PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (*) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard 8½''×11'' or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, "no later than 1 December 1996. They may also be sent by email to cruxeditor@cms.math.ca. (It would be appreciated if email proposals and solutions were written in \TeX\, preferably in \TeX\2e). Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.

2138. Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.

\(ABC\) is an acute angle triangle with circumcentre \(O\). \(AO\) meets the circle \(BOC\) again at \(A'\), \(BO\) meets the circle \(COA\) again at \(B'\), and \(CO\) meets the circle \(AOB\) again at \(C'\).

Prove that \([A'B'C'] \geq 4[ABC]\), where \([XYZ]\) denotes the area of triangle \(XYZ\).

2139. Proposed by Waldemar Pompe, student, University of Warsaw, Poland.

Point \(P\) lies inside triangle \(ABC\). Let \(D, E, F\) be the orthogonal projections from \(P\) onto the lines \(BC, CA, AB\), respectively. Let \(O'\) and \(R'\) denote the circumcentre and circumradius of the triangle \(DEF\), respectively. Prove that

\[
[ABC] \geq 3\sqrt{3R'}\sqrt{R'^2 - (O'P)^2},
\]

where \([XYZ]\) denotes the area of triangle \(XYZ\).

2140. Proposed by K. R. S. Sastry, Dodballapur, India.

Determine the quartic \(f(x) = x^4 + ax^3 + bx^2 + cx - c\) if it shares two distinct integral zeros with its derivative \(f'(x)\) and \(abc \neq 0\).
2141. Proposed by Toshio Seimiy a, Kawasaki, Japan.

A_1A_2A_3A_4 is a quadrilateral. Let B_1, B_2, B_3 and B_4 be points on the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 respectively, such that

A_1B_1 : B_1A_2 = A_4B_3 : B_3A_3 and A_2B_2 : B_2A_3 = A_1B_4 : B_4A_4.

Let P_1, P_2, P_3 and P_4 be points on B_4B_1, B_1B_2, B_2B_3 and B_3B_4 respectively, such that

P_1P_2||A_1A_2, P_2P_3||A_2A_3 and P_3P_4||A_3A_4.

Prove that P_4P_1||A_4A_1.

2142. Proposed by Victor Oxman, Haifa, Israel.

In the plane are given an arbitrary quadrangle and bisectors of three of its angles. Construct, using only an unmarked ruler, the bisector of the fourth angle.

2143. Proposed by B. M***y, Devon, Switzerland.

My lucky number, 34117, is equal to 166^2 + 81^2 and also equal to 159^2 + 94^2, where |166 - 159| = 7 and |81 - 94| = 13; that is,

it can be written as the sum of two squares of positive integers in two ways, where the first integers occurring in each sum differ by 7 and the second integers differ by 13.

What is the smallest positive integer with this property?

2144. Proposed by B. M***y, Devon, Switzerland.

My lucky number, 34117, has the interesting property that 34 = 2 · 17 and 341 = 3 · 117 - 10, that is,

it is a 2N + 1-digit number (in base 10) for some N, such that

(i) the number formed by the first N digits is twice the number formed by the last N, and

(ii) the number formed by the first N + 1 digits is three times the number formed by the last N + 1, minus 10.

Find another number with this property.

2145. Proposed by Robert Geretschlager, Bundesrealgymnasium, Graz, Austria.

Prove that \prod_{k=1}^{n} (ak + b^{k-1}) \leq \prod_{k=1}^{n} (ak + b^{n-k}) for all a, b > 1.
2146. **Proposed by Toshio Seimiya, Kawasaki, Japan.**

$ABC$ is a triangle with $AB > AC$, and the bisector of $\angle A$ meets $BC$ at $D$. Let $P$ be an interior point on the segment $AD$, and let $Q$ and $R$ be the points of intersection of $BP$ and $CP$ with sides $AC$ and $AB$ respectively. Prove that $PB - PC > RB - QC > 0$.

2147. **Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.**

Let $S$ be the set of all positive integers $x$ such that there exist positive integers $y$ and $m$ satisfying $x^2 + 2^m = y^2$.

(a) Characterize which positive integers are in $S$.

(b) Find all positive integers $x$ so that both $x$ and $x + 1$ are in $S$.

2148. **Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.**

Suppose that $AD$, $BE$ and $CF$ are the altitudes of triangle $ABC$. Suppose that $L$, $M$, $N$ are points on $BC$, $CA$, $AB$, respectively, such that $BL = DC$, $CM = EA$, $AF = NB$.

Prove that:

1. the perpendiculars to $BC$, $CA$, $AB$ at $L$, $M$, $N$, respectively are concurrent;

2. the point of concurrency lies on the Euler line of triangle $ABC$.

2149. **Proposed by Juan-Bosco Moreno Marquez, Universidad de Valladolid, Valladolid, Spain.**

Let $ABCD$ be a convex quadrilateral and $O$ is the point of the intersection of the diagonals $AC$ and $BD$. Let $A'B'C'D'$ be the quadrilateral whose vertices, $A'$, $B'$, $C'$, $D'$, are the feet of the perpendiculars drawn from the point $O$ to the sides $BC$, $CD$, $DA$, $AB$, respectively.

Prove that $ABCD$ is an inscribed (cyclic) quadrilateral if and only if $A'B'C'D'$ is a circumscribing quadrilateral ($A'B'$, $B'C'$, $C'D'$, $D'A'$ are tangents to a circle).

2150. **Proposed by Šefket Arslanagić, Berlin, Germany.**

Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}.$$