Packing Boxes with N–tetracubes

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Introduction

With the popularity of the video game *Tetris*, most people are aware of the five connected shapes formed by four unit squares joined edge to edge. They are called the I–, L–, N–, O– and T–tetrominoes, after the letter of the alphabet whose shapes they resemble. They form a subclass of the polyominoes, a favourite topic in research and recreational mathematics founded by Solomon Golomb.

Here is a problem from his classic treatise, *Polyominoes*. Is it possible to tile a rectangle with copies of a particular tetromino? Figure 1 shows that the answer is affirmative for four of the tetrominoes but negative for the N–tetromino, which cannot even fill up one side of a rectangle.

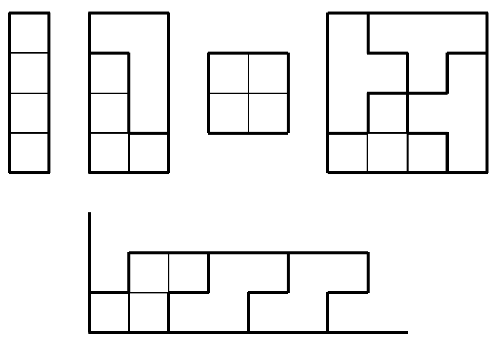

Figure 1

Getting off the plane into space, we can join unit cubes face to face to form polycubes. By adding unit thickness to the tetrominoes, we get five tetracubes, but there are three others. They are shown in Figure 2, along with the N–tetracube.

Is it possible to pack a rectangular block, or box, with copies of a particular tetracube? The answer is obviously affirmative for the I–, L–, O– and T–tetracubes, and it is easy to see that two copies of each of the three tetracubes not derived from tetrominoes can pack a $2 \times 2 \times 2$ box. Will the N–tetracube be left out once again? Build as many copies of it as possible and experiment with them.

If the $k \times m \times n$ box can be packed with the N–tetracube, we call it an N–**box**. Are there any such boxes? Certain types may be dismissed immediately.

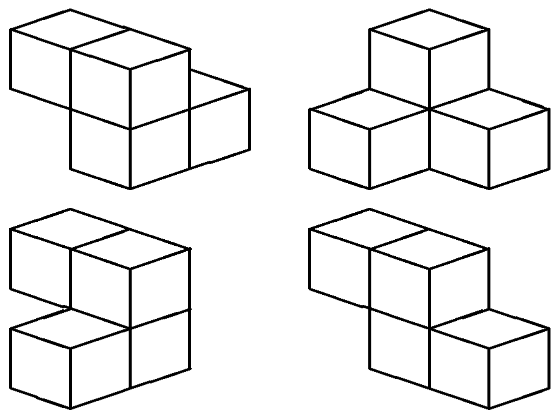

Figure 2

Observation 1.
The $k \times m \times n$ box cannot be an N–box if it satisfies at least one of the following conditions:

(a) one of k, m and n is equal to 1;
(b) two of k, m and n are equal to 2;
(c) kmn is not divisible by 4.

It follows that the $2 \times 3 \times 4$ box is the smallest box which may be an N–box. Figure 3 shows that this is in fact the case. The box is drawn in two layers, and two dominoes with identical labels form a single N–tetracube.

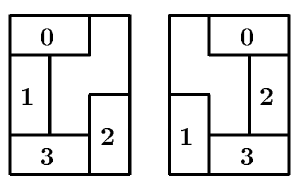

Figure 3

So there is life in this universe after all! The main problem is to find all N–boxes.

N–cubes

If $k = m = n$, the $k \times m \times n$ box is called the k–cube, and a cube which can be packed by the N–tetracube is called an N–**cube**. We can easily assemble the 12–cube with the $2 \times 3 \times 4$ N–box, which makes it an N–cube. This is a special case of the following result.

Observation 2.

Suppose the $k \times m \times n$ and $\ell \times m \times n$ boxes are N–boxes. Let a, b and c be any positive integers. Then the following are also N–boxes:

(a) $(k + \ell) \times m \times n$;
(b) $ak \times bm \times cn$.

The 12–cube is not the smallest N–cube. By Observation 1, the first candidate is $k = 4$. It turns out that this is indeed an N–cube. It can be assembled from the $2 \times 4 \times 4$ N–box, whose construction is shown in Figure 4.

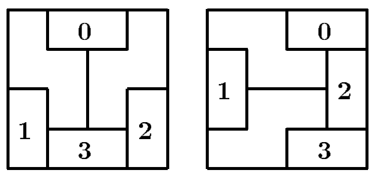

Figure 4

The next candidate, the 6–cube, is also an N–cube, but a packing is not that easy to find. In Figure 5, we begin with a packing of a $2 \times 6 \times 6$ box, with a $1 \times 2 \times 4$ box attached to it. To complete a packing of the 6–cube, add a $2 \times 3 \times 4$ N–box on top of the small box, flank it with two $2 \times 4 \times 4$ N–boxes and finally add two more $2 \times 3 \times 4$ N–boxes.

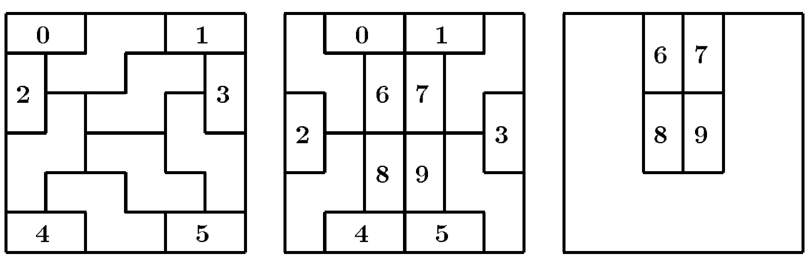

Figure 5

Can we pack the 8–cube, the 10–cube, or others? It would appear that as size increases, it is more likely that we would have an N–cube. However, it is time to stop considering one case at a time. We present a recursive construction which expands N–cubes into larger ones by adding certain N–boxes.

Theorem 1.
The k–cube is an N–cube if and only if k is an even integer greater than 2.

Proof:

That this condition is necessary follows from Observation 1. We now prove that it is sufficient by establishing the fact that if the k–cube is an N–cube, then so is the $(k+4)$–cube. We can then start from either the 4–cube or the 6–cube and assemble all others.

From the $2\times3\times4$ and $2\times4\times4$ N–boxes, we can assemble all $4\times m\times n$ boxes for all even $m,n\geq4$, via Observation 2. By attaching appropriate N–boxes from this collection, we can enlarge the k–cube first to the $(k+4)\times k\times k$ box, then the $(k+4)\times(k+4)\times k$ box and finally the $(k+4)$–cube. This completes the proof of Theorem 1.

Further Necessary Conditions

Observation 1 contains some trivial necessary conditions for a box to be an N–box. We now prove two stronger results, one of which supersedes (c) in Observation 1.

Lemma 1.
The $k\times m\times n$ box is not an N–box if at least two of k,m and n are odd.

Proof:
We may assume that m and n are odd. Place the box so that the horizontal cross-section is an $m\times n$ rectangle. Label the layers L_1 to L_k from bottom to top. Colour the unit cubes in checkerboard fashion, so that in any two which share a common face, one is black and the other is white. We may assume that the unit cubes at the bottom corners are black. It follows that L_i has one more black unit cube than white if i is odd, and one more white unit cube than black if i is even.

Suppose to the contrary that we have a packing of the box. We will call an N–tetracube **vertical** if it intersects three layers. Note that the intersection of a layer with any N–tetracube which is not vertical consists of two or four unit cubes, with an equal number in black and white. The intersection of a vertical N–tetracube with its middle layer consists of one unit cube of each colour.

Since L_1 has a surplus of one black unit cube, it must intersect ℓ_1 vertical N–tetracubes in white and ℓ_1+1 vertical N–tetracubes in black, for some non-negative integer ℓ_1. These N–tetracubes intersect L_3 in ℓ_1 black unit cubes and ℓ_1+1 white ones. Hence the remaining part of L_3 has a surplus of two black. They can only be packed with ℓ_3 vertical N–tetracubes intersecting L_3 in white, and ℓ_3+2 vertical N–tetracubes in black, for some non-negative integer ℓ_3. However, the surplus in black unit cubes in L_5 is now three, and this surplus must continue to grow. Thus the $k\times m\times n$ box cannot be packed with the N–tetracube. This completes the proof of Lemma 1.

Lemma 2.
The $k\times m\times n$ box is not an N–box if kmn is not divisible by 8.

Proof:

Suppose a $k \times m \times n$ box is an N–box. In view of Lemma 1, we may assume that at least two of k, m and n are even. Place the packed box so that the horizontal cross-section is an $m \times n$ rectangle, and label the layers L_1 to L_k from bottom to top. Define vertical N–tetracubes as in Lemma 1 and denote by t_i the total number of those which intersects L_i, L_{i+1} and $L_{i+2}, 1 \leq i \leq k-2$.

Since at least one of m and n is even, each layer has an even number of unit cubes. It follows easily that each t_i must be even, so that the total number of vertical N–tetracubes is also even. The same conclusion can be reached if we place the box in either of the other two non-equivalent orientations. Hence the total number of N–tetracubes must be even, and kmn must be divisible by 8. This completes the proof of Lemma 2.

N–Boxes of Height 2

We now consider $2 \times m \times n$ boxes. By Observation 1, $m \geq 3$ and $n \geq 3$. By Lemma 2, mn is divisible by 4. We may assume that m is even. First let $m = 4$. We already know that the $2 \times 4 \times 3$ and $2 \times 4 \times 4$ boxes are N–boxes. Figure 6 shows that so is the $2 \times 4 \times 5$ box. If the $2 \times 4 \times n$ box is an N–box, then so is the $2 \times 4 \times (n+3)$ box by Observation 2. It follows that the $2 \times 4 \times n$ box is an N–box for all $n \geq 3$.

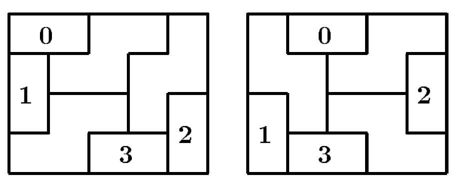

Figure 6

Now let $m = 6$. Then n is even. We already know that the $2 \times 6 \times 4$ box is an N–box. However, the $2 \times 6 \times 6$ box is not. Our proof consists of a long case-analysis, and we omit the details. On the other hand, the $2 \times 6 \times 10$ box is an N–box. In Figure 7, we begin with the packing of a $2 \times 3 \times 6$ box with a $2 \times 2 \times 3$ box attached to it. We then build the mirror image of this solid and complete the packing of the $2 \times 6 \times 10$ box by adding a $2 \times 4 \times 3$ N–box.

If the $2 \times 6 \times n$ box is an N–box, then so is the $2 \times 6 \times (n+4)$ box by Observation 2. It follows that the $2 \times 6 \times n$ box is an N–box for $n = 4$ and all even $n \geq 8$.

Theorem 2.
The $2 \times m \times n$ box is an N–box if and only if $m \geq 3, n \geq 3$ and mn is divisible by 4, except for the $2 \times 6 \times 6$ box.

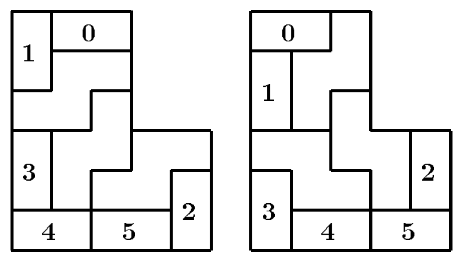

Figure 7

Proof:

If m is divisible by 4, the result follows immediately from Observation 2. Let $m = 4\ell + 2$ for some positive integer ℓ. We already know that the $2 \times 10 \times 6$ box is an N–box. If the $2 \times (4\ell + 2) \times n$ box is an N–box, then so is the $2 \times (4\ell + 6) \times n$ box by Observation 2. This completes the proof of Theorem 2.

The Main Result

Theorem 3.
The $k \times m \times n$ box, $k \leq m \leq n$, is an N–box if and only if it satisfies all of the following conditions:

(a) $k \geq 2$;
(b) $m \geq 3$;
(c) at least two of k, m and n are even;
(d) kmn is divisible by 8;
(e) $(k, m, n) \neq (2, 6, 6)$.

Proof:

Necessity has already been established, and we deal with sufficiency. We may assume that $k \geq 3$, since we have taken care of N–boxes of height 2. Consider all $3 \times m \times n$ boxes. By (c), both m and n must be even. By (d), one of them is divisible by 4. All such boxes can be assembled from the $3 \times 2 \times 4$ box.

Consider now the $k \times m \times n$ box. We may assume that m and n are even. If k is odd, then one of m and n is divisible by 4. Slice this box into one $3 \times m \times n$ box and a number of $2 \times m \times n$ boxes. Since these are all N–boxes, so is the $k \times m \times n$ box.

Suppose k is even. Slice this box into a number of $2 \times m \times n$ boxes, each of which is an N–box unless $m = n = 6$. The $4 \times 6 \times 6$ box may be assembled from the $4 \times 2 \times 3$ box, and we already know that the 6–cube is an N–cube. If the $k \times 6 \times 6$ box is an N–box, then so is the $(k + 4) \times 6 \times 6$ box. This completes the proof of Theorem 3.

Research Projects

Problem 1.
Try to prove that the $2 \times 6 \times 6$ box is not an N–box. It is unlikely that any elegant solution exists.

Problem 2.
An N–box which cannot be assembled from smaller N–boxes is called a *prime* N–box. Find all prime N–boxes.

Problem 3.
Prove or disprove that an N–box cannot be packed if we replace one of the N–tetracubes by an O–tetracube.

Problem 4.
For each of the other seven tetracubes, find all boxes which it can pack.

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