SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


For \( n > 1 \) and \( i = \sqrt{-1} \), prove or disprove that

\[
\frac{1}{4i} \sum_{k=1}^{4n} i^k \tan \left( \frac{k\pi}{4n} \right)
\]

is an integer.

Solution by G.P. Henderson, Campbellcroft, Ontario; (modified slightly by the editor).

Let \( S(n) \) denote the given sum. We show that \( S(n) \) is always an integer. First we need a lemma:

Lemma. Let \( n \) be a positive integer, and let \( g(n) = \sum_{j=1}^{n} \tan \left( \frac{4j - 3}{4n} \right) \).

Then \( g(n) = (-1)^{n+1}n \).

Proof. We first obtain an expansion of \( \tan(n\theta) \). [Ed: see also [1997: 233].] Using the following well-known formulae:

\[
\sin(n\theta) = \binom{n}{1} \sin \theta \cos^{n-1} \theta - \binom{n}{3} \sin^3 \theta \cos^{n-3} \theta + \ldots \quad (1)
\]

\[
\cos(n\theta) = \cos^n \theta - \binom{n}{2} \sin^2 \theta \cos^{n-2} \theta + \binom{n}{4} \sin^4 \theta \cos^{n-4} \theta - \ldots \quad (2)
\]

Dividing both the numerator and the denominator of the fraction \( \frac{(1)}{(2)} \) by \( \cos^n \theta \), and setting \( x = \tan \theta \), we get

\[
\tan(n\theta) = \frac{P(x)}{Q(x)},
\]

where \( P(x) \) and \( Q(x) \) are polynomials in \( x \). Specifically, if \( n \) is odd, then

\[
P(x) = \binom{n}{1} x - \binom{n}{3} x^3 + \ldots + (-1)^{(n-1)/2} \binom{n}{n} x^n,
\]

\[
Q(x) = 1 - \binom{n}{2} x^2 + \ldots + (-1)^{(n-1)/2} \binom{n}{n-1} x^{n-1};
\]

and if \( n \) is even, then the last terms of \( P(x) \) and \( Q(x) \) are

\[
-(-1)^{n/2} \binom{n}{n-1} x^{n-1} \quad \text{and} \quad (-1)^{n/2} \binom{n}{n} x^n,
\]
respectively.

Now consider the equation

\[ \tan(n\theta) = 1 = \frac{P(x)}{Q(x)}. \]  

(3)

The roots of \( \tan(n\theta) = 1 \) are clearly given by \( \theta = (4j - 3)/4n \); that is \( x = \tan((4j - 3)/4n), j = 1, 2, \ldots, n. \) Thus \( g(n) \) is simply the sum of the roots of \( \tan(n\theta) = 1. \)

On the other hand, the roots of \( \frac{P(x)}{Q(x)} = 1 \) are the roots of \( P(x) - Q(x) = 0. \) Hence, when \( n \) is odd, the sum of the roots is

\[ \frac{-[\text{coefficient of } x^{n-1} \text{ in } (-Q(x))]}{[\text{coefficient of } x^n \text{ in } P(x)]} = n = (-1)^{n+1}n; \]

and when \( n \) is even, the sum of the roots is

\[ \frac{[\text{coefficient of } x^{n-1} \text{ in } P(x)]}{[\text{coefficient of } x^n \text{ in } -Q(x)]} = -n = (-1)^{n+1}n. \]

Therefore, \( g(n) = (-1)^{n+1}n. \) The proof of the lemma is complete.

Continuing, set \( A = \{ k \mid 1 \leq k \leq 4n, \gcd(k, n) = 1 \}. \) Note that if \( k \in A, \) then \( 4n - k \in A \) and \( 4n - k \neq k. \) If \( k \) is even, then the sum of the terms corresponding to \( k \) and \( 4n - k \) is zero. If \( k \) is odd, then one of \( k \) and \( 4n - k \) is congruent to 1 and the other to \(-1, \) (all congruences are modulo 4, unless otherwise stated), and the corresponding terms are equal. Therefore

\[ S(n) = \frac{1}{2} \sum_{k \in B} \tan\left(\frac{k\pi}{4n}\right), \]

where \( B = \{ k \mid 1 \leq k \leq 4n, \gcd(k, n) = 1 \} \) and \( k \equiv 1 \pmod{4}. \)

Let \( n = 2^m a \) where \( a \) is odd. If \( a = 1, \) then \( n = 2^m, m \geq 1, \) and \( k \in B \) if and only if \( k = 4j - 3, j = 1, 2, \ldots, n. \) Hence

\[ S(n) = \frac{g(n)}{2} = -\frac{n}{2}, \]

which is an integer.

If \( a \geq 3, \) let \( p_1, p_2, \ldots, p_r, \) denote the distinct (odd) prime divisors of \( a, \) and suppose that \( p_j = b_j, \) where \( b_j = \pm 1 \) for \( j = 1, 2, \ldots, r. \)

To get \( S(n) \) from \( g(n), \) we must subtract the terms with \( k \) divisible by at least one of \( p_1, p_2, \ldots, p_r. \)

Set \( h = h(j_1, j_2, \ldots, j_s) = \sum_k \tan\left(\frac{k\pi}{4n}\right), \) where the summation is over all \( k \) such that \( k \equiv 1, \) and \( k \) is a multiple of \( c = p_{j_1}p_{j_2}\ldots p_{j_s}, \) where \( 1 \leq s \leq r \) and all the \( p \)'s are distinct.
If $c \equiv 1$, then the values of $k$ in the sum are $c, 5c, \ldots, (4d-3)c$, where $d = n/c$. Hence

$$h = \sum_{j=1}^{d} \tan \left( \frac{(4j-3)\pi}{4d} \right) = g(d) = (-1)^{d+1}d$$

$$= (-1)^{n+1} \frac{n}{p_j, p_{j_2} \cdots p_{j_s}}, \quad (4)$$

since $n = cd$ and $c$ being odd, together imply that $d \equiv n \pmod{2}$.

If $c \equiv -1$, then the values of $k$ in the sum are $3c, 7c, \ldots, (4d-1)c$ and

$$h = \sum_{j=1}^{d} \tan \left( \frac{(4j-1)\pi}{4d} \right).$$

Replacing $j$ by $d - j + 1$, we find that

$$h = -g(d) = -(-1)^{d+1}d$$

$$= -(-1)^{n+1} \frac{n}{p_j, p_{j_2} \cdots p_{j_s}}. \quad (5)$$

Since

$$b_{j_1} b_{j_2} \ldots b_{j_s} = \begin{cases} 1 & \text{if } c \equiv 1, \\ -1 & \text{if } c \equiv -1, \end{cases}$$

the answers in formulae (4) and (5) can be combined in the formula:

$$h = (-1)^{n+1} \frac{n b_{j_1} b_{j_2} \ldots b_{j_s}}{p_j, p_{j_2} \cdots p_{j_s}}.$$

It then follows from the Inclusion–Exclusion Principle that:

$$S(n) = \frac{1}{2} \left( g(n) - \sum_{j=1}^{n} h(j) + \sum_{j_1 < j_2} h(j_1, j_2) - \ldots \right)$$

$$= \frac{(-1)^{n+1}n}{2} \left( 1 - \sum_{j=1}^{n} \frac{b_j}{p_j} + \frac{b_{j_1} b_{j_2}}{p_{j_1} p_{j_2}} - \ldots \right)$$

$$= \frac{(-1)^{n+1}n}{2} \left( 1 - \frac{b_1}{p_1} \left( 1 - \frac{b_2}{p_2} \right) \ldots \left( 1 - \frac{b_r}{p_r} \right) \right)$$

$$= \frac{(-1)^{n+1}n (p_1 - b_1)(p_2 - b_2) \ldots (p_r - b_r)}{2p_1 p_2 \ldots p_r}$$

which is clearly an integer. (In fact, it is divisible by $2^{n+2r-1}$ since $p_j - b_j$ is divisible by 4.) This completes the proof.
Also solved by KURT GIRSTMAIR, University of Innsbruck, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA, and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA.

Girstmair, using character coordinates, which play an important role in the study of cyclotomic fields, shows that

\[
S(n) = \begin{cases} 
-T(n) & \text{if } n \text{ is even,} \\
T(n)/2 & \text{if } n \text{ is odd,}
\end{cases}
\]

where \( T(n) = n \prod_{p \mid n} \frac{p-1}{p} \prod_{p \neq n} (p+1) \) with \( p \) running through all prime divisors of \( n \) and the congruences are taken modulo 4. It is not difficult to show that his answer is equivalent to the one given in the solution above.


Similar non-square rectangles are placed outwardly on the sides of a parallelogram \( \pi \). Prove that the centres of these rectangles also form a non-square rectangle if and only if \( \pi \) is a non-square rhombus.

As Florian Herzig and Václav Konečný both noted, the similar non-square rectangles must be placed “nicely”, that is, they must either all have the longest side or all have the shortest side in common with the parallelogram. Otherwise, the conclusion of the problem does not hold!

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let \( i, j \) be unit vectors making an angle \( \theta \) with one another, and let \( i^* \) and \( j^* \) be the images of \( i \) and \( j \) under a \( 90^\circ \) anticlockwise rotation.

Then

\[
i \cdot j = \cos \theta \quad i^* \cdot i = 0 \quad i^* \cdot j = \sin \theta
\]

\[
(\ast)
\]

\[
i^* \cdot j^* = \cos \theta \quad j^* \cdot j = 0 \quad j^* \cdot i = \sin \theta
\]

(The proof is trivial, but these relations will be much used in what follows.)

Let the parallelogram be \( OACB \) with \( \overrightarrow{OA} = 2ai \) and \( \overrightarrow{OB} = 2bj \) and let \( W, X, Y, Z \) be the centres of the non-square similar rectangles (see the figure).
Those centred at $X$ and $Z$ have sides $2a, 2ka$ and the other two $2b, 2kb$, where $k \neq 1$, since they are non-square. Let the position vectors of $W, X, Z$ relative to $O$ be $w, x, z$, respectively. Then it is easy to see that

$$w = bj + k bj^*, x = 2bj + ai = kai^*$$

and

$$z = ai - kai^*.$$ 

Hence

$$\overrightarrow{ZW} = w - z = bj + k bj^* - ai + kai^*$$

and

$$\overrightarrow{WX} = x - w = bj + ai + kai^* - k bj^*.$$ 

From which, using relation (*) repeatedly, we see that

$$\overrightarrow{ZW} \cdot \overrightarrow{WX} = (b^2 - a^2)(1 - k^2)$$

and

$$ZW^2 = WX^2$$ if and only if $2ab \cos \theta(1 - k^2) = 0.$

But $k \neq 1$, so $ZW$ and $WX$ are at right angles if and only if $a = b$ and $ZW = WX$ if and only if $\theta = 90^\circ$.

So $ZWXY$ is a non-square rectangle if and only if $OACB$ is a non-square rhombus.

The proposer notes that this is related to two well-known results.

**Napoleon's Theorem:** The centres of equilateral triangles, placed on the sides of any triangle, form an equilateral triangle.

**Thébault's Theorem:** The centres of squares placed on any parallelogram form a square.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALther JANOUS, Ursulangymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.


$A_1A_2A_3A_4$ is a quadrilateral. Let $B_1, B_2, B_3$ and $B_4$ be points on the sides $A_1A_2, A_2A_3, A_3A_4$ and $A_4A_1$ respectively, such that

$$A_1B_1 : B_1A_2 = A_4B_3 : B_3A_3$$

and

$$A_2B_2 : B_2A_3 = A_1B_4 : B_4A_4.$$
Let \( P_1, P_2, P_3 \) and \( P_4 \) be points on \( B_4B_1, B_1B_2, B_2B_3 \) and \( B_3B_4 \) respectively, such that

\[
P_1 P_2 \parallel A_1A_2, \quad P_2 P_3 \parallel A_2A_3 \quad \text{and} \quad P_3 P_4 \parallel A_3A_4.
\]

Prove that \( P_4 P_1 \parallel A_4A_1 \).

\[
\text{Solution by D.J. Smeenk, Zaltbommel, the Netherlands.}
\]

We denote:

\[
\begin{align*}
A_1A_2 &= a_1, \quad A_2A_3 = a_2, \quad A_3A_4 = a_3, \quad A_4A_1 = a_4; \\
A_1B_1 &= \lambda a_1, \quad B_1A_2 = \mu a_1, \quad \lambda + \mu = 1; \\
A_2B_2 &= \rho a_2, \quad B_2A_3 = \tau a_2, \quad \rho + \tau = 1; \\
A_3B_3 &= \mu a_3, \quad B_3A_4 = \lambda a_3; \\
A_4B_4 &= \tau a_4, \quad B_4A_1 = \rho a_4; \\
\triangle A_4A_1A_2 &= \alpha_1, \quad \triangle A_1A_2A_3 = \alpha_2, \\
\triangle A_2A_3A_4 &= \alpha_3, \quad \triangle A_3A_4A_1 = \alpha_4.
\end{align*}
\]

The distance from \( P_1 \) and \( P_2 \) to \( A_1A_2 \) is \( d_1 \); from \( P_2 \) and \( P_3 \) to \( A_2A_3 \) is \( d_2 \); from \( P_3 \) and \( P_4 \) to \( A_3A_4 \) is \( d_3 \); from \( P_4 \) to \( A_4A_1 \) is \( d_1 \) and from \( P_4 \) to \( A_4A_1 \) is \( d_4 \). It suffices to show: \( d_4 = d_4 \) which implies \( P_4 P_1 \parallel A_4A_1 \).

Consider \( \triangle B_1A_2B_2 \). We see that \([B_1A_2P_2] + [A_2B_2P_2] = [B_1A_2B_2]\) or

\[
\mu a_1 d_1 + \rho a_2 d_2 = \mu \rho a_1 a_2 \sin \alpha_2,
\]

(1)

Similarly, for \( \triangle B_2A_3B_3, \triangle B_3A_4B_4 \) and \( \triangle B_4A_1B_1 \):

\[
\begin{align*}
\tau a_2 d_2 + \mu a_3 d_3 &= \mu \tau a_2 a_3 \sin \alpha_3, \\
\lambda a_3 d_3 + \tau a_4 d_4 &= \lambda \tau a_3 a_4 \sin \alpha_4, \\
\rho a_4 d_4 + \lambda a_1 d_1 &= \lambda \rho a_1 a_4 \sin \alpha_1.
\end{align*}
\]

(2) \quad (3) \quad (4)

Eliminating \( d_2 \) out of (1) and (2), we find:

\[
\mu \tau a_1 d_1 - \mu \rho a_3 d_3 = \mu \rho \tau (a_1 a_2 \sin \alpha_2 - a_2 a_3 \sin \alpha_3).
\]

(5)
Eliminating $d_3$ out of (3) and (5):

$$\lambda a_1 d_1 + \rho a_4 d_4 = \lambda \rho (a_1 a_2 \sin \alpha_2 - a_2 a_3 \sin \alpha_3 + a_3 a_4 \sin \alpha_4).$$

(6)

We rewrite (4):

$$\lambda a_1 d_1 + \rho a_4 d_4' = \lambda \rho a_1 a_4 \sin \alpha_1.$$  

(7)

(6) and (7) imply

$$\rho a_4 (d_4 - d_4') = \lambda \rho (a_1 a_2 \sin \alpha_2 + a_3 a_4 \sin \alpha_4) - \lambda \rho (a_2 a_3 \sin \alpha_3 + a_1 a_4 \sin \alpha_1).$$

(8)

Then

$$2[A_1 A_2 A_3 A_4] = a_1 a_2 \sin \alpha_2 + a_3 a_4 \sin \alpha_4 = a_2 a_3 \sin \alpha_3 + a_1 a_4 \sin \alpha_4.$$  

(9)

(8) and (9) imply $d_4' = d_4$ and $P_4 P_1 \parallel A_4 A_1$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (for a parallelogram); WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; and the proposer.


In the plane are given an arbitrary quadrangle and bisectors of three of its angles. Construct, using only an unmarked ruler, the bisector of the fourth angle.

Solution by Toshio Seimiya, Kawasaki, Japan.

$ABCD$ is a given quadrangle. We denote the bisectors of angles $A, B, C, D$, by $a, b, c, d$, respectively. We assume that $a, b, c$ are given. We shall construct $d$ using only an unmarked ruler.

Case I. $ABCD$ is not a parallelogram. We may assume that $AB$ is not parallel to $CD$.

Construction: Let $O$ be the intersection of $AB, CD$, and let $P$ be the intersection of $b$ and $c$. Join $O$ and $P$, and let $Q$ be the intersection of $a$ and $OP$. Join $D$ and $Q$; then $DQ$ is $d$.

Proof: $P$ is either the incentre or the excentre of $\triangle OBC$ [depending on whether or not the given quadrangle is convex and how its angles are arranged], so that $OP$ is a bisector of $\angle BOC$. Thus $Q$ is the excentre or incentre of $\triangle OAD$ so that $QD$ bisects $\angle ADC$.

Case II. $ABCD$ is a parallelogram. If $ABCD$ is a rhombus then $b$ coincides with $d$ and is the bisector of $\angle D$. We shall assume that $ABCD$ is not a rhombus.

Construction. Let $P$ be the intersection of $b$ and $c$. Draw $AC$ and $BD$, and let $O$ be their intersection. Join $P, O$, and let $Q$ be the intersection of $PO$ with $a$. Join $Q$ and $D$; then $QD$ is the bisector of $\angle D$. 
[Editor's comment. Although Seimiya provides a brief argument that his construction is correct, it simply proves that a parallelogram is symmetric about its centre, which can be left to the reader.]

Also solved by CLAUDIO ARCONCER, Jundiaí, Brazil; NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (Case I only); and the proposer.

2143. [1996: 170] Proposed by B. M***y, Devon, Switzerland.

My lucky number, 34117, is equal to 1662 + 812 and also equal to 1592 + 942, where |166 - 159| = 7 and |81 - 94| = 13; that is, it can be written as the sum of two squares of positive integers in two ways, where the first integers occurring in each sum differ by 7 and the second integers differ by 13.

What is the smallest positive integer with this property?

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let \( N = a^2 + b^2 = (a - 13)^2 + (b + 7)^2 \), where \( a \geq 14 \) and \( b \geq 1 \), be the smallest natural number having the desired property. Simplifying yields

\[
\begin{align*}
 a^2 + b^2 &= a^2 - 26a + 169 + b^2 + 14b + 49, \\
 13a &= 7b + 109 \\
 b &= \frac{13a - 109}{7}.
\end{align*}
\]

As \( b \) is a natural number, \( 13a - 109 \equiv 0 \mod 7 \) which implies \( a \equiv 3 \mod 7 \) [for example, since \( 109 \equiv 39 \mod 7 \)]. Hence the smallest \( a \) is 17, and then \( b = 16 \) and

\[
 N = 17^2 + 16^2 = 4^2 + 23^2 = 545.
\]

We get all solutions by setting \( a = 10 + 7k \) for \( k \geq 1 \):

\[
(10 + 7k)^2 + (3 + 13k)^2 = (7k - 3)^2 + (10 + 13k)^2 = 109(2k^2 + 2k + 1).
\]

The solution mentioned in the proposal is obtained for \( k = 12 \).

From the above equation we see that 109 divides all solutions. And incidentally, the smallest solution which is also a square is

\[
885^2 + 1628^2 = 872^2 + 1635^2 = 1853^2.
\]

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; IAN JUNE L. GARCES, Ateneo de Manila University, Manila, The Philippines, and GIOVANNI MAZZARELLO, Ferrovie
Janous notes that 545 is a palindrome and wonders if there are any other palindromic solutions! Könčný replaces the digits 5 and 4 in the solution by the corresponding letters E and D respectively, and therefore wonders if the proposer’s name is MEDEY. (It is not—in fact, each star represents two letters.)

\[2144\text{. [1996: 170] Proposed by B. M*****y, Devon, Switzerland.}]

My lucky number, 34117, has the interesting property that \(34 = 2 \cdot 17 \) and \(341 = 3 \cdot 117 - 10\); that is,

it is a \(2N + 1\)-digit number (in base 10) for some \(N\), such that

(i) the number formed by the first \(N\) digits is twice the number formed by the last \(N\), and

(ii) the number formed by the first \(N + 1\) digits is three times the number formed by the last \(N + 1\), minus 10.

Find another number with this property.

\[1\text{ Solution by David Hankin, Hunter College High School, New York, NY, USA.} \]

Let \(A\) be the number formed by the first \(N\) digits of the required number, let \(B\) be the number formed by the last \(N\) digits, and let \(x\) be the middle digit. Then

\[A = 2B \text{ and } 10A + x = 3(x \cdot 10^N + B) - 10.\]

From these, we obtain

\[17B = 3x \cdot 10^N - x - 10,\]

so \(x + 10 \equiv 3x \cdot 10^N \mod 17\). Since \(A = 2B\) has \(N\) digits, \(B < 5 \cdot 10^{N-1}\) and so \(17B < 8.5 \cdot 10^N\). Thus by (1), \(1 \leq x \leq 2.\)
When \( x = 1 \), we have \( 3 \cdot 10^N \equiv 11 \mod 17 \), so \( 10^N \equiv 15 \mod 17 \). This is satisfied by \( N = 2 \), and since \( 10^{16} \equiv 1 \mod 17 \) [by Fermat's little theorem], it is satisfied by \( N = 2 + 16k \) for non-negative \( k \). Note that \( x = 1 \), \( N = 2 \) yields the lucky number referred to in the problem.

When \( x = 2 \), we have \( 6 \cdot 10^N \equiv 12 \mod 17 \), so \( 10^N \equiv 2 \mod 17 \). A little arithmetic yields \( N = 10 + 16k \) this time, so the next smallest number with the required property is the one given by \( x = 2 \) and \( N = 10 \). These values yield \( 17B = 6 \cdot 10^{10} - 12 \), from which \( B = 3\,529\,411\,764 \), \( A = 7\,058\,823\,528 \), and a lucky number of

\[
705\,882\,352\,823\,529\,411\,764
\]

(which coincidentally happens to be my lucky number too).

II Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

[Godin first obtained equation (1) and that \( x \) must be 1 or 2, and then his solution, with Hankin's notation, continues as follows.]

When \( x = 1 \), we need to find a natural number of the form

\[
B = \frac{3 \cdot 10^N - 11}{17}.
\]

Now we can calculate

\[
\frac{3}{17} = 0.1764705882352941,
\]

and if we look in the long division for the first occurrence of a remainder of 11, we can construct a solution. It occurs after the first 7, which yields \( B = 17 \), \( A = 34 \) and thus the lucky number 34117 which is the proposer's lucky number. It is also the first in an infinite family of solutions. We can create new ones by appending a full period of the repeating decimal for \( 3/17 \) to the left of the \( B \) of our last solution. Thus the next solution in this family has

\[
B = 1764705882352941\,17, \quad A = 3529411764705882\,34,
\]

and thus is

\[
3529411764705882341\,176470588235294117.
\]

There is another family of solutions with \( x = 2 \), obtained the same way. In this case

\[
B = \frac{6 \cdot 10^N - 12}{17} \quad \text{and} \quad \frac{6}{17} = 0.3529411764705882
\]

with a remainder of 12 occurring first at the second 4, so

\[
B = 3529411764, \quad A = 7058823528,
\]
and the lucky number is

7058823528 2 3529411764.

Similarly this is the first solution in an infinite family of solutions. The others are obtained by appending a full period of $6/17$ to the left of the $B$ of the last solution.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABELÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward’s School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinen-Gymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon, USA; AMIT KHETAN, Troy, Michigan, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. One incorrect solution was sent in.

Notice from Godin’s solution that the fractions with denominator 17 all have the same digits in their repeating parts, just “cycled around”. This is the same behaviour that readers will likely know from the fractions $1/7, 2/7$, etc. In both cases the reason for this behaviour is that the repeating parts have the largest possible number of digits, 16 in the case of denominator 17. The consequence for this problem is that the solutions have “repeated blocks” in them, which some solvers remarked upon.