PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 April 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX, *preferably in* LaTeX2e). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2164. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on the side BC of triangle ABC, and let E and F be the incentres of triangles ABD and ACD respectively. Suppose that B, C, E, F are concyclic. Prove that

$$\frac{AD+BD}{AD+CD} = \frac{AB}{AC}.$$

2165. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Given a triangle ABC, prove that there exists a unique pair of points P and Q such that the triangles ABC, PQC and PBQ are directly similar; that is, $\angle ABC = \angle PQC = \angle PBQ$ and $\angle BAC = \angle QPC = \angle BPQ$, and the three similar triangles have the same orientation. Find a Euclidean construction for the points P and Q.

2166. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a right-angled triangle, establish the existence of a unique interior point with the property that the line through the point perpendicular to any side cuts off a triangle of the same area.

2167. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove, **without the aid the differential calculus**, the inequality, that in a right triangle

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \geq 2+\sqrt{2},$$

where a and b are the legs and c the hypotenuse of the triangle.

2168. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside a regular tetrahedron $ABCD$, with circumradius R and let R_1, R_2, R_3, R_4 denote the distances of P from vertices of the tetrahedron. Prove or disprove that

$$R_1R_2R_3R_4 \leq \frac{4}{3}R^4,$$

and that the maximum value of $R_1R_2R_3R_4$ is attained.

2169. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

AB is a fixed diameter of circle $\Gamma_1(0,R)$. P is an arbitrary point of its circumference. Q is the projection onto AB of P. Circle $\Gamma_2(P_1PQ)$ intersects Γ_1 at C and D. CD intersects PQ at E. F is the midpoint of AQ. $FG \perp CD$, where $G \in CD$. Show that:

1. $EP = EQ = EG$,
2. A, G and P are collinear.

2170. *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Find, with justification, the positive integer which comes next in the sequence $1411, 4463, 4464, 1412, 4466, 4467, 1413, 4469, \ldots$.

[Ed.: the answer is NOT 4470.]

2171. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be an arbitrary point taken on an ellipse with foci F_1 and F_2, and directrices d_1, d_2, respectively. Draw the straight line through P which is parallel to the major axis of the ellipse. This line intersects d_1 and d_2 at points M and N, respectively. Let P' be the point where MF_1 intersects NF_2.

Prove that the quadrilateral $PF_1P'F_2$ is cyclic.

Does the result also hold in the case of a hyperbola?

2172. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x, y, z \geq 0$ with $x+y+z=1$. For fixed real numbers a and b, determine the maximum $c = c(a,b)$ such that $a + bxyz \geq c(yz+zx+xy)$.

2173. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$.

Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k) \sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{x - 1})^{n+1} (\sqrt{n})^n / (n + 1)^{n-1}.$$

1. Show $l_2 \leq r_2$.
2. Prove or disprove: $l_n \geq r_n$ for $n \geq 3$.

2174. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let A be an $n \times n$ matrix. Prove that if $A^{n+1} = 0$ then $A^n = 0$.

2175. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \quad \frac{1}{6} = \frac{1}{3} - \frac{1}{6}; \quad \frac{1}{6} = \frac{1}{4} - \frac{1}{2}; \quad \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y},$$

where x and y are positive integers?

2176. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that

$$\sqrt[n]{\prod_{k=1}^{n} (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k}$$

where $a_1, a_2, \ldots, a_n > 0$ and $n \in \mathbb{N}$.