THE ACADEMY CORNER

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In the February 1996 issue, we gave the first set of problems in the Academy Corner. Here we present solutions to the last three questions, as sent in by Šefket Arslanagić, Berlin, Germany.

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4. If a, b, c, d are positive integers such that $ad = bc$, prove that $a^2 + b^2 + c^2 + d^2$ is never a prime number.

   We will give a generalization: If a, b, c, d, r, s are positive integers such that $rad = sbc$, prove that $r(a^2 + d^2) + s(b^2 + c^2)$ is never a prime number.

   Solution. All positive integers greater than or equal to 2 can be written as products of finitely many prime numbers. Therefore $rad = sbc = p_1 \dots p_\ell$ (p_i, p_j not necessarily different) are the prime factors of r, a, d and similarly of factors s, b, c. Also, there exist positive integers $x_{11}, x_{12}, \dots, x_{33}$ such that

$$r = x_{11}x_{12}x_{13} = y_1; \quad a = x_{21}x_{22}x_{23} = y_2; \quad d = x_{31}x_{32}x_{33} = y_3;$$

$$s = x_{11}x_{21}x_{31} = z_1; \quad b = x_{12}x_{22}x_{32} = z_2; \quad c = x_{13}x_{23}x_{33} = z_3.$$

   We write this as

$$\begin{array}{cccl} x_{11} & x_{12} & x_{13} & \to \prod = r = y_1 \\ x_{21} & x_{22} & x_{23} & \to \prod = a = y_2 \\ x_{31} & x_{32} & x_{33} & \to \prod = d = y_3 \\ \downarrow & \downarrow & \downarrow & \\ \prod = s & \prod = b & \prod = c & \\ = z_1 & = z_2 & = z_3 & \end{array}$$

Let

$$x_{iji} = \prod_{\substack{px \in y_i \\ px \in z_j \\ k = 1, \ldots, \ell}} px$$

Now, we get

$$\begin{aligned} & r(a^2 + s^2) + s(b^2 + c^2) \\ &= x_{11}x_{12}x_{13}(x_{21}^2 x_{22}^2 x_{23}^2 + x_{31}^2 x_{32}^2 x_{33}^2) \\ &\quad + x_{11}x_{21}x_{31}(x_{12}^2 x_{22}^2 x_{32}^2 + x_{13}^2 x_{23}^2 x_{33}^2 \\ &= x_{11}x_{12}x_{13}x_{21}^2 x_{22}^2 x_{23}^2 + x_{11}x_{12}x_{13}x_{31}^2 x_{32}^2 x_{33}^2 \\ &\quad + x_{11}x_{21}x_{31}x_{12}^2 x_{22}^2 x_{32}^2 + x_{11}x_{21}x_{31}x_{12}^2 x_{23}^2 x_{33}^2 \\ &= x_{11}x_{12}x_{21}x_{22}^2(x_{13}x_{21}x_{23}^2 + x_{12}x_{31}x_{32}^2) \\ &\quad + x_{11}x_{13}x_{31}x_{33}^2(x_{12}x_{31}x_{32}^2 + x_{12}x_{21}x_{23}^2) \\ &= (x_{11}x_{12}x_{21}x_{22}^2 + x_{11}x_{13}x_{31}x_{33}^2) \\ &\quad \times (x_{13}x_{21}x_{23}^2 + x_{12}x_{31}x_{32}^2). \end{aligned}$$

Thus, $r(a^2 + d^2) + s(b^2 + c^2)$ is the product of two factors of positive integers, and is greater than or equal to 2, because $x_{ij} \geq 1$ for all i, j.

Therefore, $r(a^2 + d^2) + s(b^2 + c^2)$ is never a prime number.

For $r = s + 1$, that is, $ad = bc$, the sum $a^2 + b^2 + c^2 + d^2$ is never a prime number.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

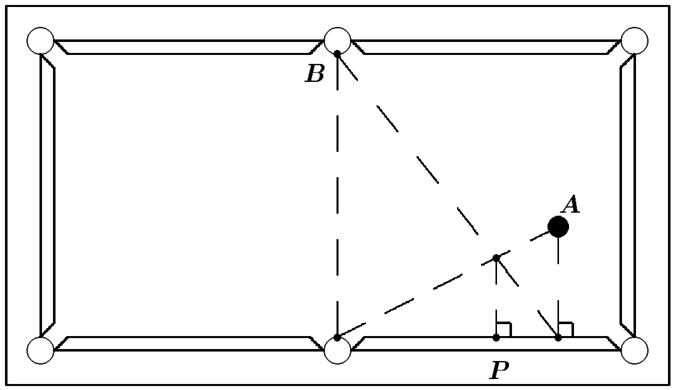
for all real numbers x, y.

Solution. This has already appeared in *CRUX* [1993: 41–42]. The problem was used in the *XLI Mathematics Olympiad in Poland*.

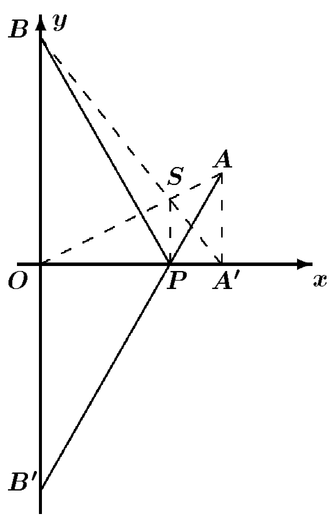
6. Assume that when a snooker ball strikes a cushion, the angle of incidence equals the angle of reflection.

For any position of a ball A, a point P on the cushion is determined as shown.

Prove that if the ball A is shot at point P, it will go into the pocket B.



Solution. We will use analytic geometry. We let the axes be $Ox \equiv OP$, $Oy = OB$, $Ox \perp Oy$; and give coordinates: $O(0,0)$, $A(a,b)$, $B(0,c)$, $A'(a,0)$; we have



OA: $y = \frac{b}{a}x$
BA': $y = -\frac{c}{a}x + c$,
$\{S\} = OA \cap BA'$, i,e,
$S\left(\frac{ac}{b+c}, \frac{bc}{b+c}\right)$ and $P\left(\frac{ac}{b+c}, 0\right)$
AP: $y = \frac{b+c}{a}x - c$,
that is, $B'(0, -c)$.
Therefore $|OB| = |OB'| = c$ and
$\Delta OBP \simeq \Delta OB'P$,
that is, $\angle BPO = \angle B'PO$.

Because $\angle B'PO = \angle APA'$,
we obtain $\angle BPO = \angle APA'$.