THE ACADEMY CORNER
No. 3

Bruce Shawyer

All communications about this column should be sent to Professor Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Spring is approaching in parts of Canada, and, as undergraduate students in North America prepare for final examinations, some find time to try their skills on some mathematics problems. Memorial University offers some modest prizes for the best students. How do you compare? Please send me your best solutions to these problems.

MEMORIAL UNIVERSITY OF NEWFOUNDLAND UNDERGRADUATE MATHEMATICS COMPETITION
March, 1996

1. Prove that if \( n \) is a positive integer, then \( \frac{n^2 + 3n + 1}{n^2 + 4n + 3} \) is an irreducible fraction.

2. A jar contains 7 blue balls, 9 red balls and 10 white balls. Balls are drawn at random one by one from the jar until either four balls of the same colour or at least two of each colour have been drawn. What is the largest number of balls that one may have to draw?

3. Find all functions \( u(x) \) satisfying \( u(x) = x + \int_0^{\frac{1}{2}} u(t) \, dt \).

4. Show that \((\sqrt{5} + 2)^{\frac{x}{2}} - (\sqrt{5} - 2)^{\frac{x}{2}}\) is a rational number and find its value.

5. In a quadrilateral \( ABCD \) (vertices named in clockwise order), \( AC \) and \( BD \) intersect in \( X \). You are given that \( AB \parallel DC \), that \( AB \) is twice as long as \( CX \) and that \( AC \) is equal in length to \( DC \). Show that \( AB \) and \( CD \) are equal in length (and hence \( ABCD \) is a parallelogram).

6. Prove that among any thirteen distinct real numbers it is possible to choose two, \( x \) and \( y \), such that \( 0 < \frac{x - y}{1 + xy} < 2 - \sqrt{3} \).
7. A coast guard boat is hunting a bootlegger in a fog. The fog rises disclosing the bootlegger 4 miles distant and immediately descends. The speed of the boat is 3 times that of the bootlegger, and it is known that the latter will immediately depart at full speed on a straight course of unknown direction. What course should the boat take in order to overtake the bootlegger?

**Historical Titbit**

*Taken from a 1950’s University Scholarship Paper.*

*A, C, B are collinear points.*

Prove that there is one and only one point *D* such that *(ACBD)* is harmonic.

The following result is stated in a book on geometry:

if the pencils *O*(ACBD) and *O’*(ACBD) are harmonic and if *A, C, B* are collinear, then *D* lies on the line *ACB*.

Give an example to show that this result is not always true.

What alteration is required in order to make it true?

Prove the result after making this alteration.