THE OLYMPIAD CORNER

No. 174

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As a first Olympiad set this number we give the third Team Competition, the Baltic Way 1994. The contest was written in Vilnius, Lithuania. Teams from Denmark, St. Petersburg, Poland, Latvia, Iceland, Lithuania, Estonia and Sweden participated. My thanks go to Georg Gunther for collecting this problem set when he was Canadian Team Leader to the IMO at Istanbul.

MATHEMATICAL TEAM CONTEST
"BALTIC WAY — 92"
Vilnius, 1992 — November 5–8

1. Let p, q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least 3 positive integers > 1 (not necessarily different).

2. Denote by $d(n)$ the number of all positive divisors of a positive integer n (including 1 and n). Prove that there are infinitely many n such that $\frac{n}{d(n)}$ is an integer.

3. Find an infinite non-constant arithmetic progression of positive integers such that each term is neither a sum of two squares, nor a sum of two cubes (of positive integers).

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

5. Given that $a^2 + b^2 + (a+b)^2 = c^2 + d^2 + (c+d)^2$, prove that $a^4 + b^4 + (a+b)^4 = c^4 + d^4 + (c+d)^4$.

6. Prove that the product of the 99 numbers $\frac{k^3-1}{k^3+1}$, $k = 2, 3, \ldots, 100$, is greater than $\frac{2}{3}$.

7. Let $a = \sqrt[1992]{1992}$. Which number is greater:

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\right\}1992 \quad \text{or} \quad 1992?$$

8. Find all integers satisfying the equation

$$2^x \cdot (4 - x) = 2x + 4.$$

9. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial has three different real roots.

10. Find all fourth degree polynomials $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$, for all x,
(ii) $p(x) \geq 0$, for all x,
(iii) $p(0) = 1$,
(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

11. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Show that there exists one and only one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:

(i) If $0 < q < \frac{1}{2}$ then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$.
(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q-1)$.
(iii) $f(q) \cdot f(\frac{1}{q}) = 1$ for all $q \in \mathbb{Q}^+$.

12. Let $\mathbb{N}$ denote the set of positive integers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijective function and assume that there exists a finite limit

$$\lim_{n \to \infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

13. Prove that for any positive $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n} (x_i + y_i)^2}$$

holds.

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accommodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

19. Let C be a circle in the plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2, touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

20. Let $a \leq b \leq c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that $p(p-c) = (p-a)(p-b) = S$ (the area of the triangle).

As a second Olympiad set to give you recreation over the summer months, we give the 8th Iberoamerican Mathematical Olympiad written September 14–15, 1993 in Mexico.

8th IBEROAMERICAN MATHEMATICAL OLYMPIAD
September 14–15, 1993 (Mexico)

First Day — 4.5 hours

1. (*Argentina*) Let $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ be all the palindromic natural numbers, and for each i, let by $y_i = x_{i+1} - x_i$. How many distinct prime numbers belong to the set $\{y_1, y_2, y_3, \dots\}$?

2. (*Mexico*) Show that for any convex polygon of unit area, there exists a parallelogram of area 2 which contains the polygon.

3. (*Mexico*) Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$. Find all the functions

$$f : \mathbb{N}^* \to \mathbb{N}^*$$

such that

(i) If $x < y$, then $f(x) < f(y)$

(ii) $f(yf(x)) = x^2 \cdot f(xy)$, for all x, y belonging to $\mathbb{N}^*$.

Second Day — 4.5 hours

4. (*Spain*) Let ABC be an equilateral triangle, and Γ its incircle. If D and E are points of the sides AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

5. (*Mexico*) Let P and Q be distinct points of the plane. We denote $m(PQ)$ the perpendicular bisector of the segment PQ. Let S be a finite subset of the plane, with more than one element, which satisfies the following properties:

(i) If P and Q are distinct points of S, then $m(PQ)$ intersects S.

(ii) If P_1Q_1, P_2, Q_2 and P_3Q_3 are three distinct segments with extreme points belonging to S, then no point of S belongs simultaneously to the three lines $m(P_1Q_1)$, $m(P_2Q_2)$, $m(P_3Q_3)$.

Determine the number of possible points of S.

6. (*Argentina*) Two non-negative integer numbers, a and b, are "cuates" (friends in Mexican) if the decimal expression of $a+b$ is formed only by 0's and 1's. Let A and B be two infinite sets of non-negative integers, such that B is the set of all the numbers which are "cuates" of all the elements of A, and A is the set of all the numbers which are "cuates" of all the elements of B. Show that in one of the sets A or B there are infinitely many pairs of numbers x, y such that $x - y = 1$.

We now turn to the readers' comments and solutions to problems given in the December 1994 number of the *Corner* and the Nordic Mathematical Contest, 1992 [1994: 277].

1. [1994: 277] Determine all real numbers x, y, z greater than 1, satisfying the equation

$$x+y+z+\frac{3}{x-1}+\frac{3}{y-1}+\frac{3}{z-1}=2\left(\sqrt{x+2}+\sqrt{y+2}+\sqrt{z+2}\right).$$

Solutions by Šefket Arslanagić, Berlin, Germany; by Cyrus C. Hsia, student, Woburn Collegiate Institute, Toronto; by Chandan Reddy, Rochester, Michigan; Michael Selby, University of Windsor, Windsor, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Hsia's solution.

For $a > 1$, $a \in \mathbb{R}$ we have the Arithmetic Mean–Geometric Mean inequality

$$a-1+\frac{a+2}{a-1}\geq 2\sqrt{a+2},$$

with equality if and only if $a-1=\frac{a+2}{a-1}$, or $a^2-3a-1=0$, giving $a=(3+\sqrt{13})/2$, since $a>1$.

For each of $a = x, y, z$ we have this same result and adding them gives

$$x+y+z-3+\frac{x+2}{x-1}+\frac{y+2}{y-1}+\frac{z+2}{z-1}\geq 2(\sqrt{x+2}+\sqrt{y+2}+\sqrt{z+2})$$

whence

$$x+y+z+\frac{3}{x-1}+\frac{3}{y-1}+\frac{3}{z-1}\geq 2(\sqrt{x+2}+\sqrt{y+2}+\sqrt{z+2})$$

with equality if and only if $x = y = z = (3+\sqrt{13})/2$. Since there is equality, the unique solution is $x = y = z = (3+\sqrt{13})/2$.

2. [1994: 277] Let n be an integer greater than 1 and let $a_1, a_2, \ldots, a_n$ be n different integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)\ldots(x - a_n) - 1$ is not divisible by any polynomial of positive degree less than n and with integer coefficients and leading coefficient 1.

Solutions by Cyrus C. Hsia, Woburn Collegiate Institute, Toronto, Ontario; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Chandan Reddy; Rochester, Michigan; and by Michael Selby, University of Windsor, Windsor, Ontario. We give the solution sent in by Selby and Klamkin's comment.

Suppose $p(x)$ divides $f(x)$, where $1 \leq \deg p < n$, and p is monic with integral coefficients. Therefore $f(x) = p(x)q(x)$ where $q(x)$ is also monic, with integral coefficients and $1 \leq \deg q < n$.

Now $p(a_i)q(a_i) = -1$, $i = 1, 2, \ldots, n$. Since $p(a_i)$, $q(a_i)$ are integers, $p(a_i) = 1$ and $q(a_i) = -1$ or $p(a_i) = -1$ and $q(a_i) = 1$ for each $i = 1, 2, \ldots, n$.

In either case $p(a_i) + q(a_i) = 0$ for $i = 1, 2, \ldots, n$.

Consider $p(x) + q(x)$. This is a polynomial of positive degree less than n, since both $p(x)$ and $q(x)$ are monic with degree less than n. However $p(a_i) + q(a_i) = 0$ for n distinct values, but has positive degree less than n, an impossibility. Therefore no such $p(x)$ exists.

[Editor's Note.] Klamkin (whose solution was similar) points out that the problem is well-known.

3. [1994: 277] Prove that among all triangles with given incircle, the equilateral one has the least perimeter.

Solutions by Šefket Arslanagić, Berlin, Germany; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Chandon Reddy, Rochester, Michigan. We first give Reddy's solution.

Let the three sides be a, b, c. Then the inradius r times the semiperimeter is the area $A = rs$, $r = 1$ so, using Heron's formula

$$\sqrt{s(s-a)(s-b)(s-c)} = s$$

or

$$\sqrt{(s-a)(s-b)(s-c)} = \sqrt{s}$$

so

$$s = (s-a)(s-b)(s-c). \qquad (1)$$

Also

$$s = (s-a) + (s-b) + (s-c). \qquad (2)$$

We minimize the perimeter and therefore s by the AM–GM inequality and (1) and (2)

$$\sqrt{(s-a)(s-b)(s-c)} \le \frac{(s-a)+(s-b)+(s-c)}{3}$$

The RHS is minimized when we have equality, that is $s-a=s-b=s-c$, and $a=b=c$.

Next we give Klamkin's comments and alternative approaches.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. This is a well-known result. It follows immediately from the isoperimetric theorem for polygons, that is, of all n-gons with given perimeter, the regular one has the maximum area and, dually, of all polygons of given area, the regular one has the least perimeter. For triangles we have the inequality

$$\frac{p^2}{F} \ge \frac{p_0^2}{F_0} = 12\sqrt{3}$$

where p and F denote the perimeter and area of a general triangle and p_0, F_0 correspond to the same for an equilateral triangle. Since $F = rp/2$ ($r =$ inradius), we have

$$p \ge 6\sqrt{3}\, r$$

and with equality **if and only if** the triangle is equilateral.

For another proof in terms of the angles A, B, C of the triangle, it is equivalent to establishing the known triangle inequality

$$2r\left(\frac{\cot A}{2} + \frac{\cot B}{2} + \frac{\cot C}{2}\right) \ge 6\sqrt{3}\, r$$

It also follows that of all n-gons circumscribed to a given circle, the regular one has the least perimeter (and area as well).

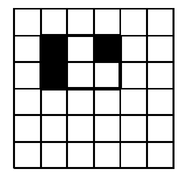
Comment: These results generalize to simplexes. In particular for the tetrahedron, we have the following known inequalities, where $E =$ the sum of the 6 edges $\sum E_i$, $F =$ the sum of the areas of the 4 faces $\sum F_i$, $V =$ the volume and $r =$ the inradius,

$$E \ge 6\prod E_i^{1/6} \ge k_1 V^{1/3}, \qquad (1)$$

$$F \ge 4\prod F_i^{1/4} \ge k_2 V^{2/3}, \qquad (2)$$

and there is equality if the tetrahedron is regular so that the constants k_1, k_2 are determined by taking $E_i = 1$. Then, $F_i = \sqrt{3}/4$ and $V = \sqrt{2}/12$. On multiplication of (1) and (2), we get $EF \ge k_1 k_2 V = \frac{k_1 k_2 r F}{3}$ or that $E \ge \frac{k_1 k_2 r}{3}$. This proves a 3-dimensional extension of the given result.

4. [1994: 277] Peter has a great number of squares, some of them are black, some are white. Using these squares, Peter wants to construct a square, where the edge has length n, and with the following property: The four squares in the corners of an arbitrary subrectangle of the big square, must never have the same colour. How large a square can Peter build?



$n = 6$

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

That the answer is 4×4 is given in the solution of problem #1 of the U.S.A. 1976 Mathematical Olympiad [1].

Reference

[1] M.S. Klamkin, U.S.A. Mathematical Olympiads 1972–1986, M.A.A., Washington, D.C. 1988, pp. 93–94.

To complete this number we given two comments by Murray Klamkin about earlier solutions.

3. [1994: 279; 1993: 131] The abscissa of a point which moves in the positive part of the axis Ox is given by $x(t) = 5(t+1)^2 + a/(t+1)^5$, in which a is a positive constant. Find the minimum a such that $x(t) \geq 24$ for all $x \geq 0$.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

A simpler non-calculus solution is given by applying the AM–GM inequality, that is,

$$\frac{5(t+1)^2 + 2\left\{\frac{\frac{1}{2}a}{(t+1)^5}\right\}}{[5+2]} \geq \left(\frac{a^2}{4}\right)^{1/7}.$$

Then we have

$$7\left(\frac{a^2}{4}\right)^{1/7} \geq 24,$$

so that

$$\min a = 2\left(\frac{24}{7}\right)^7.$$

5. [1994: 281; 1993: 132]
For each natural number n, let $(1+\sqrt{2})^{2n+1} = a_n + b_n\sqrt{2}$ with a_n and b_n integers.

(a) Show that a_n and b_n are odd for all n.
(b) Show that b_n is the hypotenuse of a right triangle with legs

$$\frac{a_n + (-1)^n}{2} \quad \text{and} \quad \frac{a_n - (-1)^n}{2}.$$

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

A more direct solution of part (a) is as follows:

$$\begin{aligned} a_n &= \frac{(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1}}{2} \\ &= \binom{2n+1}{1} 2^n + \binom{2n+1}{3} s^{n-1} + \cdots + 1, \\ b_n &= \frac{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}}{2\sqrt{2}} \\ &= 2^n + \binom{2n+1}{2} 2^{n-1} + \cdots + \binom{2n+1}{2n}. \end{aligned}$$

That completes the Corner for this issue. Have a good summer – spend some time solving problems and send me your nice solutions as well as Olympiad Contest materials.

Book wanted!

Bruce Shawyer would like to purchase a copy of the out-of-print book:

On Mathematics and Mathematicians (Memorabilia Mathematica)

by Robert Edouard Moritz.

Dover Edition published 1942.

Originally published as:

Memorabilia Mathematica or The Philomath's Quotation-Book.

Original Edition published 1914.

Anyone willing to part with a copy please send him details. Thank you.