SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2035. [1995: 130] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

If the locus of a point E is an ellipse with fixed foci F and G, prove that the locus of the incentre of triangle EFG is another ellipse.

Solution by the Science Academy Problem Solvers, Austin, Texas.

Let $F = (-c, 0)$ and $G = (c, 0)$ so that the locus of $E = (x, y)$ satisfies $GE + EF = 2a$ (where a is the length of the semimajor axis). Let I be the incentre and I' be the point where the incircle is tangent to FG (so that $II' \perp FG$). We have

$$FI' = a + c - GE = \frac{EF - GE}{2} + c$$

[since $s = a + c$ is half the perimeter of ΔEFG, and the tangents to the incircle from the vertices are equal in pairs and sum to $2s$]. If O is the origin then $FO = c$ and

$$\begin{aligned} OI' = FI' - c &= \frac{EF - GE}{2} = \frac{EF^2 - EG^2}{4a} \\ &= \frac{(x^2 + 2cx + c^2 + y^2) - (x^2 - 2cx + c^2 + y^2)}{4a} = \frac{cx}{a}. \end{aligned}$$

Since $r = II'$ is the radius of the incircle [and the area of ΔEFG is rs]

$$II' = \frac{\text{area}(EFG)}{a + c} = \frac{cy}{a + c}.$$

Thus, $I = \left(\frac{c}{a}x, \frac{c}{a + c}y\right)$. The mapping $(x, y) \to \left(\frac{c}{a}x, \frac{c}{a + c}y\right)$ is therefore an affine transformation that maps each point E of the given ellipse to the point I. Affine transformations map ellipses to ellipses so that I traces out an ellipse.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2040. [1995: 130] *Proposed by Frederick Stern, San Jose State University, San Jose, California.*

Let $a < b$ be positive integers, and let

$$t = \frac{2^a - 1}{2^b - 1}.$$

What is the relative frequency of 1's (versus 0's) in the binary expansion of t?

[Ed. My interpretation of the question asked is to find the ratio of the number of 1's to the number of 0's; most solvers also read it this way.

The proposer was the only solver to actually mention that the ratio we are to compute must be done asymptotically. I feel that should have been part of the problem statement.]

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x = .\overline{000\ldots01}$, where there are $b-1$ zeros (the pattern below the superbar repeats infinitely often). Then $(2^b-1)x = .111\ldots1 = 1$, so $x = 1/(2^b-1)$. Therefore, $(2^a-1)/(2^b-1) = .\overline{00\ldots01\ldots11}$, where there are $b-a$ zeros and a ones in every period, so there are a zeros for every $b-a$ ones when counted from the left.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; Science Academy Problem Solvers, Austin, Texas, USA; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Janous generalizes the problem to show that in base $z+1$ we have

$$\frac{(z+1)^a}{(z+1)^b - 1} = 000\ldots0zz\ldots z$$

where there are $b-a$ zeros and a copies of z when $a < b$.

2041. [1995: 157] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of triangle ABC. AP, BP, CP meet BC, CA, AB at D, E, F respectively. Let M and N be points on segments BF and CE respectively so that $BM : MF = EN : NC$. Let MN meet BE and CF at X and Y respectively. Prove that $MX : YN = BD : DC$.

Solution by Jari Lappalainen, Helsinki, Finland.

First, we apply Menelaus' theorem to triangles CNY and YFM with line EB. Dividing the results, we get

$$\frac{-1}{-1} = \frac{\frac{CE}{EN} \cdot \frac{NX}{XY} \cdot \frac{YP}{PC}}{\frac{FB}{BM} \cdot \frac{MX}{XY} \cdot \frac{YP}{PF}} = 1$$

and substituting $CE : EN = FB : BM$ (which follows directly from $BM : MF = EN : NC$), we get

$$\frac{NX}{MX} = \frac{PC}{PF},$$

or equivalently

$$\frac{FC}{PF} = \frac{MN}{XM}. \tag{1}$$

In a similar way using Menelaus's theorem for triangles BXM and XNE with line FC, and substituting $MF : FB = NC : CE$, we get

$$\frac{EB}{PE} = \frac{NM}{YN}. \tag{2}$$

Finally, applying Ceva's theorem to triangle ABC and Menelaus's theorem to triangles CEP and BEA, with lines AB and CF respectively, we find

$$\frac{1}{-1 \times (-1)} = \frac{\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}}{\frac{CA}{AE} \cdot \frac{EB}{BP} \cdot \frac{PF}{FC} \times \frac{BP}{PE} \cdot \frac{EC}{CA} \cdot \frac{AF}{FB}} = \frac{BD}{DC} \cdot \frac{PE}{EB} \cdot \frac{FC}{PF} = 1.$$

Using (1) and (2)

$$\frac{BD}{DC} \cdot \frac{YN}{NM} \cdot \frac{MN}{XM} = -\frac{BD}{DC} \cdot \frac{YN}{XM} = 1,$$

or $MX : YN = BD : DC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2042. [1995: 157] *Proposed by Jisho Kotani, Akita, Japan, and K.R.S. Sastry, Dodballapur, India.*

If A and B are three-digit positive integers, let $A * B$ denote the six-digit integer formed by placing them side by side. Find A and B such that

$$A, \quad B, \quad B - A, \quad A * B \quad \text{and} \quad \frac{A * B}{B}$$

are all integer squares.

Solution by Kathleen E. Lewis, SUNY Oswego, New York, NY, USA.

Since A, $B - A$ and B are all integer squares, their square roots form a Pythagorean triple. Thus, the easiest place to look for numbers meeting the stated conditions is in the multiples of the $3, 4, 5$ triple. So, letting $A = 9t^2$ and $B = 25t^2$ for an integer t, we see that $B - A = 16t^2$,

$$A * B = 1000A + B = 9000t^2 + 25t^2 = 9025t^2 = (95t)^2$$

and

$$\frac{A * B}{B} = \frac{(95t)^2}{(5t)^2} = 19^2.$$

Any choice of t would yield A and B satisfying all the other conditions, but in order to make A and B three–digit numbers, t would have to be 4, 5 or 6, yielding values of

$$(144, 400), \qquad (225, 625), \qquad \text{and} \quad (324, 900)$$

for (A, B).

These are in fact the only possibilities. Suppose $A = x^2$, $B - A = y^2$ and $B = z^2$ satisfy the given conditions. Let $q = \gcd(x, y, z)$, $x' = x/q$, $y' = y/q$ and $z' = z/q$.

Then ${x'}^2 + {y'}^2 = {z'}^2$ and ${z'}^2$ divides $(A * B - B)/q^2 = 1000{x'}^2$. Since x' and z' are relatively prime, this means that ${z'}^2$ must divide 1000, so z' must be 1, 2, 5 or 10. By inspection, none of the others are possible values, so z' must be 5. We can also rule out the remaining case of $x' = 4$, $y' = 3$ and $z' = 5$ [that is, $A = 16t^2$ and $B = 25t^2$] since $(A * B)/t^2 = 16025$ is not a perfect square.

Editor's note. As some other solvers point out, the case $B - A = 0$ should be considered. This is the case $y' = 0$, $x' = z' = 1$, which is impossible because $A * B = 1001x^2$ is not a perfect square.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; J.K. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G.

HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposers.

Most solvers found all three solutions.

The two proposers actually had sent the editor this problem, or something quite similar, independently and at almost the same time.

2043. [1995: 158] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

What is the locus of a point interior to a fixed triangle that moves so that the sum of its distances to the sides of the triangle remains constant?

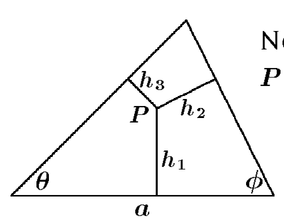
Editor's Comment.

The solution to the problem as stated in the proposal was quite easy using analytic/trigonometric arguments and this was the path chosen by most of the solvers. Some contributors, anxious no doubt to uphold the impeccable standards of CRUX, tried to do something more with the proposal - much to the delight of the Editor. One such example is to look at the locus for different triangles as portrayed in the featured solution by Hess. Just the observation that all points in the interior is the locus when the given triangle is equilateral can do wonders for the morale of the Editor. The second solution, while restricted to the interior of the triangle, did offer a respite from the analytic. Several solvers made comments of varying substance relating to the case when the point was exterior to the triangle. The third solution by Fritsch did the job nicely and was novel.

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*

From the figure, where the largest side is the base, we have

$$
\begin{aligned}
h_1 &= y, \\
h_2 &= (a-x)\sin\phi - y\cos\phi, \\
h_3 &= x\sin\theta - y\cos\theta.
\end{aligned}
$$



Note:
$P = P(x, y)$

From $h_1 + h_2 + h_3 = k$ (a constant), we get

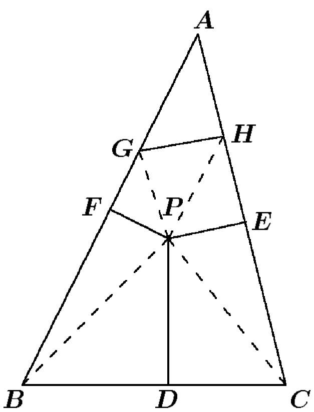
$$y(1 - \cos\phi - \cos\theta) = k - a\sin\phi + x(\sin\phi - \sin\theta).$$

There are several cases:

1. When $\phi = \theta = 60°$, the whole interior of the triangle has $h_1+h_2+h_3 = a\sqrt{3}/2$.
2. When $\cos\theta + \cos\phi = 1$ but $\theta \neq \phi$, the locus is $x = \dfrac{k - a\sin\phi}{\sin\theta - \sin\phi}$.
3. When $\theta = \phi \neq 60°$, the locus is $y = \dfrac{k - a\sin\phi}{1 - 2\cos\phi}$.
4. Otherwise, the locus is a straight line:
$$y = \frac{k - a\sin\phi}{1 - \cos\theta - \cos\phi} + \frac{x(\sin\phi - \sin\theta)}{1 - \cos\theta - \cos\phi}.$$

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let P be a point interior to triangle ABC, so that the sum of its distances to the sides of triangle ABC is constant, and let D, E, F be the feet of the perpendiculars to BC, CA, AB respectively.
We assume that triangle ABC is non-equilateral and that BC is the least side.
We put $PD + PE + PF = k$, where k is a constant.



Let G, H be the points on the side AB, AC respectively, such that $BG = CH = BC$. Then

$$[PBC] + [PCH] + [PBG] = \tfrac{1}{2}a(PD + PE + PF) = \tfrac{1}{2}ak,$$

which is constant ($[XYZ]$ denotes the area of triangle XYZ).
Since the area of quadrilateral $BCHG$ is constant, so also is the area of triangle PGH. Therefore P lies on a fixed line ℓ, parallel to GH.
Hence the locus of P is the segment of ℓ contained within the triangle ABC.

III. *Solution by Rudolf Fritsch, Ludwig-Maximilians-Universität, München, Germany.*

Let ABC be a triangle in the real plane; without loss of generality we assume $a \geq b \geq c$. Then we may choose cartesian coordinates such that

$$A = (0, h), \quad B = (-p, 0), \quad C = (q, 0)$$

with h, p, $q > 0$. The signed distance of a point in the plane from a side of this triangle is taken positive if the point is in the same half plane as the vertex opposite to this side, thus, the signed distance of $P = (x, y)$ from the line BC is just y.

For the other sides of triangle we choose the following equations

$$\begin{aligned} AB &\equiv x \cdot \sin\beta - y \cdot \cos\beta + h \cdot \cos\beta = 0\,, \\ AC &\equiv -x \cdot \sin\gamma - y \cdot \cos\gamma + h \cdot \cos\gamma = 0\,. \end{aligned}$$

These equations are called the Hessian normal forms of the lines under consideration. The namesake is OTTO LUDWIG HESSE (born in Königsberg/East Prussia 1811/4/22, died in München 1874/8/4) although this form has already been used by Gauss in a paper published in 1810. The general idea is to normalize the line equation

$$g \equiv ux + vy + w = 0$$

by $u^2 + v^2 = 1$, $w \geq 0$ (or $w \leq 0$), the line g being unique if it does not pass through the origin. The advantage of this form is that for any point $P = (x, y)$ in the plane the expression

$$d(x, y) = ux + vy + w$$

gives the signed distance of P from g, positive if and only if P is on the same side of g as the origin (origin not on g).

Thus, the sum of the signed distances of $P = (x, y)$ from the reference triangle is

$$s(x, y) = x \cdot (\sin\beta - \sin\gamma) + y \cdot (1 - \cos\beta - \cos\gamma) + h \cdot (\cos\beta + \cos\gamma)\,.$$

Since this expression is linear, the equation $s(x, y) = k$ describes a line, for any $k \in R$. Since the slope is independent of k, all lines obtained in this way form a pencil of parallel lines. A distinguished member of this pencil is obtained by taking $k = 0$ giving the line connecting the points where the exterior angle bisectors meet the opposite sides.

Taking absolute distances we get buckled lines for the desired locus. If the problem is restricted to the interior points of the triangle ABC, then the result gives parallel line segments.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIELS BEJLEGAARD, Stavanger, Norway; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; ASHISH KR. SINGH, Kanpur, India; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Klamkin comments: since $a_1r_1 + a_2r_2 + a_3r_3 = 2\Delta$ (where r_i is the distance to side a_i, and Δ is the area of the triangle), the constant sum of the distances must be bounded by

$$\frac{2\Delta}{a_3} \geq r_1 + r_2 + r_3 \geq \frac{2\Delta}{a_1},$$

where it is assumed that $a_1 \geq a_2 \geq a_3$. Otherwise, there are no points in the locus. This is also true, if, for example, a_3 is smaller than the other sides and the constant sum is $\frac{2\Delta}{a_3}$.

2044. [1995: 158] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that $n \geq m \geq 1$ and $x \geq y \geq 0$ are such that

$$x^{n+1} + y^{n+1} \leq x^m - y^m.$$

Prove that $x^n + y^n \leq 1$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

There is nothing to prove if $x = y$ since in this case the given inequality implies $x = y = 0$. Suppose $0 \leq y < x$, then $x^m - y^m > 0$. From $x^{n+1} \leq x^{n+1} + y^{n+1} \leq x^m - y^m \leq x^m$ and $n + 1 > m$, it follows that $x \leq 1$. Using $n - m \geq 0$ and $m \geq 1$, we now obtain

$$\begin{aligned}(x^n + y^n)(x^m - y^m) &= x^{n+m} - y^{n+m} - (xy)^m(x^{n-m} - y^{n-m}) \\ &\leq x^{n+m} \leq x^{n+1} \\ &\leq x^{n+1} + y^{n+1} \leq x^m - y^m.\end{aligned}$$

Dividing through by $x^m - y^m$ gives $x^n + y^n \leq 1$.

II. *Solution by the proposer.*

It is easy to see that $1 \geq x \geq y \geq 0$, and thus $x^m \leq x$, $xy^n \leq y^m$ and $x^n y \leq y$. Therefore,

$$\begin{aligned}(x^n + y^n)(x + y) &= x^{n+1} + y^{n+1} + x^n y + xy^n \\ &\leq x^m - y^m + y + y^m \leq x + y.\end{aligned}$$

Dividing through by $x + y$ [the case when $x + y = 0$ being trivial – Ed.] yields $x^n + y^n \leq 1$. Equality holds if and only if $x = 1$, $y = 0$.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

Janous showed that if $1 \le m \le n-1$, *then the condition can be relaxed to* $x^{n+1}+y^{n+1} \le x^m - y^{n-1}$. *Flanigan obtained the stronger result that if* $k = \lfloor \frac{n+1}{m} \rfloor$, *then* $x^{(k-1)m} + (k-1)y^{(k-1)m} \le 1$ *if* $m|n+1$ *and* $x^{km} + ky^{km} \le 1$ *if* $m \not| n+1$.

2045. [1995: 158] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Show that there are an infinite number of Pythagorean triangles (right-angled triangles with integer sides) whose hypotenuse is an integer of the form $3333\ldots3$.

Once again, our readers have been extremely inventive! Most provided a way to construct an infinite sequence (or more than one sequence) of suitable Pythagorean triples (a, b, c), where $a^2 + b^2 = c^2$. We summarize the results below.

I. *Solution by: Niels Bejlegaard, Stavanger, Norway; Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA; Christopher J. Bradley, Clifton College, Bristol, UK; Miguel Angel Cabezón Ochoa, Logroño, Spain; Toby Gee, student, the John of Gaunt School, Trowbridge, England; David Hankin, Hunter College Campus Schools, New York, NY, USA; Richard I. Hess, Rancho Palos Verdes, California, USA; Friend H. Kierstead Jr., Cuyahoga Falls, Ohio, USA; Kathleen E. Lewis, SUNY Oswego, NY, USA; Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; P. Penning, Delft, the Netherlands; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; David R. Stone, Georgia Southern University, Statesboro, USA; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.*

First, note that neither 3 nor 33 is the hypotenuse of a Pythagorean triangle.

Let (a, b, c) be a Pythagorean triple, where $c = 33\ldots3$ is a k-digit integer, $k > 2$. Then (ma, mb, mc) is also a Pythagorean triple, for all $m = 10^k + 1, 10^{2k} + 10^k + 1, \ldots, \sum_{i=0}^{n} 10^{ik}, \ldots$ and

$mc = 33\ldots3$ has $2k, 3k, \ldots, (n+1)k, \ldots$ digits.

Each of the following triples can each be used in this way to generate an infinite sequence of triples:

$(108, 315, 333)$
$(660, 3267, 3333)$
$(7317, 32520, 33333)$
$(128205, 307692, 333333)$
$(487560, 3297483, 3333333)$
$(25114155, 21917808, 33333333)$.

II. *Solution by: Heinz-Jürgen Seiffert, Berlin, Germany.*

$$\left(\frac{2}{3}10^n(10^{2n}-1), \frac{1}{3}(10^{2n}-1)^2, \frac{1}{3}(10^{4n}-1)\right), \text{ where } n \in \mathrm{N},$$

gives an infinite sequence of Pythagorean triples with hypotenuse the $4n$-digit integer $33\ldots3$. Note that the first triple, $(660, 3267, 3333)$, is the same as one given above, but the rest of the sequence is different.

Also solved (in a non-constructive way) by ASHISH KR. SINGH, student, Kanpur, India; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and CHRIS WILDHAGEN, Rotterdam, the Netherlands. There was one partial solution.

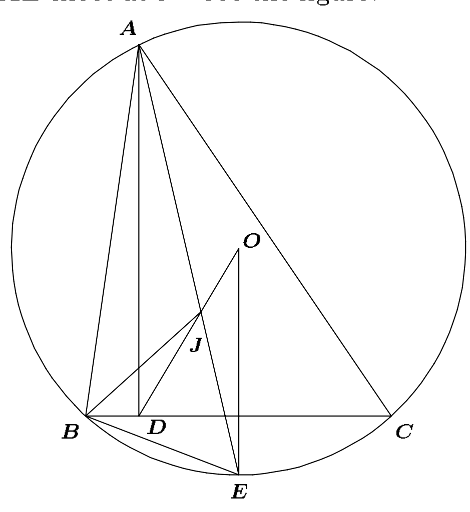
2047. [1995: 158] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

ABC is a non-equilateral triangle with circumcentre O and incentre I. D is the foot of the altitude from A to BC. Suppose that the circumradius R equals the radius r_a of the excircle to BC. Show that O, I and D are collinear.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

If $AB = AC$, the I, O lie on AD; hence O, I and D are collinear. Henceforth assume $AB \neq AC$.

Let the angle bisector of the angle $\angle A$ intersect the circumcircle of ABC at E and let DO and AE meet at J - see the figure.



OE is perpendicular to BC, so AD and OE are parallel. Therefore, since $R = r_a$

$$\frac{AJ}{JE} = \frac{AD}{OE} = \frac{h_a}{R} = \frac{h_a}{r_a} = \frac{2S}{ar_a} = \frac{2r_a(s-a)}{ar_a} = \frac{b+c-a}{a}, \tag{1}$$

where h_a, S, s denote the altitude AD, the area, and the semiperimeter of ABC, respectively. To complete the solution it is enough to show that J is the incentre of ABC. Using Ptolemy's theorem on the quadrilateral $ACEB$, and the fact that $BE = CE$, we get

$$(AJ + JE)a = BE(b + c). \qquad (2)$$

Since (1) is equivalent to $(AJ + JE)a = JE(b + c)$, comparing it with (2) we obtain $JE(b + c) = BE(b + c)$, which gives $JE = BE$. Hence

$$\angle JBA + \angle BAJ = \angle BJE = \angle JBE = \angle CBJ + \angle CBE = \angle CBJ + \angle EAC.$$

Since $\angle BAJ = \angle EAC$, we get $\angle JBA = \angle CBJ$, which means that BJ bisects $\angle B$. And since AJ bisects $\angle A$, J is the incentre of ABC, as we wished to prove.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, student, Kanpur, India; and the proposer.

Several solvers noted that the result was mentioned in the Editor's comments following the solution to problem 1918 (*CRUX, Vol. 21, No. 1*).

2046. [1995: 158] *Proposed by Stanley Rabinowitz, Westford, Massachusetts, USA.*

Find integers a and b so that

$$x^3 + xy^2 + y^3 + 3x^2 + 2xy + 4y^2 + ax + by + 3$$

factors over the complex numbers.

Solution by the proposer.

If the cubic is to factor in $\mathbb{C}[x, y]$, then one of the factors must be linear. Without loss of generality, we may assume this factor is of the form $x - py - q$ where p and q are complex numbers.

Substituting $x = py + q$ in the original cubic, we get a polynomial in y that must be identically 0. Thus each of its coefficients must be 0. This gives us the four equations:

$$\begin{aligned} 1 + p + p^3 &= 0 \\ 4 + 2p + 3p^2 + q + 3p^2q &= 0 \\ 3 + aq + 3q^2 + q^3 &= 0 \\ b + ap + 2q + 6pq + 3pq^2 &= 0. \end{aligned}$$

Solving these equations simultaneously, yields $a = 4$ and $b = 5$. (*The editor being a mere mortal needed Maple to verify this claim!*)

As a check we note that the resulting polynomial can be written as $(x+y+2)(y+1)^2+(x+1)^3$. The reducibility of this polynomial will not change if we let $x = X-1$ and $y = Y-1$. This produces the polynomial $X^3+XY^2+Y^3$. Letting $z = X/Y$ shows that this polynomial factors over $\mathbb{C}[x, y]$ since z^3+z+1 factors over $\mathbb{C}[z]$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA. Rabinowitz also remarks that,

"except for a few exceptional cases, if $f(x,y)$ is a cubic polynomial in $\mathbb{C}[x, y]$, there will be unique complex constants a and b such that $f(x,y)+ax+by$ factors over $\mathbb{C}[x, y]$."

2049*. [1995: 158] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1 respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1}+\frac{1}{GB_1}+\frac{1}{GC_1}+\frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right).$$

Equality holds if $ABCD$ is regular. (This inequality, if true, would be a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad; see [1994: 41].)

Solution to right hand inequality by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We consider a generalization of the right hand inequality:

Let G and O be the centroid and circumcentre of an n-dimensional simplex $A_0A_1\ldots A_n$ inscribed in a sphere of radius R. Let the lines A_iG meet the sphere again in points A_i' : $i = 0, 1, \ldots, n$. If the edges are denoted by $e_j; j = 1, 2, \ldots, n(n+1)/2$, then

$$\sum_{i=0}^{n}\frac{1}{GA_i'} \leq \sqrt{\frac{8(n+1)}{n^3}}\sum_{j=0}^{n(n+1)/2}\frac{1}{e_j}.$$

By the Power-of-a Point Theorem, we have $A_1'G \cdot A_iG = R^2 - OG^2$. By the power mean inequality, we have

$$\sum\frac{A_iG}{n+1} \leq \left(\sum\frac{(A_iG)^2}{n+1}\right)^{\frac{1}{2}}.$$

So it suffices to prove the stronger inequality:

$$\left(\sum (A_iG)^2(n+1)\right)^{\frac{1}{2}} \le \sqrt{\frac{8(n+1)}{n^3}}\left(R^2-(OG)^2\right)\sum\frac{1}{e_j}. \qquad (1)$$

It is known that

$$\sum (A_iG)^2 \quad = \quad (n+1)\left(R^2-(OG)^2\right) \qquad (2)$$

and

$$\left(R^2-(OG)^2\right) \quad = \quad \sum\frac{e_j^2}{(n+1)^2}. \qquad (3)$$

Using (2) and (3), we see that (1) becomes, after raising both sides to the power $2/3$,

$$\frac{n(n+1)}{2} \le \left(\sum e_j^2\right)^{1/3}\left(\sum\frac{1}{e_j}\right)^{2/3},$$

and the result follows immediately from Hölder's inequality.

There is equality only if the simplex is regular.

Comment: Corresponding to the given left hand inequality, the analogous one for the simplex (and not as yet proved) is

$$\frac{n+1}{R} \le \sum\frac{1}{A_i'G}$$

or, equivalently

$$R\sum A_iG \ge (n+1)\left(R^2-(OG)^2\right) = \sum (A_iG)^2.$$

Even more generally, I conjecture that, for $p \ge 1$, we have

$$\frac{2R\,(n^p+1)}{(n+1)\,(n^{p-1}+1)} \ge \frac{\sum (A_iG)^{p+1}}{\sum (A_iG)^p}.$$

Except for the case $p = 1$, there is no equality for a regular simplex, but for a degenerate one with n vertices coinciding at one end of a diameter and the remaining vertex at the other end of the diameter.

No other solutions were received.

2050. [1995: 158] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Find all real numbers x and y satisfying the system of equations

$$2^{x^2+y} + 2^{x+y^2} = 128, \qquad \sqrt{x}+\sqrt{y} = 2\sqrt{2}.$$

Solution. Essentially identical solutions were submitted by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA; Christopher J. Bradley, Clifton College, Bristol, UK; Toby Gee, student, the John of Gaunt School, Trowbridge, England; David Hankin, Hunter College Campus Schools, New York, NY, USA; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Kee-Wai Lau, Hong Kong; Waldemar Pompe, student, University of Warsaw, Poland; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.

Squaring $\sqrt{x} + \sqrt{y} = 2\sqrt{2}$, we obtain that

$$x + y = 8 - 2\sqrt{xy} \geq 8 - (x + y),$$

whence

$$x + y \geq 4. \tag{1}$$

Further

$$x^2 + y^2 \geq \tfrac{1}{2}(x + y)^2 \geq 8. \tag{2}$$

From (1) and (2), we have

$$\begin{aligned} 64 &= \frac{2^{x^2+y} + 2^{x+y^2}}{2} \\ &\geq 2^{\frac{1}{2}(x^2+y+x+y^2)} \geq 2^6 = 64. \end{aligned}$$

Thus $x + y = 4$ and $x^2 + y^2 = 8$. These are easily solved to obtain that $x = y = 2$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, MARTHA BELL and JIM BRASELTON; Georgia Southern University, Statesboro, Georgia, USA; STAN WAGON, Macalester College, St. Paul, Minnesota, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA. One other submission was received which assumed that x and y were integers.

Janous pointed out that the result can be generalized to:

Let $x_1, \ldots, x_n$ ($n \geq 2$) be non-negative real numbers such that $x_1 + \ldots + x_n = nw$, and let $b \neq 1$ be a positive real number.

Let $\alpha_1, \ldots, \alpha_n$ be real numbers, each greater than or equal to 1.
Then the equation

$$b^{x_1^{\alpha_1}+x_2^{\alpha_2}+\ldots+x_n^{\alpha_n}} + b^{x_2^{\alpha_1}+x_3^{\alpha_2}+\ldots+x_n^{\alpha_n}} + \ldots + b^{x_n^{\alpha_1}+x_1^{\alpha_2}+\ldots+x_{n-1}^{\alpha_n}} = b^{w^{\alpha_1}+w^{\alpha_2}+\ldots+w^{\alpha_n}+\log_b(n)}$$

has as its only solution $x_1 = x_2 = \ldots = x_n = w$.

This can be proved by applications of the AM–GM and power mean inequalities.

2051. [1995: 202] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex quadrilateral $ABCD$ is inscribed in a circle Γ with centre O. P is an interior point of $ABCD$. Let O_1, O_2, O_3, O_4 be the circumcentres of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of O_1O_3, O_2O_4 and OP are collinear.

Combination of solutions by Jordi Dou, Barcelona, Spain and the proposer.

The result holds without restriction on the point P. The proof is in two steps.

Step 1. The result is trivially true when P is on Γ (and all the circumcentres coincide with O), so let P be a point off Γ and let Ω be the conic with foci O and P whose major axis has length R (the radius of Γ). Ω is an ellipse when P is interior to Γ and a hyperbola when P is exterior. Let Q be the intersection of O_1O_2 and OB. Because $O_1B = O_1P$ and $O_2B = O_2P$, we have that O_1O_2 is the perpendicular bisector of BP. Therefore $BQ = PQ$ and $\angle BQO_2 = \angle PQO_2$. When P is interior we conclude that $OQ + PQ = OQ + BQ = R$; when exterior, $OQ - PQ = OQ - BQ = R$. Thus, in either case, Q is a point on Ω. Moreover, as O_1O_2 is a bisector of $\angle OQP$, O_1O_2 is tangent to Ω at Q. Similarly O_2O_3, O_3O_4, and O_4O_1 are tangent to Ω. [*Editor's comment by Chris Fisher.* Dou refers to Γ as the *focal circle* of Ω. I was unable to confirm that terminology in any handy reference, but I did find the circle mentioned as the basis of a construction of a central conic by folding; see, for example, E.H. Lockwood, *A Book of Curves*: Draw a circle Γ on a sheet of paper and mark an arbitrary point P not on Γ. For any number of positions B on Γ fold P onto B and crease the paper. The creases (i.e. the perpendicular bisectors of PB) envelope a conic Ω.]

Step 2. By Newton's theorem the midpoints of O_1O_3, O_2O_4, and the centre of Ω (which is the midpoint of OP) are collinear as desired. As a bonus, also on that line is the midpoint of the segment joining $O_5 := O_1O_4 \cap O_2O_3$ to $O_6 := O_1O_2 \cap O_3O_4$. Here is a simple projective proof of this theorem. Consider the pencil of dual conics tangent to the sides of the complete quadrilateral $O_1O_2O_3O_4$. As a consequence of the Desargues' involution theorem (see, for example, Dan Pedoe, *A Course of Geometry for Colleges and Universities*, Theorem II, page 342), the poles of a line l with respect to the conics of the pencil lie on a line. In particular, when l is the line at infinity the line of poles is the line of centres of these conics. The centre of Ω is one such pole. Furthermore, the line of centres passes through the midpoints of the degenerate dual conics of the pencil (consisting of pairs of opposite vertices of the quadrilateral), namely O_1 and O_3, O_2 and O_4, O_5 and O_6.

Also solved by MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain.

2053. [1995: 202] *Proposed by Jisho Kotani, Akita, Japan.*

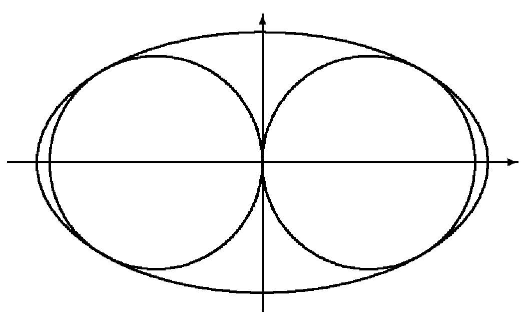
A figure consisting of two equal and externally tangent circles is inscribed in an ellipse. Find the eccentricity of the ellipse of minimum area.

Solution by David Hankin, Hunter College Campus Schools, New York, NY, USA.

[Editor's note: By symmetry, we have:

(a) the centre of the circumscribing ellipse, $\mathcal{E}$, must be the point of tangency of the two given circles;

(b) an axis of $\mathcal{E}$ passes through the centres of the two given circles.



All solvers assumed this, most without stating that they had done so.]

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Let the equations of the circles be $(x \pm r)^2 + y^2 = r^2$, and let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solving the first and third of these equations gives

$$x = \frac{-ar^2 \pm a\sqrt{a^2r^2 + b^4 - a^2b^2}}{b^2 - a^2}.$$

Therefore, at the two points of tangency (in the right half-plane), we have $a^2r^2 + b^4 - a^2b^2 = 0$. From this we get

$$a = \frac{b^2}{\sqrt{b^2 - r^2}}.$$

Since the area of the ellipse is given by $K = \pi ab$, we have

$$K = \frac{\pi b^3}{\sqrt{b^2 - r^2}}.$$

Therefore $\dfrac{dK}{db} = \dfrac{\pi b^2(2b^2 - 3r^2)}{(b^2 - r^2)^{3/2}}$. From this, we obtain that K is minimal when $b^2 = \dfrac{3r}{2}$. At this point $a^2 = 3b^2$, and from

$$e^2 = \frac{a^2 - b^2}{a^2},$$

we obtain that $e = \sqrt{\dfrac{2}{3}}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; and the proposer. Two other submissions were received that were almost correct: they used an incorrect formula for the eccentricity.

2054. [1995: 202] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Are there any integral solutions of the Diophantine equation

$$(x + y + z)^3 = 9\left(x^2y + y^2z + z^2x\right)$$

other than $(x, y, z) = (n, n, n)$?

I. *Solution by Adrian Chan, student, Upper Canada College, Toronto, Ontario.*

No, there are no integral solutions other than $(x, y, z) = (n, n, n)$.

Without loss of generality, let $x \leq y$ and $x \leq z$. Let $y = x + a$ and $z = x + b$, where a and b are non-negative integers. Then the given equation becomes

$$(3x + a + b)^3 = 9\left(x^2(x + a) + (x + a)^2(x + b) + (x + b)^2x\right).$$

After expanding and simplifying, this is $(a+b)^3 = 9a^2b$, or

$$a^3 - 6a^2b + 3ab^2 + b^3 = 0. \qquad (1)$$

Let $a = kb$, where k is a rational number. Then (1) becomes

$$b^3(k^3 - 6k^2 + 3k + 1) = 0.$$

By the Rational Root Theorem, $k^3 - 6k^2 + 3k + 1 = 0$ does not have any rational roots. So, since k is rational, $k^3 - 6k^2 + 3k + 1 \neq 0$. Therefore $b = 0$ and $a = 0$, so $x = y = z$.

II. *Solution by the proposer.*

Letting $y = x + u$ and $z = x + v$, the equation reduces to

$$(u+v)^3 = 9u^2v \qquad (2)$$

where u and v are integers. We now show that the only solution to (2) is $u = v = 0$ so that $(x, y, z) = (n, n, n)$ is the only solution of the given equation. Letting $u + v = w$, (2) becomes

$$w^3 = 9u^2(w - u). \qquad (3)$$

Hence $w = 3w_1$ where w_1 is an integer, and (3) is $3w_1^3 = u^2(3w_1 - u)$. It follows that $u = 3u_1$ for some integer u_1, and we get

$$w_1^3 = 9u_1^2(w_1 - u_1).$$

Comparing this equation to (3), we see by infinite descent that the only solution to (3) is $u = w = 0$, which gives the negative result.

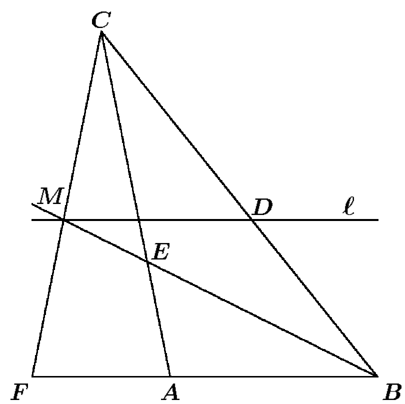
Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and HOE TECK WEE, student, Hwa Chong Junior College, Singapore. There was also one incorrect solution sent in.

2055. [1995: 202] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In triangle ABC let D be the point on the ray from B to C, and E on the ray from C to A, for which $BD = CE = AB$, and let ℓ be the line through D that is parallel to AB. If $M = \ell \cap BE$ and $F = CM \cap AB$, prove that

$$(BA)^3 = AE \cdot BF \cdot CD.$$

Solution by Toshio Seimiya, Kawasaki, Japan; essentially identical solutions were submitted by Jordi Dou, Barcelona, Spain; Mitko Christov Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; and Gottfried Perz, Pestalozzigymnasium, Graz, Austria.



By Menelaus' Theorem applied to triangle ACF and transverse line BEM, we have

$$\frac{AB}{BF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EA} = -1.$$

Since $DM \| BF$, we have $\dfrac{FM}{MC} = \dfrac{BD}{DC}$, giving

$$\frac{AB}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$

Hence we have

$$AE.BF.CD \;=\; -AB.BD.CE \;=\; BA.AB.AB \;=\; (BA)^3.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; P. PENNING, Delft, the Netherlands; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2057***.** [1995: 203] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside an equilateral triangle ABC, and let R_a, R_b, R_c and r_a, r_b, r_c denote the distances of P from the vertices and edges, respectively, of the triangle. Prove or disprove that

$$\left(1+\frac{r_a}{R_a}\right)\left(1+\frac{r_b}{R_b}\right)\left(1+\frac{r_c}{R_c}\right) \geq \frac{27}{8}\,.$$

Equality holds if P is the centre of the triangle.

Solution by G.P. Henderson, Campbellcroft, Ontario. We will prove that the inequality is true.

We have
$$\sin(PAB) = \frac{r_c}{R_a}, \quad \sin(PAC) = \frac{r_b}{R_a}.$$
Therefore
$$\cos(A) = \frac{1}{2} = \sqrt{\left(1 - \frac{r_b^2}{R_a^2}\right)\left(1 - \frac{r_c^2}{R_a^2}\right)} - \frac{r_b r_c}{R_a^2}.$$
From this, we have
$$R_a = \frac{2}{\sqrt{3}}\sqrt{r_b^2 + r_b r_c + r_c^2},$$
and similar expressions for R_b and R_c.

We can assume, without loss of generality, that P is in the section of the triangle defined by $r_a \le r_b \le r_c$. Set $x = r_b/r_c$ and $y = r_a/r_c$, so that $0 \le y \le x \le 1$. The left side of the given inequality is now
$$\left(1 + \frac{\sqrt{3}y}{2\sqrt{x^2+x+1}}\right)\left(1 + \frac{\sqrt{3}x}{2\sqrt{y^2+y+1}}\right)\left(1 + \frac{\sqrt{3}}{2\sqrt{x^2+xy+y^2}}\right). \qquad (1)$$
We will replace these factors by smaller quantities which do not involve square roots. [*Ed. This is a brilliant move. I recommend that the reader draws some graphs to see how effective this is. In fact* $\frac{6-x}{7+3x}$ *is "almost" the Čebyšev-Padé approximant of* $\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}$, *which is* $\frac{6.08409 - 1.33375x}{7 + 2.53143x}$.]
We will show that, for $0 \le x \le 1$,
$$\frac{\sqrt{3}}{2\sqrt{x^2+x+1}} \ge \frac{6-x}{7+3x}. \qquad (2)$$
This is equivalent to
$$(1-x)^2\left(3 + 36x - 4x^2\right) = (1-x)^2\left(4x(1-x) + 32x + 3\right) \ge 0.$$
Similarly, we have, for $0 \le y \le 1$,
$$\frac{\sqrt{3}}{2\sqrt{y^2+y+1}} \ge \frac{6-y}{7+3y}.$$
Replacing x by $y/x(\le 1)$ in (2), we have
$$\frac{\sqrt{3}}{2\sqrt{x^2+xy+y^2}} \ge \frac{6x-y}{x(7x+3y)}.$$
Using these three expression in (1), it is sufficient to prove that
$$\left(1 + \frac{y(6-x)}{7+3x}\right)\left(1 + \frac{x(6-y)}{7+3y}\right)\left(1 + \frac{6x-y}{x(7x+3y)}\right) \ge \frac{27}{8}.$$

Now, this is equivalent to

$$f(x,y) = Py^3 + Qy^2 + Ry + S \geq 0,$$

where

$$\begin{aligned}
P = P(x) &= 8(x-6)(x-3)(3x-1), \\
Q = Q(x) &= 56x^4 - 672x^3 + 663x^2 + 427x - 504, \\
R = R(x) &= -504x^4 + 35x^3 + 714x^2 - 273x - 392, \\
S = S(x) &= 1008x^4 + 423x^3 - 3493x^2 + 2352x.
\end{aligned}$$

We will prove this by showing that f is a decreasing function of y for $0 \leq y \leq x$ [for fixed x] and that $f(x,x) \geq 0$. We find that

$$f(x,x) = 10x(1-x)^2\left(8x^3 - 124x^2 - 35x + 196\right) \geq 0.$$

[Note that $8 + 196 > 124 + 35$.]
It remains to show that

$$F(x,y) = \frac{\partial f}{\partial y} = 3Py^2 + 2Qy + R \leq 0. \tag{3}$$

First, we note that

$$\begin{aligned}
Q &= -56x^3(1-x) - 616x(1-x)^2 - \left(569x^2 - 1043x + 504\right) \ < \ 0, \\
R &= -35x(1-x^2) - 238x - \left(504x^2 - 714x + 392\right) \ < \ 0.
\end{aligned}$$

[Also, $P \leq 0$ for $0 \leq x \leq \frac{1}{3}$ and $P \geq 0$ for $\frac{1}{3} \leq x \leq 1$.] In equation (3), when $P \leq 0$, all three terms are negative, and so $F(x,y) < 0$. When $P > 0$, $F(x,y) \leq 0$ provided that

$$F(x,0) \leq 0 \quad \text{and} \quad F(x,x) \leq 0,$$

[since $\frac{\partial^2 F}{\partial y^2} > 0$].
The first of these terms is R, which is negative. The second is

$$\begin{aligned}
&(1-x)\left(-184x^4 + 2336x^3 - 537x^2 - 1673x - 392\right) \\
&= \ (1-x)\left(-1184x^4 - 537x^2(1-x) - 1673x(1-x^2)\right. \\
&\qquad\qquad\qquad\qquad \left. -392(1-x^3) - 266x^3\right) \\
&\leq \ 0.
\end{aligned}$$

Also solved by MARCIN E. KUCZMA, Warszawa, Poland.