THE ACADEMY CORNER

No. 4

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In the February 1996 issue, we gave the first set of problems in the Academy Corner. Here we present solutions to the first three questions, as sent in by Sefket Arslanagić, Berlin, Germany.

Memorial University Undergraduate Mathematics Competition 1995

1. Find all integer solutions of the equation \( x^4 = y^2 + 71 \).

Solution. From the equation \( x^4 = y^2 + 71 \), we get

\[
(x^2 \Leftrightarrow y)(x^2 + y) = 71 \cdot 1.
\]

Since 71 is prime, we have

\[
\begin{cases}
  x^2 \Leftrightarrow y = 1 \\
  x^2 + y = 71
\end{cases}
\]

or

\[
\begin{cases}
  x^2 \Leftrightarrow y = 71 \\
  x^2 + y = 1
\end{cases}
\]

so that

\[
2x^2 = 72, \quad \text{or} \quad 2x^2 = 72,
\]

giving

\[x_{1,2} = \pm 6, \quad y_{1,2} = 35, \quad x_{3,4} = \pm 6 \quad y_{3,4} = 35.\]

That is,

\[
(x, y) \in \{(6, 35), (6, 35), (6, -35), (-6, -35)\}.
\]

The other possibility is \((x^2 \Leftrightarrow y)(x^2 + y) = (71 \cdot 1)\), which gives:

\[
x = \pm 6 \notin \mathbb{Z}.
\]

Thus we have found all solutions.

2. (a) Show that \( x^2 + y^2 \geq 2xy \) for all real numbers \( x, y \).

(b) Show that \( a^2 + b^2 + c^2 \geq ab + bc + ca \) for all real numbers \( a, b, c \).
Solution.

(a) \(x^2 + y^2 \geq 2xy \Leftrightarrow (x \Leftrightarrow y)^2 \geq 0\) for all \(x, y \in \mathbb{R}\).

(b) From (a), we get

\[
a^2 + b^2 \geq 2ab; \quad a^2 + c^2 \geq 2ac; \quad b^2 + c^2 \geq 2bc,
\]

that is

\[
2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc)
\]
or

\[
a^2 + b^2 + c^2 \geq ab + ac + bc, \quad \text{for all } a, b \in \mathbb{R}.
\]

Remark: Let \(a, b, c \in \mathbb{C}\). We observe that the equality

\[
x^3 \Leftrightarrow 1 = 0,
\]

has roots \(x_1 = 1, x_2, x_3 = \frac{1}{2}(\Leftrightarrow \pm i\sqrt{3})\), so that \(x_1 \neq x_2 \neq x_3\). Let

\[
a = x_1, \ b = x_2, \ c = x_3,
\]
giving \(a + b + c = 0, \ ab + ac + bc = \frac{1}{2}(\Leftrightarrow 1 + i\sqrt{3}) + \frac{1}{2}(\Leftrightarrow i\sqrt{3}) + \frac{1}{4}(1 + 3) = 0\) and \(a^2 + b^2 + c^2 = (a + b + c)^2 \Leftrightarrow 2(ab + ac + bc) = 0\).

(i) Further, in (b) the equality holds because

\[
a^2 + b^2 + c^2 = ab + ac + bc \Leftrightarrow \frac{1}{2}[(a \Leftrightarrow b)^2 + (a \Leftrightarrow c)^2 + (b \Leftrightarrow c)^2] = 0
\]

if and only if \(a = b = c \in \mathbb{R}\).

(ii) Also, (b) holds as well, since \(a^2 + b^2 + c^2 = ab + ac + bc\) for \(a, b, c \in \mathbb{C}\) and \(a \neq b \neq c\).

3. Find the sum of the series

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{99}{100!}.
\]

Solution. We have

\[
\frac{1}{k!} \Leftrightarrow \frac{1}{(k+1)!} = \frac{1}{k!} \Leftrightarrow \frac{1}{(k+1)!} = \frac{k}{(k+1)!}.
\]

Thus

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{n}{(n+1)!} = 1 \Leftrightarrow \frac{1}{(n+1)!}.
\]

For \(n = 99\), we get

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{99}{100!} = 1 \Leftrightarrow \frac{1}{100!}.
\]