On the Sum of n Dice

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Introduction

An interesting problem can be found in Tucker [2, p. 421] which we rephrase as follows.

> Suppose n distinct fair dice are rolled and S is the sum of their faces. Show that the probability that 2 divides S is $\frac{1}{2}$.

There are several ways to solve this problem. One, which we will use, lends itself to generalization.

		Die Two					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die	3	4	5	6	7	8	9
One	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Table 1. Possible Sums of Two Dice

The result is clear for one die, and easily verified for two dice. See Table 1. The thirty-six entries in Table 1 each have the same probability. Since there are eighteen even sums among the thirty-six entries, the result follows for $n = 2$. A sub-problem is, how to represent the sum of three, or more generally n, dice while maintaining the different frequencies for different sums. That is, the sum 3 is only obtained in one way using three dice, while the sum 15 can be obtained in ten ways with three dice. Fortunately, these frequencies are unnecessary for this solution.

In Table 2, we label the rows by the possible sums of $(n - 1)$ dice and the six columns by the possible values of the nth die. We note that, unlike the entries in Table 1, the entries in Table 2 are not equi-probable. However, the table does allow us to count the number of even sums among the 6^n possible outcomes from rolling n dice. For suppose there are f_{n+2} ways to obtain the sum $(n + 2)$ when $(n - 1)$ dice are rolled. Then by looking at the row labelled by $(n + 2)$, we can account for $3f_{n+2}$ even sums among the 6^n possible outcomes for n dice, since there are three even entries in this row. Since each row of the table consists of six consecutive integers, there

		Die n					
		1	2	3	4	5	6
	$n-1$	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$
	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$
Sum of	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$
$(n-1)$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$
dice							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$6n-7$	$6n-6$	$6n-5$	$6n-4$	$6n-3$	$6n-2$	$6n-1$
	$6n-6$	$6n-5$	$6n-4$	$6n-3$	$6n-2$	$6n-1$	$6n$

Table 2. Possible Sums of n Dice

are exactly three even entries in each row. Thus, we can sum the frequencies $(f_{n-1} + f_n + f_{n+2} + \ldots + f_{6n-6})$ and multiply this sum by 3 to obtain the number of even sums among the 6^n possible outcomes. As the sum of the frequencies is 6^{n-1}, we have the probability that 2 divides S is

$$\frac{3 \cdot 6^{n-1}}{6^n} = \frac{1}{2}.$$

As mentioned above, this method generalizes easily. Let us first state two problems. Suppose n dice are rolled, and let S_n be the sum of their faces. Let $P(k|S_n)$ be the probability that k divides S_n.

1. Find values of k such that $P(k|S_n) = \frac{1}{k}$ for all n.
2. Find values of k and n for which $P(k|S_n) = \frac{1}{k}$.

Problem 1. Find values of k such that $P(k|S_n) = \frac{1}{k}$ for all n.

As mentioned, each row in Table 2 consists of six consecutive integers. Thus, in each row, two entries are divisible by 3, and one entry is divisible by 6.

It follows, in exactly the same manner as above, that $P(3|S_n) = \frac{1}{3}$ and $P(6|S_n) = \frac{1}{6}$. Clearly, $P(1|S_n) = 1$. Combining these observations with our first result, we have that, for all n, $P(k|S_n) = \frac{1}{k}$ when $k = 1, 2, 3$, and 6.

By considering $n = 1$ (or $n = 3$, if we wish to avoid the trivial case), we see that the only values of k that hold for all n are precisely $k = 1, 2, 3$, and 6.

Problem 2. Find values of k and n for which $P(k|S_n) = \frac{1}{k}$.

Let n and k satisfy $P(k|S_n) = \frac{1}{k}$. Since $6^n P(k|S_n)$ is an integer, it follows that k divides 6^n, that is, k is of the form $2^s 3^t$ for some non-negative integers s and t.

For $0 \leq s,\ t \leq 1$ we have $k = 1, 2, 3$, or 6. These values of k, as we have seen, are precisely the solutions to Problem 1. Let us consider $k = 4$.

We shall show that $P(4|S_n) = \frac{1}{4}$ if and only if $n \equiv 2 \pmod 4$. We note that in Table 2 each row does not contain the same number of multiples of 4. For this reason, we are led to consider another approach, in particular, the congruence classes depicted in Table 3.

	1	2	3	4	5	6
0	1	2	3	0	1	2
1	2	3	0	1	2	3
2	3	0	1	2	3	0
3	0	1	2	3	0	1

Table 3. Congruence Classes Modulo 4 for n Dice

We construct Table 3 in a similar fashion to Table 2, except that the congruence classes $0, 1, 2$, and 3 (modulo 4) label the rows. The columns are still labelled by the face values of the nth die. The entry in the ith row and the jth column is the congruence class of the sum of an element from the congruence class of the ith row and the face value of the jth column.

Let $P(n, \equiv\ i)$ be the probability that the sum of n dice is congruent to $i \pmod 4$ for $i = 0, 1, 2, 3$. Note that $P(4|S_n)$ is precisely $P(n, \equiv\ 0)$. The following recurrence relations are a consequence of Table 3.

$$\begin{aligned} 6P(n, \equiv 0) &= 2P(n-1, \equiv 3) + 2P(n-1, \equiv 2) \\ &\quad + P(n-1, \equiv 1) + P(n-1, \equiv 0) \\ &= P(n-1, \equiv 3) + P(n-1, \equiv 2) + 1 \end{aligned}$$

$$\begin{aligned} 6P(n, \equiv 1) &= 2P(n-1, \equiv 3) + 2P(n-1, \equiv 0) \\ &\quad + P(n-1, \equiv 1) + P(n-1, \equiv 2) \\ &= P(n-1, \equiv 3) + P(n-1, \equiv 0) + 1 \end{aligned}$$

$$\begin{aligned} 6P(n, \equiv 2) &= 2P(n-1, \equiv 0) + 2P(n-1, \equiv 1) \\ &\quad + P(n-1, \equiv 2) + P(n-1, \equiv 3) \\ &= P(n-1, \equiv 0) + P(n-1, \equiv 1) + 1 \end{aligned}$$

$$\begin{aligned} 6P(n, \equiv 3) &= 2P(n-1, \equiv 1) + 2P(n-1, \equiv 2) \\ &\quad + P(n-1, \equiv 3) + P(n-1, \equiv 0) \\ &= P(n-1, \equiv 1) + P(n-1, \equiv 2) + 1 \end{aligned}$$

From these recurrence relations, we observe that

$$P(n, \equiv 0) + P(n, \equiv 2) = \tfrac{1}{2} \quad \text{and} \quad P(n, \equiv 1) + P(n, \equiv 3) = \tfrac{1}{2}.$$

With a little iteration, we obtain $P(n, \equiv 0) = \frac{323}{1296} + \frac{4}{1296}P(n-4, \equiv 2)$.

Replacing the last term by $(\frac{1}{2} - P(n-4, \equiv 0))$ we have,

$$P(n, \equiv 0) = \tfrac{1}{4} + \left[\tfrac{1}{1296} - \tfrac{4}{1296}P(n-4, \equiv 0)\right].$$

It follows that $P(4|S_n) = \frac{1}{4}$ if and only if $P(4|S_{n-4}) = \frac{1}{4}$. Now $P(4|S_2) = \frac{1}{4}$, but $P(4|S_m) \neq \frac{1}{4}$ if $m = 1, 3$, or 4.

Thus, $P(4|S_n) = \frac{1}{4}$ if and only if $n \equiv 2$ (mod 4).

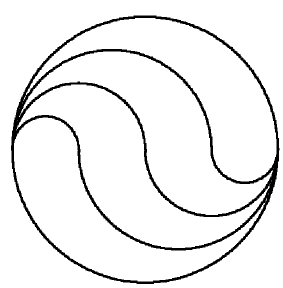
Conclusion

What about other possible k's ($k = 2^s \cdot 3^t$) for which it may be that $P(k|S_n) = \frac{1}{k}$ for some n? To be honest, we don't know. We have checked those k's which are less than 54 for $n \leq 55$. For $k = 8$ and 9 respectively, the only values of $n \leq 55$ for which $P(k|S_n) = \frac{1}{k}$ are $n = 3$ and 2 respectively. For $n \leq 55$, $P(k|S_n) = \frac{1}{k}$ never holds for $12 < k < 54$. The last value for which we could report is $k = 12$. But rather than spoil the fun of the interested reader, we say no more.

References

1. Neff, J.D. Dice tossing and Pascal's triangle. *Two-Year College Mathematics Journal*, **13** (1982), pp. 311-314.
2. Tucker, A. *Applied Combinatorics,* 2nd Ed., John Wiley and Sons, 1984.

Here are three curves that divide a circle into four equal areas.



Any other nice examples?