THE OLYMPIAD CORNER

No. 176

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the problems of the 10th Iberoamerican Mathematical Olympiad, written September 26–27, 1995 at Valparaiso (Chile). My thanks go to Professor Francisco Bellot Rosado of Valladolid, Spain for sending me an English translation of the problems.

10th IBEROAMERICAN MATHEMATICAL OLYMPIAD

September 26–27, 1995 (Valparaiso, Chile)

FIRST DAY — Time: 4.5 hours

1. (*Brazil*). Determine all the possible values of the sum of the digits of all the perfect squares.

2. (*Spain*). Let n be an integer bigger than 1. Determine the real numbers $x_1, x_2, \ldots x_n \geq 1$, and $x_{n+1} > 0$, such that the following conditions are simultaneously fulfilled:

(a) $\sqrt{x_1} + \sqrt[3]{x_2} + \cdots + \sqrt[n+1]{x_n} = n \cdot \sqrt{x_{n+1}}$

(b) $\dfrac{x_1 + x_2 + \cdots + x_n}{n} = x_{n+1}$.

3. (*Brazil*). Let r and s be two orthogonal straight lines, not belonging to the same plane. Let AB be their common perpendicular, with $A \in r$ and $B \in s$. (Note that the plane which contains B and r is perpendicular to s). Consider the sphere with diameter AB. The points $M \in r$, and $N \in s$, are variable, with the condition that MN is tangent to the sphere at some point T. Find the locus of T.

SECOND DAY — Time: 4.5 hours

4. (*Argentina*). Coins are situated on an $m \times m$ board. Each coin situated on the board "dominates" all the cells of the row (⟷), the column (↕) and the diagonal (↖↘) to which the coin belongs. Note that the coin does not "dominate" the diagonal (↙↗). Determine the smallest number of coins which must be placed in order that all the cells of the board be dominated.

5. (*Spain*). The inscribed circumference in the triangle ABC is tangent to BC, CA and AB at D, E and F, respectively. Suppose that this circumference meets AD again at its mid-point X, that is, $AX = XD$. The lines XB and XC meet the inscribed circumference again at Y and Z, respectively. Show that $EY = FZ$.

6. (*Chile–Brazil*). Let $\mathbb{N} = \{1, 2, 3, \dots\}$. A function $f : \mathbb{N} \to \mathbb{N}$ is called *circular* if for each $p \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \leq p$ such that

$$f^n(p) = \underbrace{f(f(\dots f}_{n \text{ times}}(p))\dots) = p.$$

The function f has *repulse degree* k, $0 < k < 1$, if for each $p \in \mathbb{N}$, $f^i(p) \neq p$ for all $i \leq [k \cdot p]$, in which $[x]$ is the integer part of x. Determine the biggest repulse degree that can be reached by a circular function.

The next contest we present in its official language. Here is your opportunity to brush up your French. (Of course we will accept your nice solutions in either English or French!) The questions are to the mini and maxi finals of the 19th Belgian Mathematical Olympiad, written in 1994. My thanks go to Richard Nowakowski, Canadian Team Leader at the 35th IMO in Hong Kong, for collecting this contest (and others) and forwarding them to the Olympiad Corner for its use.

DIX-NEUVIEME OLYMPIADE MATHEMATIQUE BELGE

Mini Finale 1994

1. Combien de nombres enteiers naturels de trois chiffres non nuls distincts (en base 10) sont primiers avec 10?

2. En prenant comme sommets les points d'intersection des côtés prolongés d'un hexagone régulier, Jean obtient un nouvel hexagone régulier. Il applique ensuite la même construction à ce nouvel hexagone, et recommence de même ... Combien de fois Jean doit-il effectuer cette construction pour que l'aire du dernier hexagone construit dépasse 1994 fois l'aire de l'hexagone initial?

3. Trente-huit lampes numérotées de l à 38 sont disposées en cercle autour d'une lampe centrale numérotée 0. Ces lampes forment des groupes de quatre:

$$\{0, 1, 2, 3\},\ \{0, 3, 4, 5\},\ \{0, 5, 6, 7\},$$
$$\{0, 7, 8, 9\}, \dots,\ \{0, 35, 36, 37\},\ \{0, 37, 38, 1\}$$

deux opérations seulement sont réalisables:

(α) éteindre les quatre lampes d'un même groupe;

(β) changer l'ètat de chacune des lampes d'un même groupe (c'est-à-dire, une lampe allumée est éteinte, une éteinte est allumée).

Tout état initial des 39 lampes est-il transformable par une suite de telles opératious en

(a) l'état où toutes les lampes sont allumées?

(b) l'état où seule la lampe numéro 0 est allumée?

4. Sur un terrain plat et carré de 32 ares (ou 3, 200 mètres carrés) dont les côtés sont orientés NO-SE et NE-SO se trouve une villa rectangulaire de 16 mètres sur 20 mètres, dont les quatre façades font face aux quatre points cardinaux. Le centre de la villa coïncide avec le centre du terrain. Le reste du terrain est aménagé en pelouse. Quelle fraction de la pelouse est constituée de points d'où sont visibles deux façades de la villa?

Maxi Finale 1994

1. Un pentagone plan covexe a deux angles droits non adjacents. Les deux côtés adjacents au premier angle droit out des longueurs égales. Les deux côtés adjacents au second angle droit ont des longueurs égales. En remplaçant par leur point milieu les deux sommets du pentagone situés sur un seul côté de ces angles droits, nous formons un quadrilatère. Ce quadrilatère admet-il nécessairement un angle droit?

2. Des lampes en nombre $2n$ (avec $n \geq 2$) et numérotées de 1 à $2n$ sont disposées en cercle autour d'une lampe centrale numérotée 0. Ces lampes forment des groupes de quatre:

$$\{0,1,2,3\},\ \{0,3,4,5\},\dots\ \{0,2k-3,2k-2,2k-1\},\ \{0,2k-1,2k,2k+1\},$$
$$\{0,2k+1,2k+2,2k+3\},\dots,\ \{0,2n-1,2n,1\}$$

et deux opérations seulement sont réalisables:

(α) éteindre les quatre lampes d'un même groupe;

(β) changer l'état de chacune des lampes d'un même groupe (c'est-à-dire, une lampe allumée est éteinte, une éteinte est allumée).

Pour quelles valeurs de n tout état initial des $2n+1$ lampes est-il transformable par une suite de telles opérations en

(a) l'état où toutes les lampes sont allumées?

(b) l'état où seule la lampe nuiméro 0 est allumée?

3. Existe-t-il une numérotation des arêtes d'un cube par douze nombres naturels consécutifs telle que

(a) la somme des nombres attribués aux arêtes aboutissant en un sommet soit toujours la même?

(b) la somme des nombres attribués aux arêtes d'une face soit toujours la même?

4. Le plan contient il 1994 points (distincts) non tous alignés tels que la distance entre deux quelconques d'entre eux soit un nombre entier?

As promised last issue, we now give the "official" solutions to problems of the 1996 Canadian Mathematical Olympiad. My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, for forwarding the problems and solutions. Problems with communications during summer break did not allow us to incorporate the novel solutions of some contest participants. Hopefully next year we will be on track earlier and have time to solicit permission to use the submitted solutions.

1996 CANADIAN MATHEMATICAL OLYMPIAD

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

Solution. If $f(x) = x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β, γ standard results about roots of polynomials give $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$, and $\alpha\beta\gamma = 1$.

Then

$$S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{N}{(1-\alpha)(1-\beta)(1-\gamma)}$$

where the numerator simplifies to

$$\begin{aligned} N &= 3 - (\alpha+\beta+\gamma) - (\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma \\ &= 3 - (0) - (-1) + 3(1) \\ &= 7. \end{aligned}$$

The denominator is $f(1) = -1$ so the required sum is -7.

2. Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y \\[2ex] \dfrac{4y^2}{1+4y^2} = z \\[2ex] \dfrac{4z^2}{1+4z^2} = x \end{cases}$$

Solution. For any t, $0 \le 4t^2 < 1 + 4t^2$, so $0 \le \dfrac{4t^2}{1+4t^2} < 1$. Thus x, y and z must be non-negative and less than 1.

Observe that if one of x y or z is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1+4x^2} = x.$$

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \dfrac{1}{2}$ which gives $x = y = z = \dfrac{1}{2}$.

Finally, assume that x, y and z are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \dfrac{4t^2}{1+4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$\begin{aligned} f(t) - f(s) &= \frac{4t^2}{1+4t^2} - \frac{4s^2}{1+4s^2} \\ &= \frac{4t^2 - 4s^2}{(1+4s^2)(1+4t^2)} \\ &> 0. \end{aligned}$$

So $0 < x < y < z \Rightarrow f(x) = y < f(y) = z < f(z) = x$, a contradiction.

Hence $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only real solutions.

Alternate Solution. Notice that x, y and z are non-negative. Adding the three equations gives

$$x + y + z = \frac{4z^2}{1+4z^2} + \frac{4x^2}{1+4x^2} + \frac{4y^2}{1+4y^2}.$$

This can be rearranged to give

$$\frac{x(2x-1)^2}{1+4x^2} + \frac{y(2y-1)^2}{1+4y^2} + \frac{z(2z-1)^2}{1+4z^2} = 0.$$

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\frac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only two solutions.

Alternate Solution. Notice that x, y, and z are non-negative. Multiply both sides of the inequality

$$\frac{y}{1+4y^2} \ge 0$$

by $(2y-1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1+4y^2} \geq 0,$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Alternate Solution. As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of x, y and z positive.

The arithmetic-geometric mean inequality (or direct computation) shows that $\dfrac{1+4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$ and hence $x \geq \dfrac{4x^2}{1+4x^2} = y$, with equality if and only if $1 = 4x^2$, that is, $x = \dfrac{1}{2}$. Similarly, $y \geq z$ with equality if and only if $y = \dfrac{1}{2}$ and $z \geq x$ with equality if and only if $z = \dfrac{1}{2}$. Adding $x \geq y$, $y \geq z$ and $z \geq x$ gives $x + y + x \geq x + y + z$. Thus equality must occur in each inequality, so $x = y = z = \dfrac{1}{2}$.

3. We denote an arbitrary permutation of the integers $1, \ldots, n$ by $a_1, \ldots, a_n$. Let $f(n)$ be the number of these permutations such that
(i) $a_1 = 1$;
(ii) $|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n-1$.
Determine whether $f(1996)$ is divisible by 3.

Solution. Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$ with properties (i) and (ii).

A crucial observation, needed in Case II (b) is the following: If a_k and a_{k+1} are consecutive integers (i.e. $a_{k+1} = a_k \pm 1$), then the terms to the right of a_{k+1} (also to the left of a_k) are either all less than both a_k and a_{k+1} or all greater than both a_k and a_{k+1}.

Since $a_1 = 1$, by (ii) a_2 is either 2 or 3.

CASE I: Suppose $a_2 = 2$. Then $a_3, a_4, \ldots, a_n$ is a permutation of $3, 4, \ldots, n$. Thus $a_2, a_3, \ldots, a_n$ is a permutation of $2, 3, \ldots, n$ with $a_2 = 2$ and property (ii). Clearly there are $f(n-1)$ such permutations.

CASE II: Suppose $a_2 = 3$.
(a) Suppose $a_3 = 2$. Then $a_4, a_5, \ldots, a_n$ is a permutation of $4, 5, \ldots, n$ with $a_4 = 4$ and property (ii). There are $f(n-3)$ such permutations.
(b) Suppose $a_3 \geq 4$. If a_{k+1} is the first even number in the permutation then, because of (ii), $a_1, a_2, \ldots, a_k$ must be $1, 3, 5, \ldots, 2k-1$ (in that order). Then a_{k+1} is either $2k$ or $2k-2$, so that a_k and a_{k+1} are consecutive integers. Applying the crucial observation made above, we deduce that $a_{k+2}, \ldots, a_n$ are all either greater than or smaller than a_k and a_{k+1}. But 2 must be to the right of a_{k+1}. Hence $a_{k+2}, \ldots, a_n$ are the even integers less than a_{k+1}. The only possibility then, is

$$1, 3, 5, \ldots, a_{k-1}, a_k, \ldots, 6, 4, 2.$$

Cases I and II show that
$f(n) = f(n-1) + f(n-3) + 1, \quad n \geq 4.$ $\quad (\star)$
Calculating the first few values of $f(n)$ directly gives

$$f(1) = 1,\ f(2) = 1,\ f(3) = 2,\ f(4) = 4,\ f(5) = 6.$$

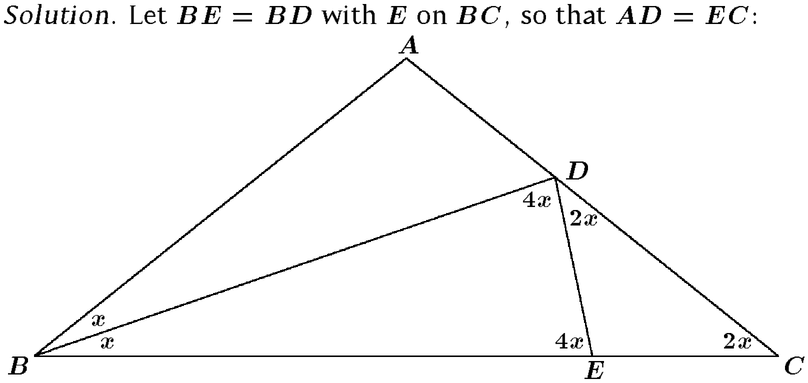
Calculating a few more $f(n)$'s using $(\star)$ and mod 3 arithmetic, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 1$, $f(5) = 0$, $f(6) = 0$, $f(7) = 2$, $f(8) = 0$, $f(9) = 1$, $f(10) = 1$, $f(11) = 2$. Since $f(1) = f(9)$, $f(2) = f(10)$ and $f(3) = f(11)$ mod 3, $(\star)$ shows that

$$f(a) = f(a \bmod 8),\ \bmod 3,\ a \geq 1.$$

Hence $f(1996) \equiv f(4) \equiv 1 \pmod 3$ so 3 does not divide $f(1996)$.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

Solution. Let $BE = BD$ with E on BC, so that $AD = EC$:



By a standard theorem, $\dfrac{AB}{CB} = \dfrac{AD}{DC}$; so in $\triangle\ CED$ and $\triangle\ CAB$ we have a common angle and

$$\frac{CE}{CD} = \frac{AD}{CD} = \frac{AB}{CB} = \frac{CA}{CB}.$$

Thus, $\triangle CED \sim \triangle CAB$, so that $\angle CDE = \angle DCE = \angle ABC = 2x$.

Hence $\angle BDE = \angle BED = 4x$, whence $9x = 180^\circ$ so $x = 20^\circ$.

Thus $\angle A = 180^\circ - 4x = 100^\circ$.

Alternate Solution. Apply the law of sines to $\triangle ABD$ and $\triangle BDC$ to get

$$\frac{AD}{BD} = \frac{\sin x}{\sin 4x} \quad \text{and} \quad 1 + \frac{AD}{BD} = \frac{BC}{BD} = \frac{\sin 3x}{\sin 2x}.$$

Now massage the resulting trigonometric equation with standard identities to get

$$\sin 2x\,(\sin 4x + \sin x) = \sin 2x\,(\sin 5x + \sin x).$$

Since $0 < 2x < 90°$, we get

$$5x - 90° = 90° - 4x \ ,$$

so that $\angle A = 100°$.

5. Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m}[r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution. Let

$$\begin{aligned} f(n) &= n - \sum_{k=1}^{m}[r_k n] \\ &= n\sum_{k=1}^{m} r_k - \sum_{k=1}^{m}[r_k n] \\ &= \sum_{k=1}^{m}\{r_k n - [r_k n]\}. \end{aligned}$$

Now $0 \leq x - [x] < 1$, and if c is an integer, $(c + x) - [c + x] = x - [x]$.

Hence $0 \leq f(n) < \sum_{k=1}^{m} 1 = m$. Because $f(n)$ is an integer, $0 \leq f(n) \leq m - 1$.

To show that $f(n)$ can achieve these bounds for $n > 0$, we assume that $r_k = \dfrac{a_k}{b_k}$ where a_k, b_k are integers; $a_k < b_k$.

Then, if $n = b_1 b_2 \ldots b_m$, $(r_k n) - [r_k n] = 0$, $k = 1, 2, \ldots, m$ and thus $f(n) = 0$.

Letting $n = b_1 b_2 \ldots b_n - 1$, then

$$\begin{aligned} r_k n &= r_k(b_1 b_2 \ldots b_m - 1) \\ &= r_k\{(b_1 b_2 \ldots b_m - b_k) + b_k - 1)\} \\ &= \text{integer} + r_k(b_k - 1). \end{aligned}$$

This gives

$$\begin{aligned} r_k n - [r_k n] &= r_k(b_k - 1) - [r_k(b_k - 1)] \\ &= \frac{a_k}{b_k}(b_k - 1) - \left[\frac{a_k}{b_k}(b_k - 1)\right] \\ &= \left(a_k - \frac{a_k}{b_k}\right) - \left[a_k - \frac{a_k}{b_k}\right] \\ &= \left(a_k - \frac{a_k}{b_k}\right) - (a_k - 1) \\ &= 1 - \frac{a_k}{b_k} = 1 - r_k. \end{aligned}$$

Hence

$$f(n) = \sum_{k=1}^{m}(1 - r_k) = m - 1.$$

Next we give reader solutions to problems of the *Second Stage Exam of the* 10*th Iranian Mathematical Olympiad* [1995: 9–10].

1. In the right triangle ABC ($A = 90°$), let the internal bisectors of B and C intersect each other at I and the opposite sides D and E respectively. Prove that the area of quadrilateral $BCDE$ is twice the area of the triangle BIC.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Clifton College, Bristol, UK; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Joseph Ling, University of Calgary, Calgary, Alberta; and by Dieter Ruoff, Department of Mathematics and Statistics, The University of Regina, Regina, Saskatchewan. We give the solution of Covas.

If b and c are the legs, a the hypotenuse, s the semiperimeter and r the inradius of the given right triangle then it is known that $r = s - a$.

The area of such a triangle is $bc/2$. On the other hand, the area of any triangle is sr. Setting the two expressions equal we have

$$bc = 2sr = 2s(s - a) \tag{1}$$

We see that (see figure 1.1)

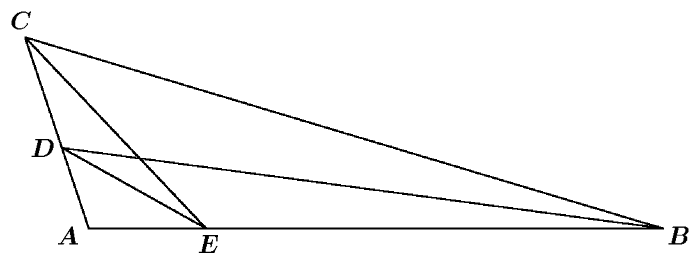


Figure 1.1

$$\text{area of quadrilateral } BCDE = \text{area } (\triangle ABC) - \text{area } (\triangle AED). \quad (2)$$

Since $AE = bc/(a+b)$ and $AD = bd/(a+c)$ we have Area $(\triangle AED) = \frac{1}{2}\frac{bc}{a+b} \cdot \frac{bc}{a+c}$. Substituting, using (2), we have

$$\begin{aligned}
&\text{area of quadrilateral } BCDE \\
&= \frac{1}{2}bc - \frac{1}{2}\frac{bc}{a+b} \cdot \frac{bc}{a+c} \\
&= \frac{1}{2}bc\left(1 - \frac{bc}{(a+b)(a+c)}\right) \\
&= \frac{1}{2}bc\frac{a(a+b+c)}{(a+b)(a+c)} = \frac{abcs}{(a+b)(a+c)}. \qquad (3)
\end{aligned}$$

Since $(a+b)(a+c) = a(a+b+c) + bc = 2as + 2s(s-a) = 2s^2$ we can write (3) in the form

$$\text{Area of quadrilateral } BCDE = \frac{abc}{2s}.$$

Finally, we substitute $2rs$ for bc from (1), simplify, and obtain Area of quadrilateral $BCDE = ar = 2(\text{area of } \triangle BIC)$, which was to be proved.

Editor's Note. Both Ling and Ruoff generalized the result proving more. Here is Ruoff's generalization.

Let $\triangle ABC$ be a triangle with angles 2α, 2β, 2γ, D the intersecting point of the angle bisector at B and AC, E the intersecting point of the angle bisector at C and AB, and I the incentre of $\triangle ABC$. Then

$$|\triangle BCI| \gtreqless \frac{1}{2}|BCDE| \quad \text{iff } \alpha \gtreqless \frac{\pi}{4}. \qquad (1)$$

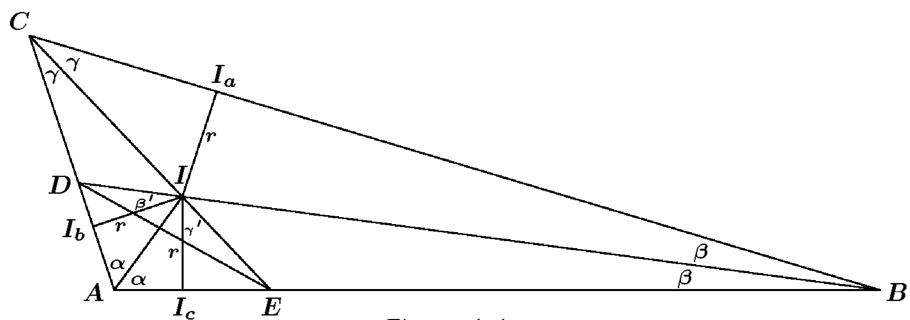


Figure 1.1a

Proof. Let I_a, I_b, I_c be the projections of I to BC, CA, AB respectively and note the LHS of (1) is equivalent to

$$|\triangle BII_c| + |\triangle CII_c| \gtreqless |BICDEB|,$$

$$|BICAB| - |II_bAI_c| \gtreqless |BICAB| - |\triangle ADE|$$

and

$$|\triangle ADE| \gtreqless |II_bAI_c|. \tag{2}$$

The essence and heuristic departure point of the following is the proof of

$$|\triangle BCI| = \frac{1}{2}|BCDE| \tag{1*}$$

respectively

$$|\triangle ADE| = |II_bAI_c| \tag{2*}$$

for triangles with right angle at A ($\alpha = \frac{\pi}{4}$). In this case II_cAI_c is obviously a square, $\triangle ADE$ a right triangle, $\angle I_cID = \beta' = \beta$, $\angle I_cIE = \gamma' = \gamma$, consequently $\beta' = \frac{\pi}{4} - \gamma'$, and (2*) becomes for $II_b = II_c = r$.

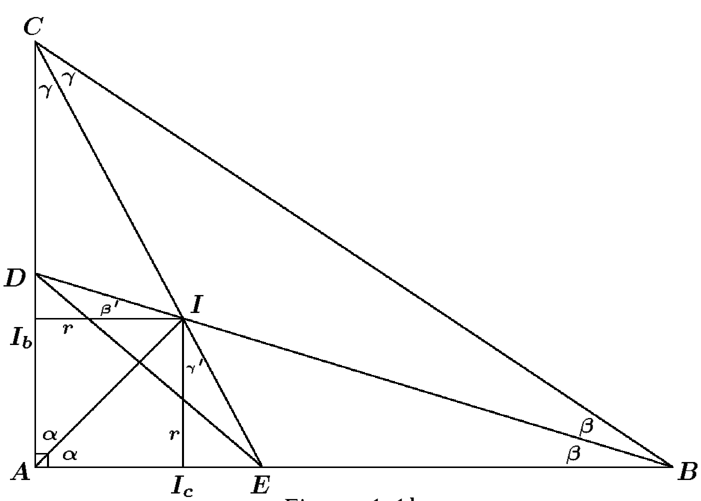


Figure 1.1b

$$\frac{r^2}{2}(1+\tan\gamma')(1+\tan(\frac{\pi}{4}-\gamma'))=r^2, \tag{3*}$$

which holds because of the trigonometric identity

$$(1+\tan x)(1+\tan(\frac{\pi}{4}-x))=2.$$

Returning to the general case, we first assume that I_c lies between A and E, and I_b between A and D, in which case (2) amounts to

$$\frac{r^2}{2}(\cot\alpha+\tan\gamma')(\cot\alpha+\tan\beta')\cdot\sin^2\alpha \gtreqless r^2\cot\alpha \tag{3}$$

respectively to

$$(1+\tan\alpha\tan\gamma')(1+\tan\alpha\tan\beta')\cdot\cos^2\alpha \gtreqless 1.$$

Multiplying this equation by $\frac{1}{\cos^2\alpha}=1+\tan^2\alpha$, we obtain

$$\frac{\tan\gamma'+\tan\beta'}{1-\tan\gamma'\cdot\tan\beta'}=\tan(\gamma'+\beta') \gtreqless \tan\alpha. \tag{4}$$

The angle sum in the quadrilateral $AEID$ is

$$2\alpha+(2\beta+\gamma)+(\gamma'+\pi-2\alpha+\beta')+(2\gamma+\beta)=2\pi \tag{5}$$

and hence

$$\begin{aligned}\gamma'+\beta' &= \pi-3(\beta+\gamma) = \pi-3\left(\frac{\pi}{2}-\alpha\right)\\ &= 3\alpha-\frac{\pi}{2} \gtreqless \alpha \text{ for } \begin{cases}\alpha>\frac{\pi}{4}\\ \alpha=\frac{\pi}{4}\\ \alpha<\frac{\pi}{4}\end{cases}.\end{aligned} \tag{6}$$

The conditions on the RHS of (6) determine the signs of (4) and one by one those of (3), (2) and (1), q.e.d.

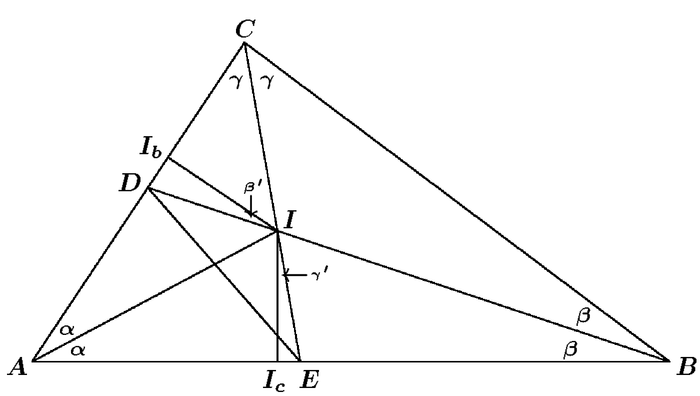


Figure 1.1c

If I_c lies between A and E, but D between A and I_b then, in (3) and in the following, β' has to be multiplied by -1, and (4), (5), (6) turn into

$$\tan(\gamma' - \beta') \gtreqless \tan\alpha, \tag{4*}$$

$$2\alpha + (2\beta+\gamma) + (\gamma' + \pi - 2\alpha - \beta') + (2\gamma+\beta) = 2\pi \tag{5*}$$

and

$$\gamma' - \beta' = 3\alpha - \frac{\pi}{2} \gtreqless \alpha \text{ for } \begin{cases} \alpha > \frac{\pi}{4} \\ \alpha = \frac{\pi}{4} \\ \alpha < \frac{\pi}{4} \end{cases}, \tag{6*}$$

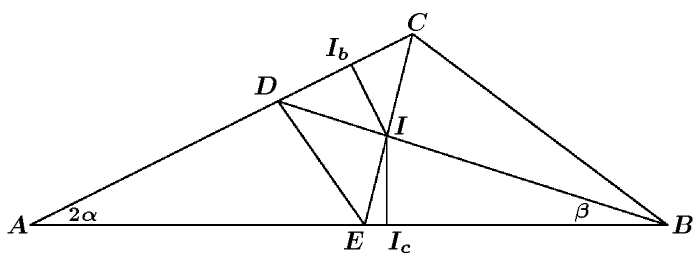


Figure 1.1d

from which, however, the same conclusions result as before. Since $2\alpha+\beta < \frac{\pi}{2}$, the only de facto applicable condition is $\alpha < \frac{\pi}{4}$. Also, if E lies between A and I_c and D between A and I_b, $\alpha < \frac{\pi}{4}$ is the only applicable condition; that the $<$-sign holds in (2) follows directly from the figure.

2. Given the sequence $a_0 = 1$, $a_1 = 2$, $a_{n+1} = a_n + \frac{a_{n-1}}{1+(a_{n-1})^2}$, $n > 1$, show that $52 < a_{1371} < 65$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Note first that "$n > 1$" in the statement should have been "$n \geq 1$" for the problem to be correct. We show that in general,

$$\sqrt{2n+1} \leq a_n \leq \sqrt{3n+2} \text{ for all } n \geq 0. \tag{1}$$

Since when $n = 1371$, $\sqrt{2n+1} = \sqrt{2743} \approx 52.374$ and $\sqrt{3n+2} = \sqrt{4115} \approx 64.148$, $52 < a_{1371} < 65$ would follow. To establish (1), we first show by induction that

$$a_n = a_{n-1} + \frac{1}{a_{n-1}} \text{ for all } n \geq 1. \tag{2}$$

This is clearly true for $n = 1$ since $a_1 = 2 = a_0 + \frac{1}{a_0}$. Suppose (2) holds for some $n \geq 1$. Then

$$a_n = \frac{(a_{n-1})^2+1}{a_{n-1}} \Rightarrow \frac{1}{a_n} = \frac{a_{n-1}}{1+(a_{n-1})^2}$$

and thus, from the given recurrence relation, we get $a_{n+1} = a_n + \frac{1}{a_n}$, completing the induction. Since clearly $a_n > 0$ for all n, we see from (2) that the sequence $\{a_n\}$ is strictly increasing. In particular, $\frac{1}{a_{n-1}^2} \le 1$ for all $n \ge 1$ and so from $a_n^2 = a_{n-1}^2 + 2 + \frac{1}{a_{n-1}^2}$ we get

$$a_{n-1}^2 + 2 < a_n^2 \le a_{n-1}^2 + 3 \text{ for all } n \ge 1. \tag{3}$$

Now we use (3) and induction to establish (1). The case when $n = 0$ is trivial since $a_0 = 1 < \sqrt{2}$. Suppose (1) holds for some $n \ge 0$. Then by (3),

$$a_{n+1} \le \sqrt{a_n^2 + 3} \le \sqrt{3n + 2 + 3} = \sqrt{3(n+1) + 2}$$

and

$$a_{n+1} > \sqrt{a_n^2 + 2} \ge \sqrt{2n + 1 + 2} = \sqrt{2(n+1) + 1}$$

and our proof is complete.

3. There is a river with cities on both of its sides. Some boat lines connect these cities in such a way that each line connects a city of one side to a city on the other side, and each city is joined exactly to k cities on the other side. One can travel between every two cities. Prove if one of the boat lines is cancelled, one can travel between every two cities.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

Originally, one can travel between every two cities. If we consider cities as vertices and boat lines as edges on a graph, then this graph is connected. We must show that if an edge is removed, then the graph is still connected.

We interpret cities on the two sides of the river as a bipartite graph since each line connects a city of one side to a city on the other side. As well, each vertex has k edges. If we count the number of edges for each vertex on one side we are counting all the vertices because each edge has exactly one end on that side. Thus, originally we have ak edges (where a is the number of vertices on one of the sides). This shows that we must have a vertices on both sides.

Now suppose, on the contrary, that by removing an edge we have two disjoint graphs. Suppose it is divided into parts U, V, X, Y where $U \cup V$ and $X \cup Y$ are the vertices of the two sides and the remaining edges are between U and X, and V and Y respectively. Further, since one edge was removed, let U and V have $k|U| - 1$ and $k|V|$ incident edges respectively. If X has $k|X| - 1$ incident edges, that would mean that originally the whole graph was not connected. Therefore X has $k|X|$ edges remaining and Y has $k|Y| - 1$ edges remaining. This is impossible! $k|U| - 1 \ne k|V|$ unless $k = 1$. If $k = 1$, $a = 1$ or else we would have disjoint lines!

Now there are cities on both sides, so $a > 1$, so $k > 1$, and it is impossible to have two disjoint bipartite graphs by removing one edge.

4. Prove that for each natural number t, 18 divides

$$A = 1^t + 2^t + \cdots + 9^t - (1 + 6^t + 8^t).$$

Remarks by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; and by Stewart Metchette, Gardena, California, USA.

There must be a misprint.

For $t = 1$, $A = 1 + 2 + 3 + \cdots + 9 - (1 + 6 + 8) = 30$,

and $t = 2$, $A = 1+2^2+3^2+\cdots+9^2-(1+6^2+8^2) = 285-101 = 184$.

Both are not divisible by 18.

5. In the triangle ABC we have $A \leq 90°$ and $B = 2C$. Let the internal bisector of C intersect the median AM (M is the mid-point of BC) at D. Prove that $\angle MDC \leq 45°$. What is the condition for $\angle MDC = 45°$?

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

It suffices to show that $\tan(\angle MDC) \leq 1$.

If $\triangle ABC$ has sides a, b, c, in the usual order, then the condition $B = 2C$ is equivalent to the condition $b^2 = c(c + a)$ (see this journal [1976: 74] and [1984: 287]). (1)

We introduce a Cartesian frame with origin at B and x-axis along BC:

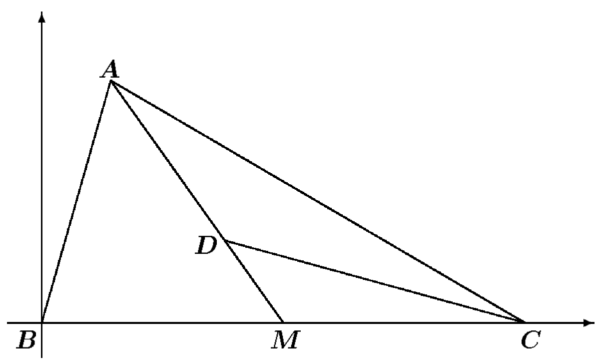


Figure 5

The coordinates of A are $(c \cdot \cos B, c \cdot \sin B)$, the coordinates of C are $(a, 0)$, and those of M are $(a/2, 0)$.

The internal angle bisector of C has slope

$$m_1 = \tan(180° - C/2) = -\tan(C/2)$$

and the slope of the median AM is

$$m_2 = \frac{c \cdot \sin B}{c \cdot \cos B - a/2}. \tag{2}$$

The law of cosines gives

$$b^2 = c^2 + a^2 - 2ca \cdot \cos B.$$

Substituting for b^2 from (1), we obtain

$$c(c+a) = c^2 + a^2 - 2ca \cdot \cos B,$$

and hence

$$c \cdot \cos B = (a-c)/2.$$

Substituting this into (2), we obtain

$$m_2 = -2 \cdot \sin B.$$

Since $B = 2C$ and $\sin 2C = 2\sin C \cos C$, this equation may be rewritten as

$$m_2 = -4 \sin C \cos C.$$

Using the formula for the tangent of the angle between two lines, we get

$$\tan(\angle MDC) = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\tan(C/2) + 4\sin C \cos C}{1 + 4(\tan(C/2)) \cdot \sin C \cos C}. \quad (3)$$

We express $\sin C$ and $\cos C$ in terms of $\tan(C/2)$. Putting $t = \tan(C/2)$, we get

$$\sin C = \frac{2t}{(1+t^2)}, \quad \cos C = \frac{(1-t^2)}{(1+t^2)}.$$

When these are substituted into (3), it becomes

$$\tan(\angle MDC) = \frac{-t + \frac{8t(1-t)^2}{(1+t^2)^2}}{1 + \frac{8t^2(1-t^2)}{(1+t^2)^2}} = \frac{-t(1+t^2)^2 + 8t(1-t^2)}{(1+t^2)^2 + 8t^2(1-t^2)}.$$

We now prove that $\tan(\angle MDC) \le 1$. This holds if and only if

$$8t(1-t^2) - t(1+t^2)^2 \le 8t^2(1-t^2) + (1+t^2)^2,$$

or, equivalently,

$$8t(1-t^2) - 8t^2(1-t^2) \le (1+t^2)^2 + t(1+t^2)^2,$$

or

$$8t(1-t^2)(1-t) \le (1+t^2)^2(1+t).$$

Dividing both sides by the positive number $1+t$, we get

$$8t(1-t)^2 \le (1+t^2)^2,$$

equivalent to

$$8t - 16t^2 + 8t^3 \le 1 + 2t^2 + t^4,$$

which is indeed true since

$$t^4 - 8t^3 + 18t^2 - 8t + 1 = (t^2 - 4t + 1)^2 \ge 0.$$

Equality $\angle MDC = 45^\circ$ occurs if and only if $t^2 - 4t + 1 = 0$, where $t = \tan(C/2)$. This is satisfied when $t = 2 - \sqrt{3}$, i.e. $C = 30^\circ + 360^\circ k$; or $t = 2 + \sqrt{3}$, i.e. $C = 150^\circ + 360^\circ k$ $(k = \ldots, -2, -1, 0, 1, 2, \ldots)$.

The only acceptable value for C is 30°.

We conclude that $\angle MDC = 45^\circ$ iff $A = 90^\circ$, $B = 60^\circ$, $C = 30^\circ$.

6. Let X be a non-empty finite set and $f : X \to X$ a function such that for all x in X, $f^p(x) = x$, where p is a constant prime. If $Y = \{x \in X : f(x) \neq x\}$, prove that the number of elements of Y is divisible by p.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and remarks.

For each $x \in Y$ consider that *orbit*, $B(x)$, of x defined by

$$B(x) = \{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}.$$

We claim that all the elements of $B(x)$ are distinct.

Suppose not. Then let j be the *least* positive integer such that $f^i(x) = f^j(x)$ for some integer i with $0 \leq i < j \leq p - 1$. (We define $f^i(x) = x$ if $i = 0$.) Then

$$x = f^p(x) = f^{p-j}(f^j(x)) = f^{p-j}(f^i(x)) = f^{p-j+i}(x)$$
$$\Rightarrow f^{j-i}(x) = f^{j-i}(f^{p-j+i}(x)) = f^p(x) = x.$$

Since $0 < j - i \leq j$, we must have $i = 0$ and thus $f^j(x) = x$. Now let $p = qj + r$, where q, r are integers with $q > 0$ and $0 \leq r < j$. Clearly $f^j(x) = x$ implies $f^{qj}(x) = x$ and hence

$$f^r(x) = f^r(f^{qj}(x)) = f^p(x) = x.$$

Since $r < j$, we must have $r = 0$ and thus $p = qj$. Since $f(x) \neq x$, $j > 1$. On the other hand, since $j < p$, $q > 1$. Hence p is a composite, a contradiction. Therefore, $f^i(x) \neq f^j(x)$ for all $i = 0, 1, 2, \ldots, p-1$, we see that $B(x) \subset Y$.

Next we show that the orbits of two elements of Y are either *disjoint* or *identical*. Let $x, y \in Y$ and suppose $B(x) \cap B(y) \neq \emptyset$. Then $f^l(x) = f^k(y)$ for some integers l and k, with $0 \leq l \leq k \leq p - 1$. Hence

$$y = f^k(y) = f^{p-k}(fk(y)) = f^{p-k}(f^l(x)) = f^{p-k+l}(x),$$

which show that $y \in B(x)$. It then follows that $B(x) = B(y)$. Therefore Y can be partitioned into disjoint orbits each having cardinality p and the result follows.

Remarks. (1) Actually, the result still holds even when $p = 1$ since in this case $Y = \emptyset$ and thus $|Y| = 0$. (2) The result need not hold if p is composite. A counterexample is given by $X = \{1, 2, 3, 4, 5, 6\}$, $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 5$, $f(5) = 6$, and $f(6) = 3$. In this case, $p = 4$ and $Y = X$, $|Y| = 6$.

That completes the Olympiad Corner for this issue. Send me your contests and nice solutions.