THE SKOLIAD CORNER

No. 22

R. E. Woodrow

We begin this number by giving the problems of the Twelfth W. J. Blundon Contest, which was written February 22, 1995. This contest is prepared at the Memorial University of Newfoundland and is sponsored by the Canadian Mathematical Society.

THE TWELFTH W. J. BLUNDON CONTEST

February 22, 1995

1. (a) From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this, 45 boys leave. There are then 5 girls for each boy. How many boys and how many girls were in the original group?

(b) A certain number of students can be accommodated in a hostel. If two students share each room then two students will be left without a room. If 3 students share each room then two rooms will be left over. How many rooms are there?

2. How many pairs of positive integers \((x, y)\) satisfy the equation

\[
\frac{x}{19} + \frac{y}{95} = 1?
\]

3. A book is to have 250 pages. How many times will the digit 2 be used in numbering the book?

4. Without using a calculator

(a) Show that \(\sqrt{7 + \sqrt{48}} + \sqrt{7 - \sqrt{48}}\) is a rational number.

(b) Determine the largest prime factor of 9919.

5. A circle is inscribed in a circular sector which is one sixth of a circle of radius 1, and is tangent to the three sides of the sector as shown. Calculate the radius of the inscribed circle.

6. Determine the units digit of the sum

\[26^{26} + 33^{33} + 45^{45}.\]
7. Find all solutions \((x, y)\) to the system of equations

\[
x + y + \frac{x}{y} = 19
\]

\[
x(x + y) \frac{y}{60} = 60.
\]

8. Find the number of different divisors of 10800.

9. Show that \(n^4 - n^2\) is divisible by 12 for any positive integer \(n > 1\).

10. Two clocks now indicate the correct time. One gains a second every hour, and the other gains 3 seconds every 2 hours. In how many days will both clocks again indicate the correct time?

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Last issue we gave the 1995 Manitoba Mathematical Contest. Here are the solutions.

THE MANITOBA MATHEMATICAL CONTEST 1995

For Students in Grade 12

Wednesday, February 22, 1995 — Time: 2 hours

1. (a) If \(a\) and \(b\) are real numbers such that \(a + b = 3\) and \(a^2 + ab = 7\) find the value of \(a\).

Solution. \(a^2 + ab = a(a + b) = a \cdot 3 = 7\) so \(a = 7/3\).

(b) Noriko's average score on three tests was 84. Her score on the first test was 90. Her score on the third test was 4 marks higher than her score on the second test. What was her score on the second test?

Solution. \(\frac{90 + s + (s+4)}{3} = 84\) so \(2s + 94 = 3 \times 84 = 252\), \(2s = 158\) and the score on the second test was 79.

2. (a) Find two numbers which differ by 3 and whose squares differ by 63.

Solution. Let the smaller number be \(x\). The larger number is then \(x + 3\). Now \(|(x + 3)^2 - x^2| = 63\). Thus \(|x^2 + 6x + 9 - x^2| = 63\) or \(|6x + 9| = 63\). Now \(6x + 9 = 63\) OR \(6x + 9 = -63\). Thus \(x = 7\) OR \(x = -8\).

(b) Find the real number which is a root of the equation:

\[27(x - 1)^3 + 8 = 0.\]

Solution. \(27(x - 1)^3 + 8 = 0\) is equivalent to \((3(x - 1))^3 = -8 = (-2)^3\). As \(-8\) has only one real cube root \(3(x - 1) = -2\) so \(x = 1/3\).
3. (a) Two circles lying in the same plane have the same centre. The radius of the larger circle is twice the radius of the smaller circle. The area of the region between the two circles is 7. What is the area of the smaller circle?

Solution. Let the radius of the smaller circle be \( r \). Then the radius of the larger circle, \( R = 2r \). The areas of the circles are \( \pi r^2 \) and \( \pi R^2 \) respectively. Thus the area of the region between them is

\[
\pi R^2 - \pi r^2 = 7,
\]

Thus \( \pi (2r)^2 - \pi r^2 = 7 \),

\[
(4-1)r^2 = 7,
\]

\[
\pi r^2 = \frac{7}{3}.
\]

The area of the smaller circle is \( \frac{7}{3} \).

(b) The area of a right triangle is 5. Also, the length of the hypotenuse of this triangle is 5. What are the lengths of the other two sides?

Solution.

Let the lengths of the two legs be \( x \) and \( y \), with \( x \geq y \). Then \( \frac{1}{2}xy = 5 \) and \( x^2 + y^2 = 25 \) from the formula for the area, and Pythagoras. So \( 2xy = 20 \) and

\[
(x+y)^2 = x^2 + y^2 + 2xy = 45, (x-y)^2 = 5,
\]

giving \( x + y = 3\sqrt{5} \), \( x - y = \sqrt{5} \), so \( x = 2\sqrt{5} \) and \( y = \sqrt{5} \). The lengths of the other two sides are \( \sqrt{5} \) and \( 2\sqrt{5} \).

4. (a) The parabola whose equation is \( 8y = x^2 \) meets the parabola whose equation is \( x = y^2 \) at two points. What is the distance between these two points?

Solution. For the intersection points \( 8y = x^2 \) and \( x = y^2 \) giving \( 8y = (y^2)^2 \) or \( y(y^3 - 8) = 0 \), so \( y = 0 \) and \( x = 0 \) or \( y = 2 \) and \( x = 4 \). The intersection points are \( (0,0) \) and \( (4,2) \). The distance is \( \sqrt{4^2 + 2^2} = 2\sqrt{5} \).

(b) Solve the equation \( 3x^3 + x^2 - 12x - 4 = 0 \).

Solution.

\[
3x^3 + x^2 - 12x - 4 = 0,
\]

\[
x^2(3x + 1) - 4(3x + 1) = 0,
\]

\[
(3x + 1)(x^2 - 4) = 0.
\]

The three solutions are \( x = -2, -1/3, 2 \).
5. (a) Find the real number $a$ such that $a^4 - 15a^2 - 16 = 0$ and $a^3 + 4a^2 - 25a - 100 = 0$.

Solution.

$$a^4 - 15a^2 - 16 = 0,$$
$$a^3 + 4a^2 - 25a - 100 = 0.$$

Now for real $a$, $a^2 + 1 \neq 0$ so $a^2 - 16 = 0$ and $a = \pm 4$.

Substituting these values in the other equation gives:

for $a = -4$, $(-4)^3 + 4(-4)^2 - 25(-4) - 100 = 0$,

and $a = -4$ is a solution;

for $a = 4$, $(4^3 + 4(4)^2 - 25(4) - 100 = 138 - 200 \neq 0$.

The only real solution is $a = -4$.

(b) Find all positive numbers $x$ such that $x^{\sqrt{x}} = (x^{\sqrt{x}})^x$.

Solution. Now $x^{\sqrt{x}} = (x^{\sqrt{x}})^x$, so $x^{\sqrt{x}} = (x^{3/2})^x = x^{3x}$.

Equating exponents:

$$x^{\sqrt{x}} = \frac{3}{2} x, \text{ so } \sqrt{x} = \frac{3}{2} \text{ and } x = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

since $x > 0$.

6. If $x$, $y$ and $z$ are real numbers prove that

$$(xy - y|x|)(y|z| - z|y|)(x|z| - z|x|) = 0.$$

Solution. Consider the left hand side. If any of $x$, $y$, $z$ equal $0$, say $x$ then $(x|y| - y|x|) = (0 - 0) = 0$, and the equation holds. If none of $x$, $y$, $z$ are zero then two must have the same sign. Without loss of generality we may consider the two cases $x > 0$ and $y > 0$ and $x < 0$, $y < 0$.

If $x, y > 0$ then $x|y| - y|x| = xy - xy = 0$.

If $x, y < 0$ then $x|y| - y|x| = x(-y) - y(-x) = -xy + xy = 0$.

In any case the left hand side has one of the terms $0$, so the product is $0$ and the identity holds.
7. \( x \) and \( y \) are integers between 10 and 100. \( y \) is the number obtained by reversing the digits of \( x \). If \( x^2 - y^2 = 495 \), find \( x \) and \( y \).

**Solution.** Let the digits of the numbers be \( a, b \), with \( x = 10a + b \) and \( y = a + 10b \) (so \( 9 \geq a > b \geq 1 \)). Now

\[
x^2 - y^2 = (10a + b)^2 - (a + 10b)^2
= ((10a + b) + (a + 10b))((10a + b) - (a + 10b))
= 11(a + b) \cdot 9(a - b)
= 99(a^2 - b^2) = 495.
\]

Thus \( a^2 - b^2 = (a + b)(a - b) = 5 \) and \( a + b = 5, a - b = 1 \) so \( a = 3, b = 2 \). The numbers \( x \) and \( y \) are 32 and 23, respectively.

8. Three points \( P, Q \) and \( R \) lie on a circle. If \( PQ = 4 \) and \( \angle PRQ = 60^\circ \), what is the radius of the circle?

**Solution.**

Let the radius of the circle be denoted by \( r \), and the centre \( O \). Now \( \angle POQ = 2\angle PRQ = 120^\circ \). Let the bisector of \( PQ \) be at \( M \). Now \( OM \perp MQ \) and \( \angle MOQ = \frac{1}{2}\angle POQ = 60^\circ \), so

\[
\frac{MQ}{OQ} = \frac{2}{r} = \sin 60^\circ = \frac{\sqrt{3}}{2}.
\]

Thus \( r = \frac{4\sqrt{3}}{3} \). The radius is \( \frac{4\sqrt{3}}{3} \).

9. Three points are located in the finite region between the \( x \)-axis and the graph of the equation \( 2x^2 + 5y = 10 \). Prove that at least two of these points are within a distance 3 of each other.

**Solution.**

Consider the rectangles \( R_1, R_2 \) with vertices \((-\sqrt{5}, 2), (-\sqrt{5}, 0), (0, 0), (0, 2)\) and \((0, 2), (0, 0), (\sqrt{5}, 0), (\sqrt{5}, 2)\) respectively. Every point of the region lies in one of the two boxes (or both). The diameter of each box is \( \sqrt{(\sqrt{5})^2 + 2^2} = 3 \). If three points in the region are given, two must lie in one of \( R_1 \) or \( R_2 \) by the Pigeonhole Principle and cannot be further than 3 units apart.
10. Three circles pass through the origin. The centre of the first circle lies in the first quadrant, the centre of the second circle lies in the second quadrant, and the centre of the third circle lies in the third quadrant. If $P$ is any point that is inside all three circles, show that $P$ lies in the second quadrant.

Solution.

The key is to consider the third circle. If its centre is in the third quadrant and it passes through the origin then the only point in common with the first quadrant is the origin itself, so no point inside could be in the first quadrant. Similarly no point inside the first circle can be in the third quadrant and no point in the second circle lies in the fourth quadrant. The result is immediate.

That completes the Skoliad Corner for this issue. Please send me contest material, submissions, and suggestions for the future direction of this feature.

Sports Writer's Math Oddity

In the issue of *Sports Illustrated* dated March 10, 1997, we find a long article about the coach of the University of Kansas Men's Basketball Coach. We quote:

*Williams is straighter than an Arrow shirt, so square that he's divisible by four, and cornier than a corncob pipe.*

Pity that the writer could not even get this little bit of math right!