PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 December 1996**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX, *preferably in* LaTeX2e). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2138. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

ABC is an acute angle triangle with circumcentre O. AO meets the circle BOC again at A', BO meets the circle COA again at B', and CO meets the circle AOB again at C'.

Prove that $[A'B'C'] \geq 4[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

2139. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3R'}\sqrt{R'^2 - (O'P)^2},$$

where $[XYZ]$ denotes the area of triangle XYZ.

2140. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Determine the quartic $f(x) = x^4 + ax^3 + bx^2 + cx - c$ if it shares two distinct integral zeros with its derivative $f'(x)$ and $abc \neq 0$.

2141. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$A_1A_2A_3A_4$ is a quadrilateral. Let B_1, B_2, B_3 and B_4 be points on the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 respectively,
such that

$$A_1B_1 : B_1A_2 = A_4B_3 : B_3A_3 \quad \text{and} \quad A_2B_2 : B_2A_3 = A_1B_4 : B_4A_4.$$

Let P_1, P_2, P_3 and P_4 be points on B_4B_1, B_1B_2, B_2B_3
and B_3B_4 respectively, such that

$$P_1P_2 \| A_1A_2, \quad P_2P_3 \| A_2A_3 \quad \text{and} \quad P_3P_4 \| A_3A_4.$$

Prove that $P_4P_1 \| A_4A_1$.

2142. *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given an arbitrary quadrangle and bisectors of three of its angles. Construct, using only an unmarked ruler, the bisector of the fourth angle.

2143. *Proposed by B. M⋆⋆⋆y, Devon, Switzerland.*

My lucky number, 34117, is equal to $166^2 + 81^2$ and also equal to $159^2 + 94^2$, where $|166 - 159| = 7$ and $|81 - 94| = 13$; that is,

it can be written as the sum of two squares of positive integers in two ways, where the first integers occurring in each sum differ by 7 *and the second integers differ by* 13.

What is the smallest positive integer with this property?

2144. *Proposed by B. M⋆⋆⋆y, Devon, Switzerland.*

My lucky number, 34117, has the interesting property that $34 = 2 \cdot 17$ and $341 = 3 \cdot 117 - 10$, that is,

it is a $2N + 1$*-digit number (in base* 10*) for some* N*, such that*

(i) *the number formed by the first* N *digits is twice the number formed by the last* N*, and*

(ii) *the number formed by the first* $N + 1$ *digits is three times the number formed by the last* $N + 1$*, minus* 10.

Find another number with this property.

2145. *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Prove that $\prod_{k=1}^{n} \left(ak + b^{k-1}\right) \leq \prod_{k=1}^{n} \left(ak + b^{n-k}\right)$ for all a, $b > 1$.

2146. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point on the segment AD, and let Q and R be the points of intersection of BP and CP with sides AC and AB respectively. Prove that $PB - PC > RB - QC > 0$.

2147. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Let S be the set of all positive integers x such that there exist positive integers y and m satisfying $x^2 + 2^m = y^2$.

(a) Characterize which positive integers are in S.

(b) Find all positive integers x so that both x and $x + 1$ are in S.

2148. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Suppose that AD, BE and CF are the altitudes of triangle ABC. Suppose that L, M, N are points on BC, CA, AB, respectively, such that $BL = DC$, $CM = EA$, $AF = NB$.

Prove that:

1. the perpendiculars to BC, CA, AB at L, M, N, respectively are concurrent;
2. the point of concurrency lies on the Euler line of triangle ABC.

2149. *Proposed by Juan-Bosco Morero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral and O is the point of the intersection of the diagonals AC and BD. Let $A'B'C'D'$ be the quadrilateral whose vertices, A', B', C', D', are the feet of the perpendiculars drawn from the point O to the sides BC, CD, DA, AB, respectively.

Prove that $ABCD$ is an inscribed (cyclic) quadrilateral if and only if $A'B'C'D'$ is a circumscribing quadrilateral ($A'B'$, $B'C'$, $C'D'$, $D'A'$ are tangents to a circle).

2150. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}.$$