THE SKOLIAD CORNER

No. 14

R.E. Woodrow

First this month, an observation by an astute reader, Derek Kisman, student, Queen Elizabeth High School, Calgary, who noticed that the figure we gave for question 25 of the Sharp U.K. Intermediate Mathematical Challenge, 1995, [1996: 14] is missing the crucial upright F in the net. It should be in the top right corner!

As a contest this month we give the European "Kangaroo" Mathematical Challenge, written Thursday 23 March, 1995. It is organized by the U.K. Mathematics Foundation and L'Association Européennes ≪Kangarou des Mathématiques≫. The contest is for students at about school year nine or below. This was written by about 5000 students in the U.K. My thanks go to Tony Gardiner of the U.K. Mathematics Foundation for sending the materials to Crux.

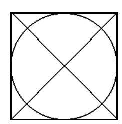
EUROPEAN "KANGAROO" MATHEMATICAL CHALLENGE

23 March 1995

1. $1 \times 9 \times 9 \times 5 - (1 + 9 + 9 + 5)$ makes:

A. 0 **B.** 381 **C.** 481 **D.** 429 **E.** 995

2. Which shape does not appear in this figure?



A. circle
B. square
C. right-angled triangle
D. isosceles triangle
E. equilateral triangle

3. The whole numbers from 1 to 1995 are alternately added and subtracted thus: $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 1993 - 1994 + 1995$. What is the result?

A. 997 **B.** 1995 **C.** 998 **D.** 0 **E.** -997

4. What is the angle between the hour hand and the minute hand of a clock at 1.30?

A. 180° **B.** 120° **C.** 130° **D.** 150° **E.** 135°

5. C_1 is a circle of radius 6cm, C_2 is a circle of radius 8cm. José wants the two circles to touch tangentially. He knows that there are two possibilities for the distance between their centres. What are these two distances?

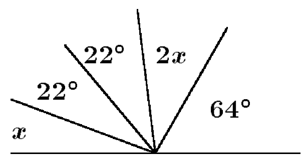
A. 3 and 4cm **B.** 2 and 8cm **C.** 2 and 14cm **D.** 6 and 8cm **E.** 6 and 14cm

6. A train 1km long is restricted to travel through a tunnel of length 1km at 1km/h. How long does it take the train to pass through the tunnel?

A. 1 hour **B.** 1h 30m **C.** 2 hours **D.** 3 hours **E.** 1/2 hour

7. The angle x in the figure is equal to:

A. 20°
B. 22°
C. 24°
D. 26°
E. 28°

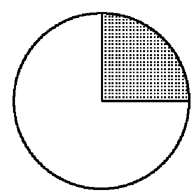


8. Which number is smallest?

A. 19^{95} **B.** 19×95 **C.** 1^{995} **D.** 199^5 **E.** 1995

9. What is the ratio of the perimeter of the shaded region to the circumference of the circle?

A. $\frac{3}{4}$ **B.** $\frac{4+\pi}{4\pi}$ **C.** $\frac{2\pi}{4+\pi}$ **D.** $\frac{4+\pi}{2\pi}$ **E.** $\frac{1}{4}$



10. After two successive 20% reductions, a coat costs \$320. What was its original price?

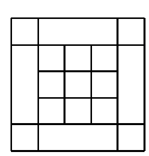
A. \$204 **B.** \$400 **C.** \$448 **D.** \$500 **E.** \$533

11. Nine people are sitting in a room; their average age is 25. In another room eleven people are gathered their average age is 45. If the two groups were to combine, what would their average age be?

A. 70 **B.** 36 **C.** 35 **D.** 32 **E.** 20

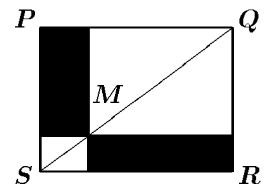
12. How many squares are there in this figure?

A. 13
B. 14
C. 19
D. 21
E. 23



13. The quadrilateral $PQRS$ is a rectangle; M is any point on the diagonal SQ. What can one say for sure about the two shaded regions?

A. the upper area is larger
B. the lower area is larger
C. they always have equal areas
D. the two areas are equal only if M is the midpoint of SQ
E. insufficient information

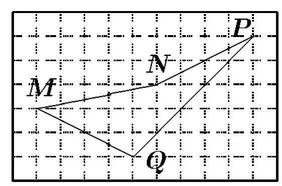


14. A metallic disc of diameter 20cm weighs 2.4kg. From it one cuts out a disc of diameter 10cm. What does it weigh?

A. 1.2kg **B.** 0.8kg **C.** 0.6kg **D.** 0.5kg **E.** 0.4kg

15. What is the area (in unit squares) of the quadrilateral $MNPQ$?

A. 9
B. 10
C. 11
D. 12
E. 13

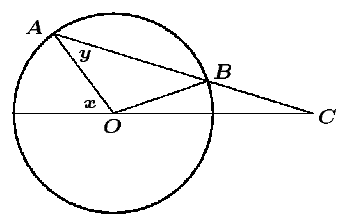


16. Six hundred and twenty-five students enter a 100m competition. The track has five lanes. At the end of each race the winner survives, while the other four are eliminated. How many races are needed to determine the champion sprinter?

A. 98 **B.** 106 **C.** 125 **D.** 126 **E.** 156

17. In the figure O is the centre of the circle and $OA = BC$. Which of the following relations holds?

A. $2x = 3y$ **B.** $x = 2y$ **C.** $x = y$
D. $x + y = 90°$ **E.** $x + 2y = 180°$



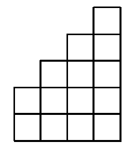
18. What is the sum of all the digits in the number $10^{95} - 95$?

A. 6 **B.** 29 **C.** 108 **D.** 663 **E.** 842

19. In a group of pupils, 40% of the pupils have poor eyesight; 70% of these wear glasses, while the other 30% wear contact lenses. If there are 21 pairs of glasses, which of these assertions is true?

A. 45 pupils have poor eyesight
B. 30 pupils have good eyesight
C. there are 100 pupils in the group
D. 10 pupils wear contact lenses
E. none of the other assertions is true

20. One has to arrange four pawns in this figure so that each column contains one pawn, and each row contains at most one pawn. How many different arrangements are possible?



A. 64 **B.** 28 **C.** 16 **D.** 8 **E.** 4

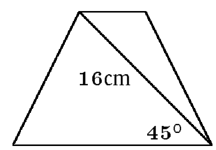
21. Let $\boxed{X} = X^4$ and $X \mid Y = X + Y$. Then $\boxed{\boxed{2} \mid \boxed{2}}$ equals:

A. 3×2^4 **B.** 2^9 **C.** 2^{12} **D.** 2^{20} **E.** 2^3

22. Bruce buys three ostriches, seven koalas and one kangaroo. Darren buys four ostriches, ten koalas and one kangaroo. Sheila buys one ostrich, one koala and one kangaroo. Bruce pays 3150 Australian dollars and Darren pays 4200 Australian dollars. How many Australian dollars must Sheila pay?

A. 1700 **B.** 1650 **C.** 1200 **D.** 1050 **E.** 950

23. The diagonal of an isosceles trapezium is 16cm long, and makes an angle of 45° with the base. What is the area of the trapezium?



A. 64cm^2 **B.** 96cm^2 **C.** 128cm^2
D. more information required **E.** 256cm^2

24. Each positive whole number which (in base 10) can be written as a string of 1's and 2's only is called *simple*. For example 22121 and 2222 are simple; 1021 is not simple. How many simple numbers are there less than one million?

A. 63 **B.** 62 **C.** 127 **D.** 128 **E.** 126

25. What is the maximum possible number of points of intersection one can get with eight circles?

A. 16 **B.** 32 **C.** 38 **D.** 44 **E.** 56

Next we give solutions to the contest given last issue, The Eleventh W.J. Blundon Contest.

THE ELEVENTH W.J. BLUNDON CONTEST

February 23, 1994

1. (a) The lesser of two consecutive integers equals five more than three times the larger integer. Find the two integers.

Solution. Let the integers be n, $n+1$. Then $n = 3(n+1) + 5$ so $n = -4$.

(b) If $4 \leq x \leq 6$ and $2 \leq y \leq 3$, find the minimum values of $(x-y)(x+y)$.

Solution. If $4 \leq x \leq 6$ and $2 \leq y \leq 3$, then $(x-y)(x+y) = x^2 - y^2 \geq 4^2 - 3^2 = 16 - 9 = 5$.

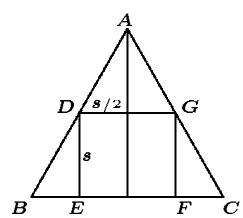
2. A geometric sequence is a sequence of numbers in which each term after the first can be obtained from the previous term by multiplying by the same fixed constant, called the **common ratio**. If the second term of a geometric sequence is 12 and the fifth term is $81/2$, find the first term and the common ratio.

Solution. Let the first term be a, and the ratio r. Then the sequence is $a, ar, arr = ar^2, ar^3, ar^4, ar^5, \ldots$, with the nth term being ar^{n-1}. So $ar = 12$ and $ar^4 = 81/2$. Dividing gives

$$r^3 = \frac{ar^4}{ar} = \frac{81/2}{12} = \frac{81}{24} = \frac{27}{8},$$

so $r = \frac{3}{2}$. Since $ar = a \cdot \frac{3}{2} = 12$, $a = 8$.

3. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.



Solution. Let the equilateral triangle ABC and the square $DEFG$ be as shown. Now $\angle ADG = \angle ABC = \angle AGD = \angle ACB = 60°$ so ADG is an equilateral triangle with side length s, the length of the side of the square. If DE is superimposed over GF a third equilateral triangle is formed with height s.

Now the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$, while the area of an equilateral triangle with *height* s is $\frac{1}{\sqrt{3}}s^2$. So the area of the square is s^2 and the area of ABC is

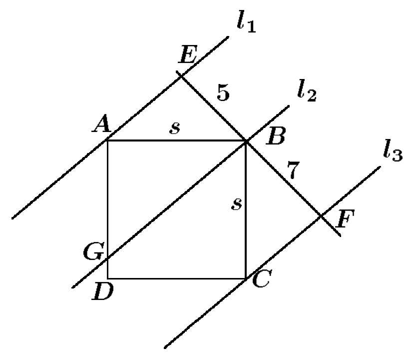
$$\frac{\sqrt{3}}{4}s^2 + s^2 + \frac{s^2}{\sqrt{3}} = \left(1 + \frac{\sqrt{3}}{4} + \frac{1}{\sqrt{3}}\right) s^2.$$

The ratio of the area of the square to the area of the triangle is then

$$\frac{1}{1 + \frac{7}{4\sqrt{3}}} = \frac{4\sqrt{3}}{4\sqrt{3} + 7} = \frac{4\sqrt{3}}{-48 + 49}(-4\sqrt{3} + 7) = 28\sqrt{3} - 48.$$

4. $ABCD$ is a square. Three parallel lines l_1, l_2 and l_3 pass through A, B and C respectively. The distance between l_1 and l_2 is 5 and the distance between l_2 and l_3 is 7. Find the area of $ABCD$.

Solution. Let the line perpendicular to l_2 through B meet l_1 at E and l_2 at F respectively. Let l_2 meet AD at G. Then $\angle BAE = \angle CBF$ and $\angle AEB = 90° = \angle BFC$ so ΔBAE is similar to ΔBCF. If the side length is s this gives



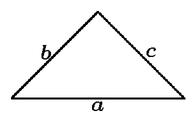
$$\frac{s}{\sqrt{s^2 - 5^2}} = \frac{s}{7}$$

so $s^2 = 5^2 + 7^2 = 25 + 49 = 74$.

5. The sum of the lengths of the three sides of a right triangle is 18. The sum of the squares of the lengths of the three sides is 128. Find the area of the triangle.

Solution.

$$a^2 + b^2 = c^2$$
$$a + b + c = 18$$
$$\text{and} \quad a^2 + b^2 + c^2 = 128.$$



So $2c^2 = 128$, $c^2 = 64$ and $c = 8$. Thus $a + b = 10$, $a^2 + b^2 = 64$, so

$$\begin{aligned} 2ab &= (a+b)^2 - (a^2 + b^2) \\ &= 100 - 64 = 36 \end{aligned}$$

The area $\frac{1}{2}ab = \frac{2ab}{4} = \frac{36}{4} = 9$.

6. A palindrome is a word or number that reads the same backwards and forwards. For example, 1991 is a palindromic number. How many palindromic numbers are there between 1 and 99, 999 inclusive?

Solution. Let us find the number n_k of the properly k digit palindromic numbers. To be a k digit number the first digit must be one of $1, 2, 3, \ldots, 9$. The answer now depends whether k is even or odd.

Now $n_1 = 9$. If $k > 1$ is odd, $k = 2l + 1$, $l \geq 1$ then $n_k = 9 \times 10^l$. If k is even, $k = 2l$, $l \geq 1$ then $n_k = 9 \times 10^{l-1}$. For the answer we want

$$n_1 + n_2 + n_3 + n_4 + n_5 = 9 + 9 + 90 + 90 + 900 = 1098.$$

7. A graph of $x^2 - 2xy + y^2 - x + y = 12$ and $y^2 - y - 6 = 0$ will produce four lines whose points of intersection are the vertices of a parallelogram. Find the area of the parallelogram.

Solution. Note that $x^2 - 2xy + y^2 - x + y = 12$ is equivalent to

$$(x - y)^2 - (x - y) - 12 = 0 \quad \text{or}$$

$$((x-y)-4)(x-y+3)=0$$

Thus its graph is the two parallel lines $x-y-4=0$ and $x-y+3=0$.

Also $y^2-y-6=0$ is equivalent to $(y-3)(y+2)=0$, giving two horizontal lines $y-3=0$ and $y+2=0$. The sides of the parallelogram lie on these lines. It has base 7 and height 5. (The distance between the horizontal lines.)

8. Determine the possible values of c so that the two lines $x-y=2$ and $cx+y=3$ intersect in the first quadrant.

Solution. For the curves to intersect in the first quadrant we need $x\geq 0$, $y\geq 0$. Adding the equations gives $(c+1)x=5$, so we require $c+1>0$, or $c>-1$. Substitution of $x=\frac{5}{c+1}$ into $x-y=2$ gives

$$y=\frac{5}{c+1}-2=\frac{3-2c}{c+1}$$

and we need $3-2c\geq 0$, so $c\leq\frac{3}{2}$. The intersection is in the first quadrant for $-1<c\leq\frac{3}{2}$.

9. Consider the function $f(x)=\frac{cx}{2x+3}$, $x\neq -3/2$. Find all values of c, if any, for which $f(f(x))=x$.

Solution.

$$f(f(x))=\frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}\quad\text{for } x\neq\frac{-3}{2}$$

and $f(x)\neq -3/2$, i.e. $cx/(2x+3)\neq -3/2$, $x\neq -9/(6+2c)$. For these excluded values $f(f(x))=x$ gives

$$\frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}=x$$

so

$$c^2x=x(2cx+3(2x+3))$$
$$c^2x=2cx^2+6x^2+9x$$

so equating coefficients of x, x^2, etc. (since the polynomials are equal for infinitely many values of x) gives $c^2=9$ and $2c=6$, so $c=3$.

10. Two numbers are such that the sum of their cubes is 5 and the sum of their squares is 3. Find the sum of the two numbers.

Solution. Let the numbers be x and y. We are given $x^3+y^3=5$ and $x^2+y^2=3$.

Let $x+y=A$.

From $(x+y)^3=x^3+3x^2y+3xy^2+y^3=x^3+y^3+3xy(x+y)$ we have $A^3=6+3xyA$.

From $(x+y)^2 = x^2+y^2+2xy$ we have $A^2 = 3+2xy$ or $xy = \frac{1}{2}(A^2-3)$.

So we get $A^3 = 5 + \frac{3}{2}(A^2-3)A$ or

$$\frac{1}{2}A^3 + \frac{9}{2}A + 5 = 0 \quad \text{and} \quad A^3 + 9A + 10 = 0$$

this gives $A = -1$.

That completes the Skoliad Corner for this issue. Send me your contest materials, comments, suggestions, and solutions.

Historical Titbit

Taken from a 1950's University Scholarship Paper.

A box contains forty-eight chocolates all exactly similar in appearance.

There are four of each of twelve different sorts.

How many must you take out to be certain of having at least one of each of five different sorts?

Suppose that, having taken this number out, you eat four of them, and find that they are all of one sort.

How many must you now put back to be certain that the box contains at least two of each of eight different sorts?