THE ACADEMY CORNER
No. 7

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In the May 1996 issue of CRUX [1996: 165], we printed the questions of the 1996 Memorial University Undergraduate Mathematics Competition, and invited readers to send in solutions. Here is a complete solution set.

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
UNDERGRADUATE MATHEMATICS COMPETITION
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Solutions

1. Prove that if \( n \) is a positive integer, then \( \frac{n^2 + 3n + 1}{n^2 + 4n + 3} \) is an irreducible fraction.

Solution by Panos E. Tsaoussoglou, Athens, Greece [Shortened by the editors.]

Assume that the fraction is reducible. Then there is a positive integer \( k \) such that \( k|\left(n^2 + 3n + 1\right) \) and \( k|(n^2 + 4n + 3) \).

Then \( k(n^2 + 3n + 1) - (n^2 + 4n + 3) = kn + 2 \), so that there is a positive integer \( l \) such that \( n + 2 = kl \).

Now, \( \frac{n^2 + 3n + 1}{k} = nl + \frac{n + 1}{k} \). The latter term is an integer only if \( k|n + 1 \). Since \( k|n + 2 \), this implies that \( k = 1 \).

This is a contradiction, proving the result.

2. A jar contains 7 blue balls, 9 red balls and 10 white balls. Balls are drawn at random one by one from the jar until either four balls of the same colour or at least two of each colour have been drawn. What is the largest number of balls that one may have to draw?

Solution by Panos E. Tsaoussoglou, Athens, Greece.

Assume that three balls of one colour are drawn, then three balls of an other colour are drawn, and then one ball of the third colour. The next ball (of any colour) satisfies the conditions of the problem.

The answer is 8 balls.
3. Find all functions $u(x)$ satisfying $u(x) = x + \int_0^x u(t) \, dt$.

"Official Solution" from Maurice Oleson.

Since $\int_0^x u(t) \, dt$ is a constant, we have that $u(x) = x + c$ for some constant $c$.

Then $\int_0^x (t + c) \, dt = c$, which gives $c = \frac{1}{4}$.

Thus $u(x) = x + \frac{1}{4}$.

4. Show that $\left( \sqrt{5} + 2 \right)^\frac{1}{3} - \left( \sqrt{5} - 2 \right)^\frac{1}{3}$ is a rational number and find its value.

Solution by Panos E. Tsoussoglou, Athens, Greece.

Let $k = \left( \sqrt{5} + 2 \right)^\frac{1}{3} - \left( \sqrt{5} - 2 \right)^\frac{1}{3}$. Then

$$k^3 = \left( \sqrt{5} + 2 \right) - \left( \sqrt{5} - 2 \right) - 3 \left( \left( \sqrt{5} + 2 \right) \left( \sqrt{5} - 2 \right) \right)^\frac{1}{3}$$

$$\times \left( \left( \sqrt{5} + 2 \right)^\frac{1}{3} - \left( \sqrt{5} - 2 \right)^\frac{1}{3} \right),$$

or

$$k^3 = 4 - 3k \quad \text{or} \quad (k^3 - 1) + 3(k - 1) = 0.$$ 

Thus $(k - 1)(k^2 + 4k + 4) = 0$, so that $k = 1$ (since $k > 0$).

5. In a quadrilateral $ABCD$ (vertices named in clockwise order), $AC$ and $BD$ intersect in $X$. You are given that $AB \parallel DC$, that $AB$ is twice as long as $CX$ and that $AC$ is equal in length to $DC$. Show that $AB$ and $CD$ are equal in length (and hence $ABCD$ is a parallelogram).

Solution by Panos E. Tsoussoglou, Athens, Greece.

We have $AB = 2CX$ and $AC = DC$.

Triangles $\triangle AXB$ and $\triangle DXC$ are similar, so that

$$\frac{AX}{CX} = \frac{AB}{DC}$$
so that \[ \frac{AX + CX}{CX} = \frac{AB + DC}{DC}. \]

Thus, we have
\[
\begin{align*}
\frac{2DC}{AB} &= \frac{AB + DC}{DC} \\
2DC^2 &= AB^2 + AB \cdot DC \\
0 &= DC^2 - AB^2 + DC^2 - AB \cdot DC \\
&= (DC - AB)(AB + 2CD),
\end{align*}
\]

so that \( AB = DC \).

6. Prove that among any thirteen distinct real numbers it is possible to choose two, \( x \) and \( y \), such that \( 0 < \frac{x - y}{1 + xy} < 2 - \sqrt{3} \).

"Official Solution" from Maurice Oleson.

We recognize the similarity of \( \frac{x - y}{1 + xy} \) with \( \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \).

Since twelve fifteen degree angles make a straight angle, the pigeonhole principle gives that \( 0 < \frac{x - y}{1 + xy} < \tan(15^\circ) \).

We note that
\[
\tan^2(15^\circ) = \frac{\sin^2(15^\circ)}{\cos^2(15^\circ)} = \frac{1 - \cos(30^\circ)}{1 + \cos(30^\circ)} = \frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} = (2 - \sqrt{3})^2,
\]

and the result follows.

7. A coastguard boat is hunting a bootlegger in a fog. The fog rises disclosing the bootlegger 4 miles distant and immediately descends. The speed of the boat is 3 times that of the bootlegger, and it is known that the latter will immediately depart at full speed on a straight course of unknown direction. What course should the boat take in order to overtake the bootlegger?
"Official Solution" from Maurice Oleson. We look at two diagrams:

The distance from A to B is 1 mile.

From $ds^2 = dr^2 + (r \, d\theta)^2$, $\frac{ds}{dt} = 3v$ and $\frac{dr}{dt} = v$, we get

$$9v^2 = v^2 + r^2 \left( \frac{d\theta}{dt} \right)^2.$$  

Thus

$$\sqrt{8v} = r \frac{d\theta}{dt} = r \frac{dr}{dr} \frac{dr}{dt} = r \frac{d\theta}{dr} v,$$

yielding $\frac{dr}{r} = \frac{1}{\sqrt{8}}d\theta$. Hence $\ln(r) = \frac{\theta}{\sqrt{8}} + c$.

Since $\theta = 0$ when $r = 1$, we have $c = 0$, giving $r = e^{\theta/\sqrt{8}}$.

This is the spiral path that the coast guard must follow if she wishes to intercept the bootlegger.

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PHOTO PROBLEM

Can you identify this regular CRUX contributor?