ in the complex plane.

(b) Show that the function w can be written in the form

$$\frac{w-1}{w-2} = \lambda\,\frac{z-1}{z-2}.$$

(c) Let $z_0 = \frac{1}{2}$ and the sequence $\{z_n\}$ be defined recursively by

$$z_n = \frac{2}{3 - z_{n-1}}, \quad n \geq 1.$$

Using the property (b) calculate the limit of the sequence $\{z_n\}$.

3. Determine all polynomials $P(x)$ with real coefficients such that for some $n \in \mathbb{N}$ we have $xP(x-n) = (x-1)P(x)$, $\forall\, x \in \mathbb{R}$.

4. In the plane five points P_1, P_2, P_3, P_4, P_5 are chosen having integer coordinates. Show that there is at least one pair (P_i, P_j), for $i \neq j$ such that the line P_iP_j contains a point Q, with integer coordinates, and is strictly between P_i and P_j.

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Additional Competition for the Olympiad

May 14, 1994

1. Find all ordered triples (a, b, c) of real numbers such that for every three integers x, y, z the following identity holds:

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|.$$

2. Construct a triangle ABC if the lengths $|AO|$, $|AU|$ and radius r of the incircle are given, where O is the orthocentre and U the centre of the incircle.

3. Let P be the set of all lines of the plane M. Does there exist a function $f : P \to M$ having the following properties:

(a) the function f is an injection:

(b) $f(p) \in p$, $\forall\, p \in P$?

That should provide some problems for your puzzling pleasure over the next couple of months. Now we return to readers' solutions to problems featured in earlier numbers of the *Corner*.

First, an apology. Somehow, in shifting my files around we misplaced solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain, to two problems that we discussed in the October number of the *Corner*. His name should be added as a solver of problems 6 and 7 of the Telecom 1993 Australian Mathematical Olympiad in the solutions given [1997: 324–325].

Last number we gave solutions by the readers to the first ten problems of the "Baltic Way — 92" contest given in the May 1996 number [1996: 157–159].

MATHEMATICAL TEAM CONTEST "BALTIC WAY — 92" Vilnius, 1992 — November 5–8

11. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Show that there exists one and only one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:

(i) If $0 < q < \frac{1}{2}$ then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$.
(ii) If $1 < q \le 2$ then $f(q) = 1 + f(q-1)$.
(iii) $f(q) \cdot f(\frac{1}{q}) = 1$ for all $q \in \mathbb{Q}^+$.

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

By a change of variable $\widetilde{q} = \frac{1}{1-2q}$, we have from (i),

$$f\left(\frac{\widetilde{q}}{1+2\widetilde{q}}\right) = 1 + f(\widetilde{q}), \quad (0 < \widetilde{q} < \infty), \quad \text{or} \quad f\left(\frac{1}{\frac{1}{\widetilde{q}}+2}\right) = 1 + f\left(\frac{1}{\frac{1}{\widetilde{q}}}\right).$$

Calling $t = \frac{1}{\widetilde{q}}$ and using (iii) we have

$$\frac{1}{f(t+2)} = 1 + \frac{1}{f(t)}, \quad 0 < t < \infty, \quad t \in \mathbb{Q}^+. \tag{1}$$

Then

$$\frac{1}{f(t+4)} = \frac{1}{f(t+2+2)} = 1 + \frac{1}{f(t+2)} \quad = \quad 1 + 1 + \frac{1}{f(t)}$$
$$= \quad 2 + \frac{1}{f(t)}.$$

Hence, we can evaluate $f(t+2k)$, $k \geq 0$, k an integer, if we know $f(t)$.

Observe that condition (ii) can be rewritten as $f(1+t) = 1 + f(t)$, $t \in \mathbb{Q}^+$, $0 < t \leq 1$.

We can now evaluate $f(2k+1+q)$ as follows: Since

$$\frac{1}{f(2+q)} = 1 + \frac{1}{f(q)}, \quad \text{we have} \quad \frac{1}{f(2+1+q)} = 1 + \frac{1}{f(1+q)}.$$

If $0 < q \leq 1$, then $\dfrac{1}{f(3+q)} = 1 + \dfrac{1}{1+f(q)}$. Hence $f(3+q)$, $0 < q \leq 1$ can be evaluated if $f(q)$ is known. Once $f(3+q)$ is known, we obtain

$$\frac{1}{f(5+q)} = \frac{1}{f(2+3+q)} = 1 + \frac{1}{f(3+q)},$$

and

$$\frac{1}{f(2k+1+q)} + 1 + \frac{1}{f(2k-1+q)}, \quad 1 \geq q > 0.$$

Therefore, we can now evaluate

$$f(2k+q), \ f(2k+1+q) \qquad 0 < q \leq 1, \tag{2}$$

for all $k \geq 0$, k an integer, if we know $f(q)$.

Furthermore, we can evaluate $f(n)$, $n \geq 1$.

First $f(1) = 1$ since putting $q = 1$ in (iii) gives $(f(1))^2 = 1$. Now $f(2) = 1 + f(1) = 2$ from (ii). We follow recursively, $f(3)$:

$$\frac{1}{f(3)} = 1 + \frac{1}{f(1)} = 2$$

and

$$\frac{1}{f(2k+1)} = 1 + \frac{1}{f(2k-1)}.$$

Similarly

$$\frac{1}{f(2k+2)} = 1 + \frac{1}{f(2k)} \text{ and } f(2) = 2.$$

Thus any such function is uniquely defined on the integers.

Finally, we can evaluate the function at any q from the values on the positive integers. Let $q = \frac{a}{b}$, where $(a, b) = 1$.

Write $a = bq_1 + r_1$ where q_1 is a non-negative integer, and $0 \le r_1 < b$ is an integer. If $r_1 = 0$, $f(q) = f(q_1)$ which is determined.

If $a \le r_1 < b$, we apply $f(\frac{a}{b}) = f(q_1 + \frac{r_1}{b})$. This is determined if the value of $f(\frac{r_1}{b})$ is known using (2). Now $0 < \frac{r_1}{b} < 1$. We now compute $f(\frac{b}{r_1})$. $b = r_1q_2 + r_2$, $r_2 < r_1$. Continuing, since $0 \le r_{k+1} < r_k$, $r_j = 0$ for some j, and we will have an expression for which f is evaluated at an integer. Hence f exists and is uniquely determined.

12. Let $\mathbb{N}$ denote the set of positive integers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijective function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

We claim L must be 1.

Consider $\max\{\varphi(1), \ldots, \varphi(n)\} = j_n$. We note that $j_n \ge n$, since φ is one-to-one. Let $i_n \in \{1, 2, \ldots, n\}$ be such that $\varphi(i_n) = j_n$. Then

$$\frac{\varphi(i_n)}{i_n} \ge 1.$$

Since

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L, \quad \lim_{n\to\infty} \frac{\varphi(i_n)}{i_n} = L \ge 1. \tag{1}$$

Now consider $S_n = \{n \in N : \varphi(n) \le n\}$. S_n must be infinite. First $S_n \ne \emptyset$ for if $S_n = \emptyset$ then $\varphi(k) > k$ for all k and there is no k_0 with $\varphi(k_0) = 1$.

Suppose S_n is finite, with k the largest value in the set. Then $\varphi(n) > n$ for $n \ge k + 1$. Consider $\{1, 2, \ldots, k + 1\}$. Since $\varphi(n) > k + 1$ for $n \ge k + 1$, the only integers which can be pre-images of $\{1, 2, \ldots, k + 1\}$ are $\{1, 2, \ldots, k\}$. This is not possible, since φ is one-to-one and onto.

Therefore $S_n = \{n \in N : \varphi(n) \le n\}$ is infinite. Choose a sequence, $n_k \in S_n$ with $n_k \to \infty$. We now have $\lim\limits_{k\to\infty} \dfrac{\varphi(n_k)}{n_k} = L$. However

$$\frac{\varphi(n_k)}{n_k} \le 1.$$

Thus

$$L \le 1. \tag{2}$$

From (1) and (2), $L = 1$.

13. Prove that for any positive $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}$$

holds.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Christopher J. Bradley, Clifton College, Bristol, UK; by Michael Selby, University of Windsor, Windsor, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We feature Bradley's solution.

Now,

$$\frac{1}{xy} \geq \frac{4}{(x+y)^2}$$

since $(x+y)^2 \geq 4xy$, as $(x-y)^2 \geq 0$. So

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \sum_{i=1}^{n} \frac{4}{(x_i + y_i)^2} \qquad (*)$$

Lemma. $(a_1 + a_2 + \cdots + a_n)(a_2 a_3 \cdots a_n + a_1 a_3 a_4 \cdots a_n + a_1 a_2 a_4 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1}) \geq n^2 a_1 a_2 \cdots a_n$.

This follows from separate applications of the AM–GM inequality to the two terms on the left. It follows that

$$a_1 + a_2 + \cdots + a_n \geq \frac{n^2}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

Now put

$$a_i = \frac{1}{(x_i + y_i)^2}, \qquad i = 1, \ldots, n$$

and then

$$\frac{1}{(x_1+y_1)^2} + \frac{1}{(x_2+y_2)^2} + \cdots + \frac{1}{(x_n+y_n)^2} \geq \frac{n^2}{\sum_{i=1}^{n}(x_i+y_i)^2}.$$

Combining this with $(*)$ shows

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}.$$

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution by Boase.

We prove the result by induction on the number, n, of towns. If $n \leq 2$ the result is immediate.

Label the towns $A_1, A_2, \ldots, A_k$. We shall prove that if the statement holds for all $n < k$, then it is also true for $n = k$, so by induction it will be true for all n.

We can split up the towns excluding A_1 into two sets M and N, M containing those towns which can be reached from A_1 and N those which cannot be reached from A_1.

Thus, every town in N can reach A_1, and there is no route from a town in M to a town in N.

If N is empty, then A_1 is the desired town.

If this is not the case, then, since for any two towns in N, one of them can be reached from the other, and there is no route from outside N into N, the routes in question must pass through towns in N.

By the induction hypothesis, since $|N| < k$, there is a town in N which can reach all other towns in N. It can also reach A_1, and thus all towns in M. Therefore, this is the town which can reach all the other towns in the country, and the result is proved.

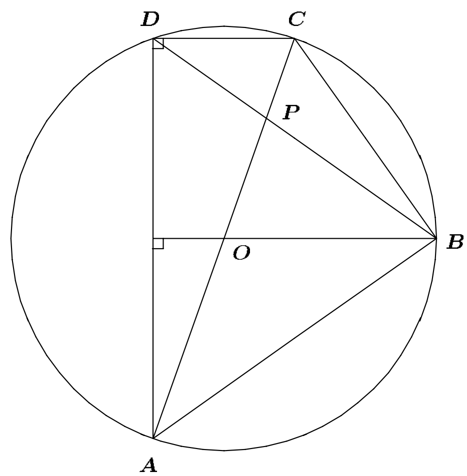
15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accommodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Let the animals be vertices of a graph. Join two animals by an edge if they are compatible. Now we have a graph with 8 vertices, and each vertex is joined to at least 4 others. So, by Dirac's theorem on Hamiltonian cycles, there must be a Hamiltonian cycle, and if we take consecutive pairs of animals in this cycle, we can put them in the same cage, and we have the required solution.

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution of Arslanagić.



The triangle ABD is isosceles because $AB = BD$. Let O be the centre of the circumcircle. Then $BO \perp AD$. Because $CD \perp AD$ (AC is a diameter), we get $CD \| BO$; that is, $\triangle PCD \sim \triangle POB$, and it follows that

$$\frac{CD}{OB} = \frac{PC}{PO}; \qquad \text{that is}$$

$$CD = \frac{OB \cdot PC}{PO} = \frac{1 \cdot \frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}.$$

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Prielipp's solution.

In this solution R, r, s will denote the circumradius, inradius and semiperimeter of a triangle. We shall show that in a non-obtuse triangle the perimeter is always greater than or equal to $2(2R) + 2r$.

Lemma. If A is an angle of triangle ABC, then $\cos A$ is a root of the equation

$$4R^2t^3 - 4R(R+r)t^2 + (s^2 + r^2 - 4R^2)t + (2R+r)^2 - s^2 = 0 \qquad (*)$$

Proof. Since $a = 2R\sin A$, and $s - a = r\cot(\frac{A}{2})$,

$$\begin{aligned} s &= a + (s-a) = 2R\sin A + r\cot\left(\frac{A}{2}\right) \\ &= 2R\sqrt{(1-\cos A)(1+\cos A)} + r\sqrt{\frac{1+\cos A}{1-\cos A}}\,. \end{aligned}$$

Thus

$$s^2 = 4R^2(1 - \cos A)(1 + \cos A) + 4Rr(1 + \cos A) + r^2\frac{1 + \cos A}{1 - \cos A}$$

so

$$4R^2(1 - \cos A)^2(1 + \cos A) + 4Rr(1 + \cos A)(1 - \cos A) \\ + r^2(1 + \cos A) - s^2(1 - \cos A) = 0.$$

Hence

$$rR^2 \cos^3 A - 4R(R + r)\cos^2 A + (s^2 + r^2 - 4R^2)\cos A \\ + (2R + r)^2 - s^2 = 0$$

making $\cos A$ a root of the equation $(*)$.

Corollary 1. If A, B, and C are the angles of triangle ABC, then $\cos A$, $\cos B$, and $\cos C$ are the roots of the equation $(*)$.

Corollary 2. If A, B, and C are the angles of triangle ABC, then

$$\cos A \cos B \cos C = \frac{s^2 - (2R + r)^2}{4R^2}.$$

Corollary 3. If A is the largest angle of triangle ABC, then

$$\begin{aligned} s &> 2R + r \quad \text{if} \quad A < 90^\circ \\ s &= 2R + r \quad \text{if} \quad A = 90^\circ \\ \text{and} \quad s &< 2R + r \quad \text{if} \quad A > 90^\circ. \end{aligned}$$

Corollary 4. In a non-obtuse triangle the perimeter of the triangle is always greater than or equal to $2(2R) + 2r$.

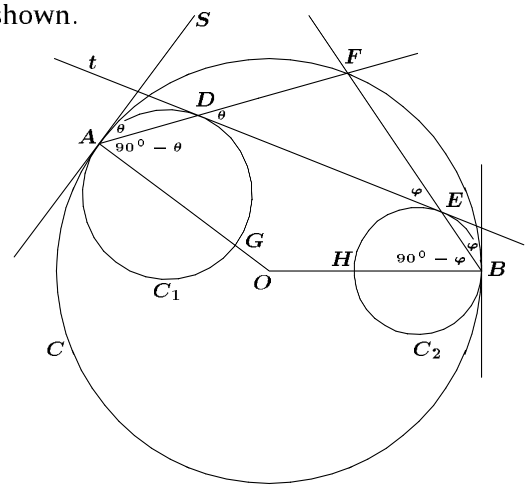
Proof. If the triangle is an acute triangle, then $s > 2R + r$, and $2s > 2(2R) + 2r$. If the triangle is a right triangle, then $s = 2R + r$. Thus $2s = 2(2R) + 2r$.

Corollary 5. In a non-obtuse triangle the perimeter of the triangle is always greater than twice the diameter of the circumcircle.

19. Let C be a circle in the plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2, touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let SA be the tangent to C_1 and C at A and TB be the tangent to C_2 and C at B, as shown.



Let $\angle SAD = \theta$ and $\angle TBE = \varphi$. Let O be the centre of C. AO meets C_1 again at G and since it is a common radius, AG is a diameter of C_1. BO meets C_2 again at H and BH is likewise a diameter of C_2.

We have $\angle DAG = 90^\circ - \theta$ and $\angle DGA = \theta$. By the alternate segment theorem $\angle GDE = 90^\circ - \theta$ and since $\angle ADG = 90^\circ$ (angle in a semicircle) it follows that $\angle FDE = \theta$. Similarly $\angle FED = \varphi$ and so $\angle DFE = 180^\circ - \theta - \varphi$. Also $\angle EBH = 90^\circ - \varphi$.

Considering the angles of the (re-entrant) quadrilateral $FAOB$ we have *reflex* $\angle AOB = 360^\circ - (90^\circ - \theta) - (90^\circ - \varphi) - (180^\circ - \theta - \varphi) = 2\theta + 2\varphi$. So $\angle AOB = 360^\circ - 2\theta - 2\varphi = 2\angle DFE$. But O is the centre of circle C, and AB is an arc of C, so F lies on C. (Converse of the angle at the centre $=$ twice angle at circumference).

20. Let $a \leq b \leq c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that $p(p-c) = (p-a)(p-b) = S$ (the area of the triangle).

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Michael Selby, University of Windsor, Windsor, Ontario; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since the triangle is a right triangle we have $c^2 = a^2 + b^2$, $p = \frac{a+b+c}{2}$, and $S = \frac{ab}{2}$.

Then

$$\begin{aligned} p(p-c) &= \frac{a+b+c}{2}\left(\frac{a+b+c}{2}-c\right) = \frac{(a+b)^2-c^2}{4} \\ &= \frac{a^2+b^2+2ab-c^2}{4} = \frac{ab}{2} = S, \end{aligned}$$

and

$$\begin{aligned} (p-a)(p-b) &= \left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right) \\ &= \frac{c+b-a}{2}\frac{c-b+a}{2} \\ &= \frac{c^2-(b-a)^2}{4} = \frac{c^2-(b^2+a^2)+2ab}{4} = \frac{ab}{2} = S, \end{aligned}$$

as required.

We conclude this number of the *Corner* with solutions to some of the problems of the 8th Iberoamerican Mathematical Olympiad, September 14–15, 1993 (Mexico) which we gave last year [1996: 159–160].

8th IBEROAMERICAN MATHEMATICAL OLYMPIAD September 14–15, 1993 (Mexico)

1. (*Argentina*) Let $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ be all the palindromic natural numbers, and for each i, let $y_i = x_{i+1} - x_i$. How many distinct prime numbers belong to the set $\{y_1, y_2, y_3, \dots\}$?

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario. We give Boase's solution.

The first few palindromic numbers are

$$1, 2, 3, \dots, 9, 11, 22, 33, \dots, 99, 101, 111, \dots.$$

Now $11 - 9 = 2$ and $22 - 11 = 11$.

We shall show that these are the only two prime values which a y_i term can take.

If x_i and x_{i+1} have different numbers of digits, then x_i will be of the form $99\dots9$ and x_{i+1} of the form $10\dots01$, so $y_i = x_{k+1} - x_i = 2$.

We can consider, without any loss of generality only y_i where x_i has more than two digits since y_i can only be prime if $y_i = 2$ or 11 for x_i with one or two digits from the above list of the first x_i.

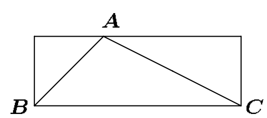
If x_i and x_{i+1} end in the same digit, then 10 divides y_i, so y_i cannot be prime. If x_i and x_{i+1} end in different digits, say r and s, then $s = r + 1$,

x_i is of the form $r999\ldots 9r$, and x_{i+1} is of the form $(r+1)0\ldots 0(r+1)$. Then $y_i = x_{i+1} - x_i = (r+1) - (10-r) = 11$. Thus only two distinct primes belong to the set $\{y_1, y_2, \ldots\}$.

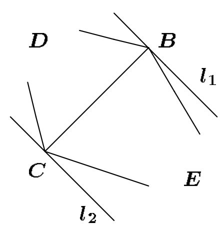
2. (*Mexico*) Show that for any convex polygon of unit area, there exists a parallelogram of area 2 which contains the polygon.

Solution by Mansur Boase, student, St. Paul's School, London, England.

We shall show more generally that there exists a rectangle of area 2 containing the polygon. The result is obviously true for a triangle. To prove this, construct a rectangle on the longest side of the triangle and circumscribing the triangle. If the area of the triangle is 1, then the rectangle will have area 2.

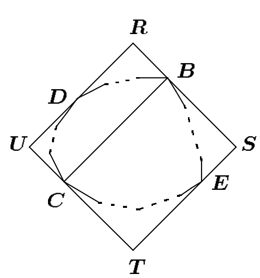


If the polygon has more than three vertices, then choose the two vertices of the polygon which are furthest apart. Call them B and C. Draw perpendiculars to the line BC at B and at C to give lines l_1 and l_2, respectively.



All the vertices of the polygon must lie between these two lines. (Otherwise there would be two vertices further apart than $|BC|$.)

Now consider the smallest rectangle which circumscribes the polygon and with one pair of opposite sides lying on l_1 and l_2. Suppose this polygon touches the polygon again at D and E.



Let the vertices of the rectangle be R, S, T and U with D on UR and E on ST. Then it is easy to see that

$$[RUCB] = 2[BCD]$$

and

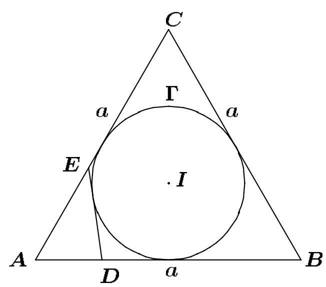
$$[BSTC] = 2[BEC]$$

since $BC \| RU$ and $BC \| ST$. $[RSUT] = 2[CDBE] \leq 2$(area of polygon)$= 2$ since the polygon is convex. We can find an even larger rectangle of area 2 containing the polygon.

4. (*Spain*) Let ABC be an equilateral triangle, and Γ its incircle. If D and E are points of the sides AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Mansur Boase, student, St. Paul's School, London, England. We give Arslanagić's solution.



Let $AB = AC = BC = a$ and $BD = p$; that is, $AD = a - p$, and $CE = q$; that is, $AE = a - q$. The circle Γ is inscribed in the quadrilateral and we get

$$ED + BC = BD + CE$$

or

$$ED + a = p + q$$

or

$$ED = p + q - a. \qquad (*)$$

By the law of cosines for the triangle ADE, it follows that

$$ED^2 = AE^2 + AD^2 - 2AE \cdot AD \cos 60°,$$

so, from $(*)$

$$(p + q - a)^2 = (a - q)^2 + (a - p)^2 - (a - q)(a - p)$$

and from this we obtain

$$a = \frac{3pq}{p + q}.$$

Now, we have

$$AD = a - p = \frac{p(2q - p)}{p + q}$$

and

$$AE = a - q = \frac{q(2p-q)}{p+q};$$

that is,

$$\frac{AD}{DB} + \frac{AE}{EC} = \frac{p}{p}\frac{(2q-p)}{p+q} + \frac{q}{q}\frac{(2p-q)}{p+q} = \frac{p+q}{p+q} = 1,$$

as required.

6. (*Argentina*) Two non-negative integer numbers, a and b, are "cuates" (friends in Mexican) if the decimal expression of $a+b$ is formed only by 0's and 1's. Let A and B be two infinite sets of non-negative integers, such that B is the set of all the numbers which are "cuates" of all the elements of A, and A is the set of all the numbers which are "cuates" of all the elements of B. Show that in one of the sets A or B there are infinitely many pairs of numbers x, y such that $x - y = 1$.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Suppose an integer of A ends with the digit 'r'. Then all integers in B must have a last digit the same as for $10 - r$ or $11 - r$ in order that they are all "cuates" of A. If B contains elements with last digits the same as for $10 - r$ and $11 - r$, then every element of A must end in the last digit r to be "cuates" with integers of B of both last digits. Thus either set A or set B has all integers ending in the same digit. Without loss, assume that all elements of A end in 'r'.

Now consider an element of B which is a "cuate" of all the integers of A. Let us say it is of type (i) if it ends with the last digit $10 - r$ and of type (ii) if it ends with the last digit of $11 - r$. If we change the last digit we obtain another number which is a "cuate" of all the elements of A, and hence in B. The difference between these pairs is 1. It follows therefore that there are equal numbers of each type in B, and as B is infinite, there are infinitely many pairs x, y in B such that $x - y = 1$.

That completes the *Corner* for this month. Send me your Olympiad contest materials and your nice solutions to problems in the *Corner*.