No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

For $0 < x < \pi/2$ prove that

$$\left( \frac{\sin x}{x} \right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

Essentially the same solution was sent in by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; Václav Konečný, Ferris State University, Big Rapids, Michigan, USA; and Heinz-Jürgen Seiffert, Berlin, Germany.

For $0 < x < \frac{\pi}{2}$, we have

$$\left( \frac{\sin x}{x} \right)^2 = \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{(n\pi)^2} \right)^2 < \left( 1 - \frac{x^2}{\pi^2} \right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; LUIS V. DIEULEFAIT, IMPA, Rio de Janeiro, Brazil; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. Two incomplete solutions were received.

Most solvers proved a better inequality, and some commented that the interval could be extended to $(-\pi, \pi)$.

Klamkin and Manes both point out the complimentary inequality (which is not as easy to prove):

$$\frac{1 - t^2}{1 + t^2} \leq \frac{\sin \pi t}{\pi t} \quad \text{for all real } t.$$


Janous gave the extension to: what is the value of $\rho$ that gives the best inequality of the type

$$\left( \frac{\sin x}{x} \right)^2 < \frac{\rho^2 - x^2}{\rho^2 + x^2}$$

which is valid for all $x \in (0, \pi/2)$? We leave this as a challenge to our other readers!
**Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.**

$AB$ is a fixed diameter of circle $\Gamma_1(0, R)$. $P$ is an arbitrary point of its circumference. $Q$ is the projection onto $AB$ of $P$. Circle $\Gamma_2(P, PQ)$ intersects $\Gamma_1$ at $C$ and $D$. $CD$ intersects $PQ$ at $E$. $F$ is the midpoint of $AQ$. $FG \perp CD$, where $G \in CD$. Show that:

1. $EP = EQ = EG$,
2. $A, G$ and $P$ are collinear.

Solution by Toshio Seimiy, Kawasaki, Japan.

1. Let $H$ be the second intersection of $PQ$ with $\Gamma_1$. Since $PH \perp AB$, $AB$ is the perpendicular bisector of $PH$, so that $PQ = QH$ and $\triangle PAQ = \triangle HAQ$. Since $PC = PQ = PD$, we get $\triangle PHC = \triangle PDC = \triangle PCD$, so that $\triangle PHC \sim \triangle PCE$, from which we have

$$PH : PC = PC : PE.$$

Thus we have $PH \cdot PE = PC^2 = PQ^2$. As $PH = 2PQ$, we have $2PQ \cdot PE = PQ^2$, so that $2PE = PQ$; thus, $PE = EQ$.

Since $FG \perp CD$ and $PQ \perp AB$, we have $\triangle GFQ = \triangle PEC$. As $\triangle PHC \sim \triangle PCE$ we get $\angle PEC = \angle PCH = \angle PAH = 2\angle PAQ$. Thus $\angle GFQ = 2\angle PAQ$. Since $F, E$ are midpoints of $AQ, PQ$, we get $FE || AP$, so that $\angle PAQ = \angle EFQ$. Thus $\angle GFQ = 2\angle EFQ$, so that $\angle GFE = \angle EFQ$. Hence we have $\angle GFE \equiv \angle QFE$, so that $EG = EQ$. Therefore $EP = EQ = EG$.

2. Since $\triangle GFE \equiv \triangle QFE$ we have $FG = FQ = AF$, so that $\angle GAF = \frac{1}{2} \angle GFQ = \angle EFQ$. Thus we have $AG || FE$. Since $AP || FE$, $A, G$ and $P$ are collinear.

Also solved by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; and the proposer.

Find, with justification, the positive integer which comes next in the sequence 1411, 4463, 4464, 1412, 4466, 4467, 1413, 4469, ....

[Ed.: the answer is NOT 4470.]

Editor's summary based on the solutions and comments submitted by the solvers whose names appear below.

Most solvers felt (and the editors agree) that the answer could be anything, as stated in the following comment by Murray Klamkin:

*Any number can be the next term! A set of numbers is a sequence mathematically if and only if a rule of formation is given. The given set of numbers is not a sequence and so the problem is meaningless. Given any finite set of n numbers, one can always find an infinite number of formulae which agree with the given n terms, and such that the (n + 1)th term is completely arbitrary.*

However, some solvers did come up with interesting formulae or reasonings to "justify" the answer that they gave. Some examples:

I. (Diminnie) Let \( g_n = n - 3 \lfloor \frac{n}{3} \rfloor \), and let

\[
x_n = \frac{1487 + \lceil n/3 \rceil}{2} - (-1)^{g(n)} \{ (1 + (-1)^{g(n)}) g(n) + 76 ((-1)^{g(n)} - 1) \}.
\]

Then \( x_1, x_2, \ldots, x_8 \) agree with the given terms, and \( x_0 = 4470 \).

In general, if \( k \) is any number and if we define

\[
y_n = x_n + \lceil n/9 \rceil (k - 4470),
\]
then \( y_1, y_2, \ldots, y_8 \) agree with the given terms, and \( y_0 = k \).

II. (Hurthig and the proposer) Squaring each of the given terms reveals

\[
199021, \quad 19918369, \quad 19927296, \quad 1993744, \quad 19945156, \quad 19954089, \quad 1996569, \quad 19971961,
\]

and the given numbers \( a_n (n = 1, 2, \ldots, 8) \), are the least positive integers whose squares begin with the digits 1990, 1991, ..., 1997. That is, \( a_n \) is the smallest positive integer \( k \) such that \( k^2 \) begins with the same digits as 1989 + \( n \).

This leads to \( a_0 = 447 \).

III. (Bradley) The given numbers are the integer parts of the square roots of

\[
1991 \times 10^3, \quad 1992 \times 10^4, \quad 1993 \times 10^4, \quad 1994 \times 10^3, \quad 1995 \times 10^4, \quad 1996 \times 10^4, \quad 1997 \times 10^3, \quad 1998 \times 10^4.
\]
Hence the next term would be \( \sqrt{1999 \times 10^4} \), or 4471.

**IV. (Hess)** Let \( f(n) = \left| n \sqrt{10} - \frac{1}{3} \right| \). The given numbers are

\[
n, f(n) + 2, f(n + 1), n + 1, f(n + 1) + 2, f(n + 2), n + 2, f(n + 2) + 2,
\]

with \( n = 1411 \). Thus the next term is \( f(n + 3) = f(1414) = 4471 \).

Other submitted answers include 4479 (Konečný) and 44610 (Ortega and Gutiérrez).

**Solved by** HAYO AHLBURG, Benidorm, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; SOLEDAD ORTEGA and JAVIER GUTIÉRREZ, students, University of La Roija, Logroño, Spain; and the proposer.

**2171. [1996: 274]** Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Let \( P \) be an arbitrary point taken on an ellipse with foci \( F_1 \) and \( F_2 \), and directrices \( d_1 \), \( d_2 \), respectively. Draw the straight line through \( P \) which is parallel to the major axis of the ellipse. This line intersects \( d_1 \) and \( d_2 \) at points \( M \) and \( N \), respectively. Let \( P' \) be the point where \( MF_2 \) intersects \( NF_1 \).

Prove that the quadrilateral \( PF_1 P' F_2 \) is cyclic.

Does the result also hold in the case of a hyperbola?

**Solution by** Richard I. Hess, Rancho Palos Verdes, California, USA, modified by the editor.

**I:** The Ellipse: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Let \( F_1, F_2 \) be \( \left( \frac{a}{\kappa}, 0 \right) \), \( \left( -\frac{a}{\kappa}, 0 \right) \), respectively, where \( \kappa > 1 \), and \( b = \frac{\sqrt{\kappa^2 - 1}a}. \) Let \( P \) be \( (a \sin \theta, b \cos \theta) \) and \( M \) be \( (\kappa a, b \cos \theta) \).

By symmetry, \( P' \) is on the \( y \)-axis. Let \( P' \) be \( (0, -d) \).
Thus

\[
\frac{d}{a/\kappa} = \frac{b \cos \theta}{\kappa a - a/\kappa},
\]

so that \( d = \frac{b \cos \theta}{\kappa^2 - 1} \).

Choose \( C \) to be the point \((0, \beta)\) such that \( CF_1 = CF_2 = CP' \). Thus

\[
(\beta + d)^2 = \beta^2 + \frac{a^2}{\kappa^2},
\]

so that

\[
\frac{2b\beta \cos \theta}{\kappa^2 - 1} + \frac{b^2 \cos^2 \theta}{(\kappa^2 - 1)^2} = \frac{a^2}{\kappa^2} = \frac{b^2}{\kappa^2 - 1},
\]

giving

\[
\beta = \frac{b}{2 \cos \theta} - \frac{b \cos \theta}{2(\kappa^2 - 1)}.\]

We now show that \( CP = CP' \). If this were true, we would have

\[
(\beta - b \cos \theta)^2 + a^2 = \sin^2 \theta = \left( \beta + \frac{b \cos \theta}{\kappa^2 - 1} \right)^2,
\]

or

\[
-2b\beta \cos \theta + b^2 \cos^2 \theta + \frac{\kappa^2 b^2}{\kappa^2 - 1} \sin^2 \theta = \frac{2b\beta \cos \theta}{\kappa^2 - 1} + \frac{b^2 \cos^2 \theta}{(\kappa^2 - 1)^2},
\]

or

\[
-b^2 + \frac{b^2 \cos^2 \theta}{\kappa^2 - 1} + b^2 \cos^2 \theta + \frac{b^2 \kappa^2}{\kappa^2 - 1} = \frac{b^2}{\kappa^2 - 1} + \frac{\kappa^2 b^2 \cos^2 \theta}{\kappa^2 - 1},
\]

or

\[
\frac{\kappa^2}{\kappa^2 - 1} - 1 = \frac{1}{\kappa^2 - 1},
\]
which is clearly true.

Thus $C$ is the centre of the cyclic quadrilateral $PF_1P'F_2$.

II: The Hyperbola: \[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

Let $F_1, F_2$ be $(\frac{a}{\kappa}, 0), (-\frac{a}{\kappa}, 0)$, respectively, where $0 < \kappa < 1$, and $b = \sqrt{\frac{1-\kappa^2}{\kappa}}a$. Let $P$ be $(a \sinh \theta, b \cosh \theta)$ and $M$ be $(ka, b \cosh \theta)$.

By symmetry, $P'$ is on the $y$-axis. Let $P'$ be $(0, d)$.

Choose $C$ to be the point $(0, \beta)$ such that $CF_1 = CF_2 = CP'$.

Follow the procedure in case I to show that $C$ is the centre of the cyclic quadrilateral $PF_1P'F_2$.

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Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, the Netherlands; and the proposer (for the ellipse only).

Smeenk notes that it is easy to verify that $PP'$ is a normal to the ellipse. Bellot Rosado refers to E.A. Maxwell’s Elementary Coordinate Geometry, Oxford University Press, 1952, which contains the two following related problems:

1. $P, Q$ are two points on an ellipse with foci $S, S'$, such that $PQ$ is perpendicular to $SS'$.
   Prove that $PS, QS, PS', QS'$ touch a circle, and identify its centre.

2. Tangents $TP, TQ$ are drawn to an ellipse with foci $S, S'$. A line through $S$ parallel to $TQ$ meets $S'T$ in $U$, and a line through $S'$ parallel to $TP$ meets $ST$ in $V$.
   Prove that $S, S', U, V$ lie on a circle.
Let \( x, y, z \geq 0 \) with \( x + y + z = 1 \). For fixed real numbers \( a \) and \( b \), determine the maximum \( c = c(a, b) \) such that \( a + bxy \geq c(yz + zx + xy) \).

Solution by Mihai Cipu, Institute of Mathematics, Romanian Academy, Bucharest, Romania.

The answer is

\[
c(a, b) = \min \left( 4a, 3a + \frac{b}{9} \right),
\]

provided that \( a \geq 0 \).

For \( x = y = 1/2, z = 0 \) one obtains \( c \leq 4a \), while by substituting \( x = y = z = 1/3 \) it follows that \( c \leq 3a + b/9 \). Thus \( c(a, b) \leq \min(4a, 3a + b/9) \).

Now to finish the proof we shall show that for all real numbers \( a \geq 0 \) and \( b \),

\[
a + bxy \geq \min \left( 4a, 3a + \frac{b}{9} \right) \cdot (xy + yz + zx)
\]

for all \( x, y, z \geq 0, x + y + z = 1 \).

To this end we shall use the fact that for any such triple \( x, y, z \) there exists a Euclidean triangle whose sides have lengths \( 1 - x, 1 - y, 1 - z \). The triangle is degenerate if \( xyz = 0 \). Let us denote by \( r \), resp. \( R \), the radius of the incircle, resp. circumcircle, of the associated triangle. Using well-known formulae and the hypothesis, one easily finds

\[
xyz = r^2 \quad \text{and} \quad xy + yz + zx = r^2 + 4Rr.
\]

Here \( r \) and \( R \) have non-negative values subject to the restrictions \( 16Rr - 5r^2 \leq 1 \) and \( R \geq 2r \). [Editor's note. The hypothesis \( x + y + z = 1 \) means that the associated triangle has semiperimeter \( s = 1 \). Thus (2) follows from the known identities

\[
(s - a_1)(s - a_2)(s - a_3) = r^2s \quad \text{and} \quad \sum (s - a_1)(s - a_2) = r^2 + 4Rr
\]

which hold for any triangle with sides \( a_1, a_2, a_3 \) — see for example equations (15) and (16), page 54 of Mitrinović, Pečarić and Volenec, Recent Advances in Geometric Inequalities. And the restriction \( 16Rr - 5r^2 \leq 1 \) is just the known inequality \( 16Rr - 5r^2 \leq s^2 \); see (3.6) on page 166 of Recent Advances, or item 5.8 of Bottema et al., Geometric Inequalities.]

We note that for \( x = y = 0, z = 1 \) one gets \( a \geq 0 \) [else there is no solution for \( c \)]. Using this fact, in the case \( b \geq 9a \) we have

\[
\min(4a, 3a + b/9) = 4a
\]

and

\[
a + br^2 \geq a(1 + 9r^2) \geq a(4r^2 + 16Rr),
\]
so that (1) holds. In the opposite case \( b \leq 9a \) we get
\[
a(1 - 3r^2 - 12br) \geq a(4Rr - 8r^2) \geq (4Rr - 8r^2)(r/9)
\]
[since \( 4Rr - 8r^2 = 4r(R - 2r) \geq 0 \)], or equivalently
\[
a + br^2 \geq (3a + b/9)(r^2 + 4Rr),
\]
which is (1) again.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and the proposer. One incorrect solution was sent in.

Most solvers noted that a solution exists only if \( a \geq 0 \).


Let \( n \geq 2 \) and \( x_1, \ldots, x_n > 0 \) with \( x_1 + \ldots + x_n = 1 \).

Consider the terms
\[
l_n = \sum_{k=1}^{n} (1 + x_k) \sqrt{1 - \frac{x_k}{x_k}}
\]
and
\[
r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - \frac{x_k}{x_k}}}
\]
where
\[
C_n = (\sqrt{n-1})^{n+1}(\sqrt{n})^n/(n+1)^{n-1}.
\]

1. Show \( l_2 \leq r_2 \).

2. Prove or disprove: \( l_n \geq r_n \) for \( n \geq 3 \).

1. Solution to Part 1 by Richard I. Hess, Rancho Palos Verdes, California, USA.

For \( n = 2 \), we get \( C_2 = 2/3 \) and [since \( x_1 + x_2 = 1 \)]
\[
l_2 = (1 + x_1) \sqrt{\frac{x_2}{x_1}} + (1 + x_2) \sqrt{\frac{x_1}{x_2}} = \frac{x_1 + x_2 + 2x_1x_2}{\sqrt{x_1x_2}},
\]
\[
r_2 = \frac{2}{3} \left( \frac{1 + x_1}{\sqrt{x_2}} \right) \left( \frac{1 + x_2}{\sqrt{x_1}} \right).
\]
Thus
\[
\sqrt{x_1 x_2 (r_2 - l_2)} = \frac{2}{3} (1 + x_1 + x_2 + x_1 x_2) - x_1 - x_2 - 2x_1 x_2
\]
\[
= \frac{1}{3} (1 - 4x_1 x_2)
\]
\[
= \frac{1}{3} [(x_1 + x_2)^2 - 4x_1 x_2]
\]
\[
= \frac{1}{3} (x_1 - x_2)^2 \geq 0.
\]
Therefore \( l_2 \leq r_2 \) with equality if and only if \( x_1 = x_2 = 1/2 \).

II. Partial solution to Part 2 by the proposer.

We show that \( l_n \geq r_n \) in the case \( n = 3 \) (which indeed was the starting point for the whole problem).

Putting \( x_1 = x \), \( x_2 = y \), \( x_3 = z \), the desired inequality \( l_3 \geq r_3 \) reads
\[
(1 + x) \sqrt{\frac{1-x}{x}} + (1 + y) \sqrt{\frac{1-y}{y}} + (1 + z) \sqrt{\frac{1-z}{z}} \geq \frac{3\sqrt{3}}{4} \cdot \frac{(1 + x)(1 + y)(1 + z)}{\sqrt{1-x} \sqrt{1-y} \sqrt{1-z}}
\]
(1)
where \( x, y, z \in (0, 1) \) such that \( x + y + z = 1 \).

We now recall the difficult Crux problem 2029 of Jun-hua Huang, solved by Kee-Wai Lau on [1996: 129]:
\[
w_b w_c + w_c w_a + w_a w_b \geq 3F \sqrt{3},
\]
(2)
where \( w_a, w_b, w_c, F \) are the angle bisectors and the area of a triangle. We claim that this inequality is equivalent to inequality (1). Indeed, let us apply the transformation \( a = y + z, b = z + x, c = x + y \) where \( x, y, z > 0 \), converting any triangle inequality into an algebraic inequality valid for positive numbers. Then it’s not difficult to see that \( x + y + z = s \) (the semiperimeter of the triangle), whence \( x = s - a, y = s - b \) and \( z = s - c \). Furthermore, due to homogeneity we may and do put \( s = 1 \), whence \( F = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz} \). Also, by the known formula
\[
w_c = \frac{2ab \cos(C/2)}{a+b} = \frac{2ab}{a+b} \sqrt{\frac{s(s-c)}{ab}} = \frac{2}{a+b} \sqrt{\frac{s(s-c)ab}{s}}
\]
(for example, see [1995: 321]) we get (using \( x + y + z = 1 \))
\[
w_c = \frac{2}{x+y+z} \sqrt{z(y+z)(z+x)} = \frac{2}{1+z} \sqrt{z(1-x)(1-y)},
\]
and similarly for \( w_a \) and \( w_b \). Hence (2) is equivalent to

\[
\sum_{\text{cyclic}} 4 \cdot \frac{\sqrt{xy\sqrt{(1-y)(1-x)(1-z)}}}{(1+x)(1+y)} \geq 3\sqrt{xyz},
\]

which is equivalent to (1). Since (2) is true, so is (1).

For \( n \geq 4 \) I do not have any idea of how to settle whether the inequality \( l_n \geq r_n \) is true. It may be interesting and useful to see a purely algebraic proof of inequality (1).

Part 1 also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and the proposer. The proposer did Part 2 only in the case \( n = 3 \) (given above). One other reader sent in a solution to Part 2 which the editor considers to be faulty. Readers are invited to try finishing off this problem completely, or even just the special case \( n = 4 \).


Let \( A \) be an \( n \times n \) matrix. Prove that if \( A^{n+1} = 0 \) then \( A^n = 0 \).

Solution by John C. Tripp, Southeast Missouri State University, Cape Girardeau, Missouri.

We consider \( A \) as a linear transformation on an \( n \)-dimensional vector space. We assume that \( A^{n+1} = 0 \). Let \( x \) be any element of the vector space. The set of vectors

\[
V = \{x, A x, A^2 x, \ldots, A^n x\}
\]

has \( n+1 \) elements, so it is linearly dependent. Let \( k \) be the smallest non-negative integer such that \( A^k x \) is a linear combination of the other vectors in \( V \). We have

\[
A^k x = c_1 A^{k+1} x + c_2 A^{k+2} x + c_3 A^{k+3} x + \cdots + c_{n-k} A^n x,
\]

for some scalars \( c_1, c_2, c_3, \ldots, c_{n-k} \), and

\[
A^n x = A^{n-k} A^k x
\]

\[
= A^{n-k} (c_1 A^{k+1} x + c_2 A^{k+2} x + c_3 A^{k+3} x + \cdots + c_{n-k} A^n x)
\]

\[
= c_1 A^{n+1} x + c_2 A^{n+2} x + c_3 A^{n+3} x + \cdots + c_{n-k} A^{2n-k})
\]

\[
= A^{n+1} (c_1 x + c_2 A x + c_3 A^2 x + \cdots + c_{n-k} A^{n-k-1} x) = 0.
\]

Since \( x \) was arbitrary, we have \( A^n = 0 \).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, Romanian Academy, Bucharest, Romania; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; LUZ M. DeALBA, Drake University, Des Moines, Iowa; F.J.
FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; E. RAPPOS, University of Cambridge, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Most solvers used the Cayley-Hamilton Theorem. Wang comments that this problem is a special case of a more general and well-known result which states that “if $A$ is an $n \times n$ complex matrix such that $A^k = 0$ for some $k \geq 1$ (that is, $A$ is nilpotent), then $A^n = 0$”. Indeed, some solvers proved this more general result.


The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \quad \frac{1}{6} = \frac{1}{3} - \frac{1}{2}; \quad \frac{1}{6} = \frac{1}{4} - \frac{1}{12}; \quad \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$  

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y},$$

where $x$ and $y$ are positive integers?

Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, Oregon, USA.

Notice that

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y}$$

so that

$$x = \frac{2175y}{y + 2175} = \frac{2175^2}{y + 2175}. \quad \therefore x = 2175 - \frac{2175^2}{y + 2175}.$$  

Thus $x$ will be an integer if and only if $y + 2175$ is a factor of $2175^2$, and $x$ will be positive whenever $y$ is, since then $2175^2/(y + 2175) < 2175$, and $y$ will be positive whenever $y + 2175 > 2175$, so we seek factors of $2175^2$ which exceed 2175. But $2175^2 = 3^2 \cdot 5^4 \cdot 29^2$ has $(2+1)(4+1)(2+1) = 45$ positive factors, one of which is its square root, 2175. Since the factors of $2175^2$ come in pairs whose product is $2175^2$, exactly half of the other 44 factors exceed 2175, giving 22 solutions in positive integers. The smallest is $x = 300, y = 348$. This immediately generalizes to the solution in positive
integers of $1/n = 1/x - 1/y$. Since $\tau(n^2)$ (the number of divisors of $n^2$) is odd for a perfect square, there will be $(\tau(n^2) - 1)/2$ solutions to the equation $1/n = 1/x - 1/y$.

Also solved by HAYO AHLBURG, Benidorm, Spain; SAM BAETHGE, Nordheim, Texas, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGEHLAUF, Franz-Ludwig-Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HES, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAIL LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; LAMARR WIDMER, Messiah College, Grantham, PA, USA; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were 4 incorrect solutions.


2176. [1996: 275] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Prove that

$$\sqrt[n]{\prod_{k=1}^{n} (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k}$$

where $a_1, a_2, \ldots, a_n > 0$ and $n \in \mathbb{N}$.

Solution by Sai C. Kwok, San Diego, CA, USA.

Using the arithmetic-geometric mean inequality, we have

$$\left( \prod_{k=1}^{n} \frac{a_k}{a_k + b_k} \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{a_k + b_k}$$

and

$$\left( \prod_{k=1}^{n} \frac{b_k}{a_k + b_k} \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{b_k}{a_k + b_k}$$

The result follows by adding the above two inequalities.
Also solved by THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; †MIHAI CIPU, Romanian Academy, Bucharest, Romania, and Concordia University, Montreal, Quebec; ROBERT GERETSCHLAGER, Bundesrealgymnasium, Graz, Austria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; †WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; †CAN ANH MINH, student, University of California Berkeley, Berkeley, CA, USA; †SOLEDAD ORTEGA and JAVIER GUTIÉRREZ, students, University of La Rioja, Logroño, Spain; WALDEMAR POMPE, student, University of Warsaw, Poland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; †HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; and the proposer. A proof for the case \( n = 4 \) was submitted by V. N. Murty. (The symbol † before a solver’s name indicates that the solver’s solution was virtually the same as the one highlighted above.)

Clearly, the condition that \( b_1, b_2, \ldots, b_n > 0 \) was inadvertently left out from the original statement. All solvers assumed, explicitly or implicitly, that this was the case. However, only Hess gave a simple example to show that the inequality need not be true without the aforementioned condition: take \( n = 2, a_1 = a_2 = 1 \) and \( b_1 = b_2 = -1 \).

Janous pointed out that equality holds if and only if the vectors \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) are proportional.

Klamkin commented that the given inequality is an immediate special case of Jensen’s generalization of Hölder’s Inequality, and referred readers to D. S. Mitrinović et al., Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, 1989, pp. 50–54.

Konečný remarked that it is known that if \( A \) and \( B \) are \( n \times n \) positive semi-definite Hermitian matrices, then

\[
\sqrt[\sqrt{n}]{\det(A + B)} \geq \sqrt[\sqrt{n}]{\det A} + \sqrt[\sqrt{n}]{\det B}
\]

(see, for example, Inequalities: Theory of Majorization and its Application by Albert W. Marshall and Ingram Olkin, Academic Press Inc., 1979, p. 475). If we let \( A \) and \( B \) be the \( n \times n \) diagonal matrices with diagonal entries \( a_k \)'s and \( b_k \)'s respectively \((i = 1, 2, \ldots, n)\), then the proposed inequality follows immediately.


$ABCD$ is a convex quadrilateral, with $P$ the intersection of its diagonals and $M$ the mid-point of $AD$. $MP$ meets $BC$ at $E$. Suppose that $BE : EC = (AB)^2 : (CD)^2$. Characterize quadrilateral $ABCD$.

Solution by Gottfried Perz, Pestalozzizgymnasium, Graz, Austria.

Let

$$
\epsilon = \angle MPA = \angle EPC,
$$

$$
\phi = \angle BPE = \angle DPM.
$$

Then, applying the Sine Rule to the triangles $\triangle APM$ and $\triangle DPM$, we get

$$
\sin \epsilon = \frac{AM \cdot \sin \angle AMP}{AP},
$$

whence

$$
\frac{\sin \epsilon}{\sin \phi} = \frac{DP}{AP}.
$$

Applying the law of sines to the triangles $\triangle CPE$, and $\triangle BPE$, we get

$$
\sin \epsilon = \frac{CE \cdot \sin \angle CEP}{CP},
$$

$$
\sin \phi = \frac{BE \cdot \sin(180^\circ - \angle CEP)}{BP} = \frac{BE \cdot \sin \angle CEP}{BP},
$$

whence

$$
\frac{\sin \epsilon}{\sin \phi} = \frac{CE \cdot BP}{BE \cdot CP} = \frac{CD^2 \cdot BP}{AB^2 \cdot CP}.
$$

From (1), (2) follows that quadrilateral $ABCD$ has the desired property if and only if

$$
\frac{AB^2}{CD^2} = \frac{AP \cdot BP}{CP \cdot DP} = \frac{[ABP]}{[CDP]} = \frac{AB \cdot d(P, AB)}{CD \cdot d(P, CD)}
$$

(where $[XYZ]$ denotes the area of $\triangle XYZ$ and $d(U, VW)$ the distance of $U$ from $VW$); that is, if

$$
$$
This implies that $ABCD$ is characterized by the fact that in the triangles $\triangle ABP$ and $\triangle CDP$, having $\angle APB = \angle CPD$ in common ($ABCD$ is convex), the ratio of the side opposite to $P$ and the altitude passing through $P$ is the same, which means that $\triangle ABP$ and $\triangle CDP$ are directly or inversely similar.

In the first case, we have $\angle BAP = \angle DCP$; that is, $ABCD$ is a trapezoid with parallel sides $AB$ and $CD$. In the second case, we have $\angle BAP (= \angle BAC) = \angle PDC (= \angle BDC)$, whence $ABCD$ is an inscribed quadrilateral ($A$ and $D$ are at the same side of $BC$).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; WALDEMAR POMPE, student, University of Warsaw, Poland; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. Four incomplete or incorrect solutions were received.


If $A, B, C$ are the angles of a triangle, prove that

$$\sin A \sin B \sin C \leq 8 (\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A + \sin^3 C \cos A \cos B)$$

$$\leq 3\sqrt{3} (\cos^2 A + \cos^2 B + \cos^2 C).$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

I shall prove a stronger version with $6 \sin A \sin B \sin C$ of the left side. We use the following known identities and inequalities:

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C \quad (1)$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C \quad (2)$$

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8} \quad (3)$$

$$\cos A \cos B \cos C \leq \frac{1}{8} \quad (4)$$

where $A, B, C$ are the angles of a triangle. Therefore

$$\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A + \sin^3 C \cos A \cos B$$

$$= \sin A (1 - \cos^2 A) \cos B \cos C + \sin B (1 - \cos^2 B) \cos C \cos A$$

$$+ \sin C (1 - \cos^2 C) \cos A \cos B$$
\[
= -\cos A \cos B \cos C (\sin A \cos A + \sin B \cos B + \sin C \cos C) \\
+ \sin A \cos B \cos C + \cos A (\sin B \cos C + \sin C \cos B) \\
= -\frac{1}{2} \cos A \cos B \cos C (\sin 2A + \sin 2B + \sin 2C) \\
+ \sin A (\cos B \cos C + \cos A) \\
= -2 \cos A \cos B \cos C \sin A \sin B \sin C + \sin A \sin B \sin C \\
= \sin A \sin B \sin C (1 - 2 \cos A \cos B \cos C) \\
= \sin A \sin B \sin C (\cos^2 A + \cos^2 B + \cos^2 C).
\]

Combining (2) and (4) yields
\[
\cos^2 A + \cos^2 B + \cos^2 C \geq \frac{3}{4}.
\]

By using this inequality and (3) we get
\[
6 \sin A \sin B \sin C \leq 8(\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A \\
+ \sin^3 C \cos A \cos B) \\
= 8 \sin A \sin B \sin C (\cos^2 A + \cos^2 B + \cos^2 C) \\
\leq 3\sqrt{3}(\cos^2 A + \cos^2 B + \cos^2 C)
\]
as we wanted to show. Equality holds as in (3) and (4) if and only if 
\[A = B = C = 60^\circ.\]

Also solved by ŞEKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous, in his solution, used a new identity, which is presented as a problem 2279 in this issue.

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