Folding the Regular Heptagon

Robert Geretschläger, Bundesrealgymnasium, Graz, Austria

Introduction

Ever since Greek antiquity, mathematicians have been considering constructions that can be done with straight-edge and compass only, the so-called Euclidean constructions. A number of famous problems, such as squaring the circle, trisecting angles and doubling the cube, were unsolvable for the Greeks, and later shown to be theoretically unsolvable by Euclidean methods. The reason for this is that only such problems that can be reduced algebraically to combinations of linear and quadratic equations are solvable in this sense. We now know that these three problems, as well as many others, cannot be represented by combinations of such equations.

One specific problem the Greeks attempted to solve in this way was the construction of regular n–gons for small n. They were successful in finding constructions for $n = 3, 4, 5, 6, 8, 10$ and 12, but not for $n = 7, 9$ or 11. Since 7 is the smallest n for which no construction could be found, it was of special interest why this particular problem should prove so stubborn. As it turned out, the construction of the regular heptagon by Euclidean methods is impossible for the same reason that angle trisection and doubling the cube are, in that each of these problems requires the graphic solution of an irreducible cubic equation in its algebraic representation.

As shown in *"Euclidean Constructions and the Geometry of Origami"* ([1]), all cubic equations can be solved graphically using elementary methods of origami[1]. This is especially interesting in light of the fact that regular n–gons are commonly used in the development of origami folding bases. A heptagon could conceivably find use in developing models of insects for instance, since six legs + one head = seven corners. In this article, I present a theoretically precise method of folding the regular heptagon from a square, derived from the results established in the above-mentioned article. The folding method is presented in standard origami notation, and the mathematical section is cross-referenced to the appropriate diagrams.

The Cubic Equation

The seven corners of a regular heptagon can be thought of as the seven solutions of the equation

$$z^7 - 1 = 0 \qquad (1)$$

[1] Origami is, of course, the art of paper folding. For readers not yet familiar with this ancient art, but interested in becoming so, there is a large amount of introductory literature easily available. I was personally introduced to origami by the books of Robert Harbin ([2]). A fine introduction to the geometry of origami is the classic "Geometric Exercises in Paper Folding" by T. Sundara Row ([3]).

in the complex plane. This implies that the unit circle is the circumcircle of the heptagon, and that one corner of the heptagon is the point $z_1 = 1$ on the real axis (Fig. 1.1.). Since one solution of (1) is known, the other six are the roots of

$$\frac{z^7 - 1}{z - 1} = z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1 = 0. \tag{2}$$

For any specific z satisfying this equation, the conjugate $\overline{z}$ is also a solution, since the real axis is an axis of symmetry of the regular heptagon. Also, since

$$|z| = |\overline{z}| = 1,$$

we have $\overline{z} = \frac{1}{z}$. Therefore we can define

$$\zeta = z + \frac{1}{z} = z + \overline{z} = 2 \cdot \mathrm{Re}\, z. \tag{3}$$

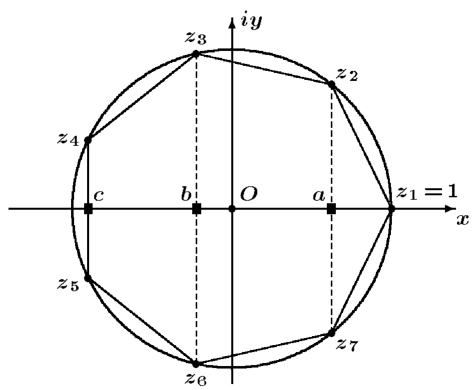


Fig. 1.1.

$$z_1 = 1 + i \cdot 0 \qquad z_2 = \cos\tfrac{2\pi}{7} + i \cdot \sin\tfrac{2\pi}{7} \qquad z_3 = \cos\tfrac{4\pi}{7} + i \cdot \sin\tfrac{4\pi}{7}$$
$$z_4 = \cos\tfrac{6\pi}{7} + i \cdot \sin\tfrac{6\pi}{7} \qquad z_5 = \overline{z_4} = \cos\tfrac{8\pi}{7} + i \cdot \sin\tfrac{8\pi}{7}$$
$$z_6 = \overline{z_3} = \cos\tfrac{10\pi}{7} + i \cdot \sin\tfrac{10\pi}{7} \qquad z_7 = \overline{z_2} = \cos\tfrac{12\pi}{7} + i \cdot \sin\tfrac{12\pi}{7}$$
$$a = \cos\tfrac{2\pi}{7} = \mathrm{Re}\, z_2 = \mathrm{Re}\, z_7 \qquad b = \cos\tfrac{4\pi}{7} = \mathrm{Re}\, z_3 = \mathrm{Re}\, z_6$$
$$c = \cos\tfrac{6\pi}{7} = \mathrm{Re}\, z_4 = \mathrm{Re}\, z_5$$

Dividing by z^3, we see that equation (2) is equivalent to

$$z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0$$

since 0 is not a root, and since

$$\begin{aligned}
\zeta^3 &= \left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \\
&= z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right) = z^3 + \frac{1}{z^3} + 3\zeta \\
\Longleftrightarrow \zeta^3 - 3\zeta &= z^3 + \frac{1}{z^3}
\end{aligned}$$

and

$$\begin{aligned}\zeta^2 &= \left(z+\frac{1}{z}\right)^2 = z^2+2+\frac{1}{z^2}\\ \Longleftrightarrow \zeta^2-2 &= z^2+\frac{1}{z^2},\end{aligned}$$

substituting yields

$$\begin{aligned}\left(z^3+\frac{1}{z^3}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1 &= 0\\ \Longleftrightarrow \quad \zeta^3-3\zeta+\zeta^2-2+\zeta+1=0&\\ \Longleftrightarrow \quad \zeta^3+\zeta^2-2\zeta-1=0.&\end{aligned}$$

From (3) ,we see that each root of the equation

$$\zeta^3+\zeta^2-2\zeta-1=0 \tag{4}$$

is real, and is equal to twice the common real component of two conjugate complex solutions of (1). It is therefore possible to find the six complex roots of (1) in the complex plane by finding the roots of (4), taking half their values, finding the straight lines parallel to the imaginary axis and at precisely these distances from it, and finally finding the points of intersection of these parallel lines with the unit circle. We shall now proceed to utilize these steps in folding the regular heptagon.

A Step-by-step Description of the Folding Process

As is usually the case in origami, we assume a square of paper to be given. We consider the edge-to-edge folds in step 1 as the x– and y–axes of a system of cartesian coordinates, and the edge-length of the given square as four units. The mid-point of the square is then the origin $M(0,0)$, and the end-points of the folds have the coordinates $(-2,0)$ and $(2,0)$, and $(0,-2)$ and $(0,2)$ respectively. For readers not familiar with origami notation, it should be mentioned that dashed lines represent so-called "valley" folds (folding up), and dot-dashed lines represent so-called "mountain" folds (folding down). Thin lines represent visible creases in the paper generated by previous folds.

As shown in [1], the solutions of the cubic equation

$$x^3+px^2+qx+r=0$$

are the slopes of the common tangents of the parabolas p_1 and p_2 defined by the foci

$$F_1\left(-\frac{p}{2}+\frac{r}{2},\frac{q}{2}\right) \qquad \text{and} \qquad F_2\left(0,\frac{1}{2}\right)$$

and directrices

$$l_1: x=-\frac{p}{2}-\frac{r}{2} \qquad \text{and} \qquad l_2: y=-\frac{1}{2}$$

respectively.

The solutions of (4) can therefore be obtained by finding the common tangents of the parabolas with foci

$$F_1(-1,-1) \qquad \text{and} \qquad F_2\left(0,\frac{1}{2}\right)$$

and directrices

$$l_1 : x = 0 \qquad \text{and} \qquad l_2 : y = -\frac{1}{2}$$

respectively. Since the slope of the common tangents is not altered by translating the parabolas parallel to the y–axis we can, for convenience, use

$$F_1\left(-1,-\frac{1}{2}\right) \qquad \text{and} \qquad F_2(0,1)$$

and

$$l_1 : x = 0 \qquad \text{and} \qquad l_2 : y = 0.$$

This is precisely what is done in steps 2 to 5. F_1 is the point A, and F_2 is the point B. The fold in step 4 is then the only common tangent of the parabolas with positive slope, and thus twice the real component of the solutions of (1) which lie to the right of the imaginary axis and are not equal to 1. In other words, the slope of this fold is $2 \cdot \cos\frac{2\pi}{7}$. Step 4, by the way, is the only step that cannot be replaced by a straight-edge and compass construction.

In steps 6 to 8, the unit-length is then transferred in such a way that point E in step 8 has y–coordinate $-2 \cdot \cos\frac{2\pi}{7}$. Since the distance from M to point 1 in step 9 is 2 units, the distances from M to points 2 and 7 are also 2 units, and so points 7, 1 and 2 are three consecutive corners of the regular heptagon. (We assume that point 1 with coordinates $(0,-2)$ is the first corner, and continue from there.)

Step 10 thus yields two sides of the heptagon, and steps 11 to 13 yield the remaining sides of the heptagon by making use of its radial symmetry, until finally step 14 shows us the completed regular heptagon. The folding process is shown at the end of the article.

Conclusion

Unlike other regular n–gons with small n, the regular heptagon is not very common in popular culture or graphics. Apart from the seven-sided star one comes across in astrology, the heptagon does not seem to show up much in public, unlike its close relatives. We come across the octagon at many a street corner, and the pentagon and hexagon can be seen on most soccer balls, just to name a few. I do not know if this (relatively) easy generation of the regular heptagon will lead to its mass popularization, but an ardent Heptagonist can certainly dream.

It should be mentioned that a similar folding method for the regular heptagon is described in the article "*Draw of a Regular Heptagon by the*

Folding" by Benedetto Scimemi ([4]) in the relatively hard to find Proceedings of the First International Meeting of Origami Science and Technology. (I have only recently gained access to a copy myself.) This volume offers a great many ideas for further research for anyone interested in the geometry of origami, and is certainly worth searching for.

References

[1] R. Geretschläger, *Euclidean Constructions and the Geometry of Origami*, Mathematics Magazine, **68** No. 5 December 1995, pp. 357-371

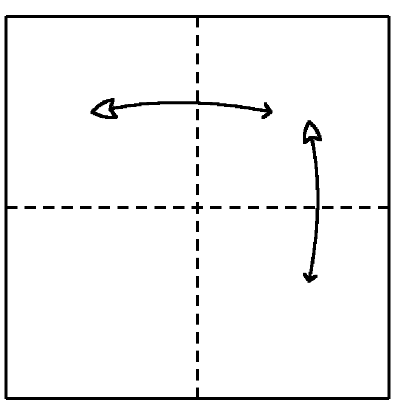
[2] R. Harbin, *Origami, The Art of Paper-Folding, Vols. 1-4*, Hodder Paperbacks, Norwich (1968)

[3] T. Sundara Row, *Geometric Exercises in Paper Folding*, Dover Publications, Inc., Mineola, NY (1966) reprint of 1905 edition

[4] B. Scimemi, *Draw of a Regular Heptagon by the Folding*, Proceedings of the First International Meeting of Origami Science and Technology, Ferarra, Italy (1989)

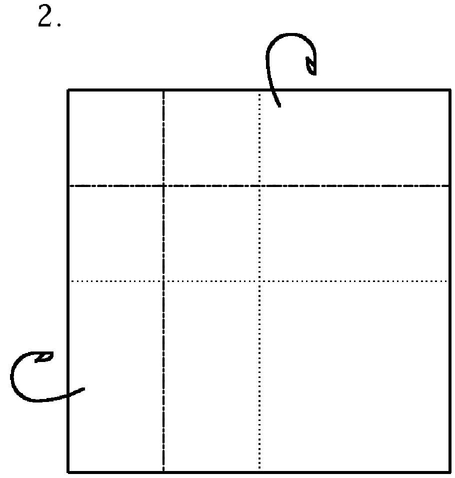
The Folding Process

1.



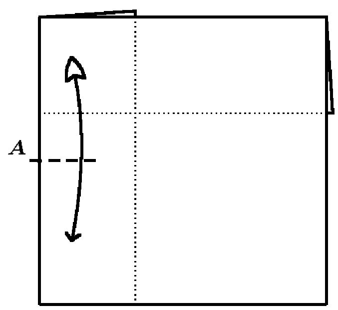
Fold and unfold twice.

2.



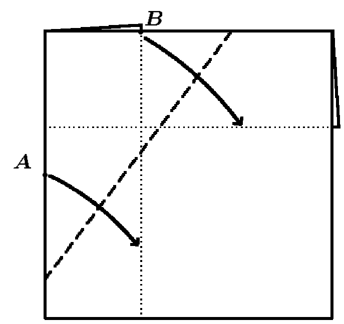
Fold back twice.

3.



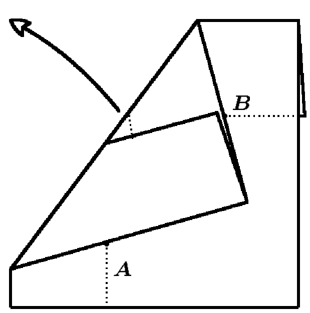
Fold and unfold, making a crease mark at point ***A*** (bisecting the side).

4.



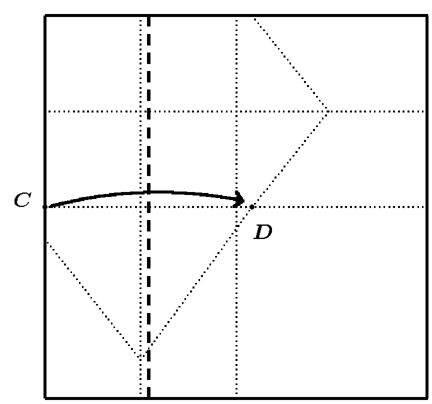
Fold such that ***A*** and ***B*** come to lie on the creases.

5.



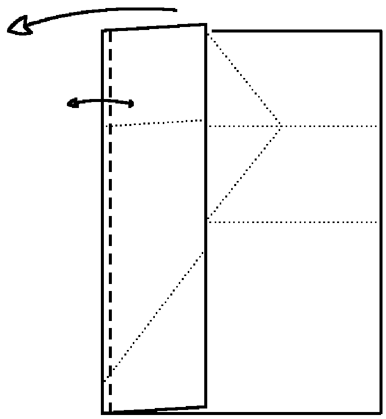
Unfold everything.

6.



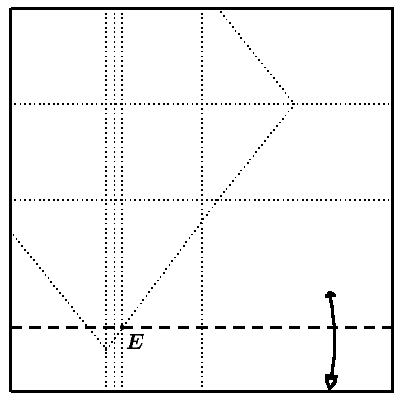
Fold ***C*** to ***D***.

7.



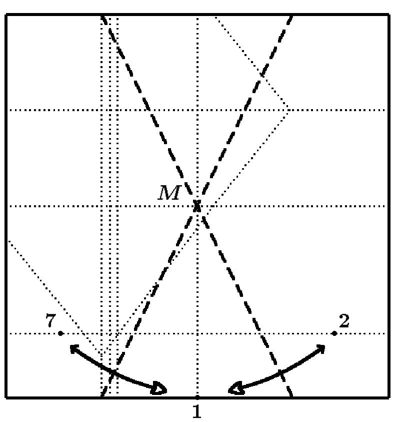
Fold and unfold both layers at crease, then unfold everything.

8.



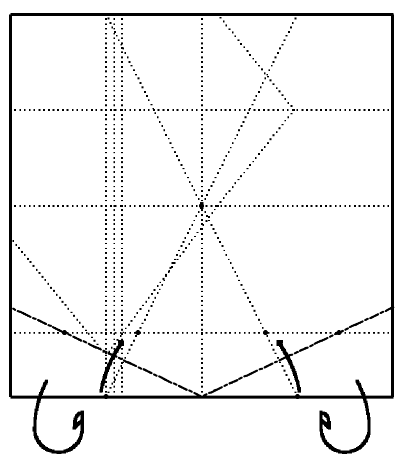
Fold horizontally through E, then unfold.

9.



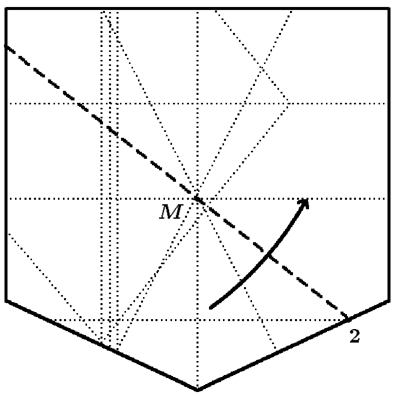
Fold through M, such that 1 lies on crease, resulting in 2 and 7 (M is the mid-point of the heptagon, 1, 2 and 7 are corners).

10.



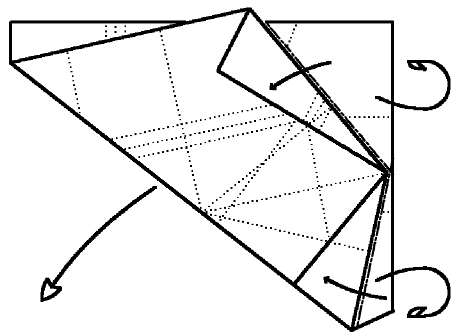
Fold back twice, so that the marked points come to lie on one another; resulting folds are first two sides of the heptagon.

11.



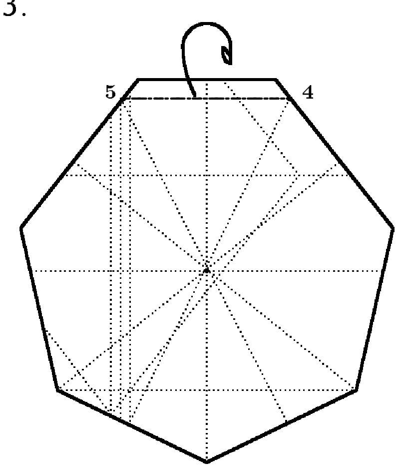
Fold through M and 2.

12.



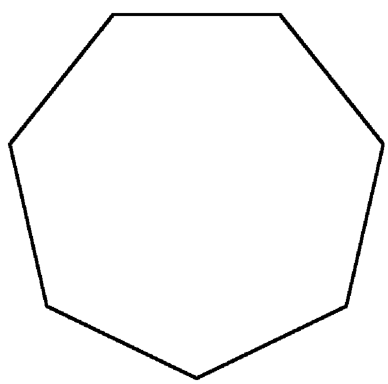
Fold back lower layers using edges
of upper layer as guidelines;
resulting folds are two
more sides of the heptagon;
open up fold from step 12 and
repeat 11 and 12 on left side.

13.



Fold back final edge of the
heptagon through 4 and 5.

14.



The finished heptagon.