Christopher Small writes:

I notice that in the Hints - 2 section of the Bernoulli Trials, the statement that the hint does not work for question 9 has been interpolated.

It seems to me that the statement is correct as I originally gave it to you. A counterexample is easily found with \( a = c \). For example, \( a = c = 1 \) and \( b = 1/2 \) is a counterexample.

This month, we present solutions to some of the problems in a university entrance scholarship examination paper from the 1940's, which appeared in the April 1997 issue of *CRUX with MAYHEM*.

1. Find all the square roots of

\[
1 - x + \sqrt{22x - 15 - 8x^2}.
\]

**Solution.**

First we note that the only way to generate the given expression as a square, is to square

\[
\sqrt{ax + b + \sqrt{cx + d}}.
\]

There are, of course, two square roots, being \( \pm \) this quantity. So we have

\[
\left(\sqrt{ax + b + \sqrt{cx + d}}\right)^2
= (a + c)x + (b + d) + 2\sqrt{acx^2 + (ad + bc)x + bd}
= -x + 1 + 2\sqrt{-2x^2 + \frac{11}{2}x - \frac{15}{4}}.
\]
Since this is an identity, we equate coefficients, yielding

\begin{align*}
a + c &= -1, & (1) \\
b + d &= 1, & (2) \\
ac &= -2, & (3) \\
ad + bc &= \frac{11}{2}, & (4) \\
bd &= -\frac{15}{4}. & (5)
\end{align*}

Solving (1) and (2) for $c$ and $d$ respectively, and substituting in (3) and (5) gives

\begin{align*}
a^2 + a - 2 &= (a + 2)(a - 1) = 0, \\
b^2 - b - \frac{15}{4} &= (b - \frac{5}{2})(b + \frac{3}{2}) = 0.
\end{align*}

so that $a = -2, 1$ and $b = \frac{5}{2}, -\frac{3}{2}$, giving the corresponding values of $c = 1, -2$ and $d = -\frac{3}{2}, \frac{5}{2}$.

We must now substitute these into (4), and this leads to the solution that the square roots of the given expression are

$$\pm \left( \sqrt{-2x + \frac{5}{2}} + \sqrt{x - \frac{3}{2}} \right).$$

2. Find all the solutions of the equations:

\begin{align*}
x + y + z &= 2, \\
x^2 + y^2 + z^2 &= 14, \\
xyz &= -6.
\end{align*}

**Solution.**

First we recall the expressions involving the roots of a cubic:

$$(P - x)(P - y)(P - z) = P^3 - (x + y + z)P^2 + (xy + yz + zx)P - xyz.$$ 

We also note that

$$\frac{(x + y + z)^2 - (x^2 + y^2 + z^2)}{2} = xy + yz + zx.$$ 

So, from two of the given three expressions, we get

$$xy + yz + zx = -5.$$ 

Thus, the solution of the given equations is the set of roots of the cubic equation

$$P^3 - 2P^2 - 5P + 6 = 0.$$
It is easy to check that \( P = 1 \) is a root: so it is easy to factor into
\[
(P - 1)(P + 2)(P - 3) = 0,
\]
giving that the other two roots are \( P = -2 \) and \( P = 3 \).
Thus, the solution set of the given equations is any permutation of 1, -2, 3.

3. Suppose that \( n \) is a positive integer and that \( C_k \) is the coefficient of \( x^k \) in the expansion of \((1 + x)^n \). Show that
\[
\sum_{k=0}^{n} (k+1)C_k^2 = \frac{(n+2)(2n-1)!}{n!(n-1)!}.
\]

Solution.
Consider \( \left( \sum_{k=0}^{n} (k+1)C_k^2 \right)x^n \). Since \( C_{n-k} = C_k \), we can write this as
\[
\sum_{k=0}^{n} ((k+1)C_kx^k)(C_{n-k}x^{n-k}).
\]
This is the term in \( x^k \) in the product
\[
\left( \sum_{k=0}^{n} (k+1)C_kx^k \right) \left( \sum_{k=0}^{n} C_kx^k \right).
\]
The right member of this product is \((1 + x)^n \). There are several ways to determine the left member; for example,
\[
x(1 + x)^n = \sum_{k=0}^{n} C_kx^{k+1},
\]
so that, on differentiating, we have that
\[
(1 + x)^n + nx(1 + x)^n - 1 = \sum_{k=0}^{n} (k+1)C_kx^k.
\]
Thus, the product above is
\[
(1 + x)^n - 1(1 + (n + 1)x) \times (1 + x)^n = (1 + (n + 1)x)(1 + x)^{2n-1}.
\]
The coefficient of \( x^n \) in this product is
\[
\frac{(2n-1)!}{(n-1)!n!(n+1)} + \frac{(2n-1)!}{n!(n-1)!} = \frac{(n+2)(2n-1)!}{n!(n-1)!}.
\]