SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1827. [1993: 78; 1994:57] *Proposed by Šefket Arslanagić, Berlin, Germany, and D.M. Milošević, Pranjani, Yugoslavia.*

In commenting on the solutions submitted, the editor asked for a short proof of

$$\sum \frac{bc}{s \Leftrightarrow a} = s + \frac{(4R+r)^2}{s}.$$

Two very nice and very different solutions have been received.

I. *Solution by TOSHIO SEIMIYA, Kawasaki, Japan.*

Since $\tan(A/2) = r_1/s$, and using a result in R.A. Johnson, *Advanced Euclidean Geometry*, p. 189, we obtain

$$\sum \tan(A/2) = \sum r_1/s = (4R+r)/s.$$

Since $A + B + C = \pi$, we then have $\sum \tan(B/2)\tan(C/2) = 1$, which leads to

$$\left(\sum \tan(A/2)\right)^2 = \Leftrightarrow 1 + \sum \sec^2(A/2).$$

Together, we now have $\sum \sec^2(A/2) = 1 + \left(\frac{4R+r}{s}\right)^2$.

Since $\frac{bc}{s \Leftrightarrow a} = s\left(\frac{bc}{s(s \Leftrightarrow a)}\right) = s\ \sec^2(C/2)$, we have

$$\sum \frac{bc}{s \Leftrightarrow a} = s\left(\sum \sec^2(A/2)\right) = s + \frac{(4R+r)^2}{s}. \quad \blacksquare$$

II. *Solution by Waldemar Pompe, student, University of Warsaw, Poland.*

Since $\sum a = 2s$, $\sum bc = s^2 + r^2 + 4Rr$, and $abc = 4sRr$, we get:

$$\sum (s \Leftrightarrow b)(s \Leftrightarrow c) = 3s^2 \Leftrightarrow 4s^2 + \sum bc = r^2 + 4Rr.$$

Using this, we obtain

$$\sum \frac{1}{s \Leftrightarrow a} = \frac{r^2 + 4Rr}{(s \Leftrightarrow a)(s \Leftrightarrow b)(s \Leftrightarrow c)} = \frac{r^2 + 4Rr}{sr^2} = \frac{r + 4R}{sr}.$$

Therefore

$$\begin{aligned}\sum \frac{bc}{s \Leftrightarrow a} &= abc \sum \frac{1}{a(s \Leftrightarrow a)} = 4Rr \sum \left(\frac{1}{a} + \frac{1}{s \Leftrightarrow a}\right) \\ &= 4Rr\left(\frac{s^2 + r^2 + 4Rr}{4sRr} + \frac{r + 4R}{sr}\right) \\ &= s + \frac{r^2 + 4Rr}{s} + \frac{4Rr + 16R^2}{s} = s + \frac{(4R + r)^2}{s}. \quad \blacksquare\end{aligned}$$

2000. [1994: 286] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
A 1000–element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to $1/5$?

Comment by Stan Wagon, Macalester College, St. Paul, Minnesota, USA.

The answer is that

$$p = \frac{1}{5} + \frac{4}{5}\frac{\binom{400}{200}}{\binom{2000}{1000}} = \frac{1}{5} + 410^{-482}.$$

See Stan Wagon and Herbert S. Wilf, *When are subset sums equidistributed modulo m?*, Electronic Journal of Combinatorics 1 (1994).

In particular, we obtained a necessary and sufficient condition that the t–subsets of $[n]$ be equidistributed $\mod m$. That condition is:

> $t* > n* \mod d$ for all d dividing m (except $d = 1$), where $*$ refers to the least non-negative residue.

In the case at hand, $m = 5$, so only $d = 5$ need be considered, and then $t* = n* = 0$, so equidistribution fails. While this does not directly answer problem 2000 (since it gives no information about the specific remainder $0 \mod 5$), the paper does discuss many intriguing open questions related to the $\mod m$ distribution of subset sums. Thus it strikes me that, because some of your readers were successful at generalizing the problem as stated, they would be interested in this reference. Indeed, perhaps some characterization of the quadruples (i, t, n, m), such that the set of t–subsets of $[n]$ whose $\mod m$ sum is i has less than average size, is possible.

The problem arose from lottery considerations. When are the tickets in a lottery equidistributed with respect to the $\mod m$ value of their sums?

2011. [1995: 52] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. BI and CI meet AC and AB at D and E respectively. P is the foot of the perpendicular from I to DE, and IP meets BC at Q. Suppose that $IQ = 2IP$. Find angle A.

Solution by the proposer.

Let X and Y be the feet of perpendiculars from I to BC and AB, then $IX = IY = r$, where r is the inradius of $\triangle ABC$.

We put $\angle ABI = \angle IBC = \beta$, $\angle ACI = \angle ICB = \gamma$, $\angle EIB = \alpha$, $\angle IDE = x$, and $\angle IED = y$. Then we have $\beta + \gamma = \alpha$ and $x + y = \alpha$. As $\angle IQC = \angle IBQ + \angle BIQ = \angle IBQ + \angle DIP = \beta + \frac{\pi}{2} \Leftrightarrow x$, we get $r = IX = IQ \sin \angle IQC = IQ \sin(\beta + \frac{\pi}{2} \Leftrightarrow x)$, so that

$$r = IQ \cos(\beta \Leftrightarrow x) \tag{1}$$

As $\angle AEI = \angle EBI + \angle EIB = \alpha + \beta$, we get

$$r = IY = IE \sin \angle AEI = IE \sin(\alpha + \beta). \tag{2}$$

Because $PI = IE \sin y$, and $IQ = 2PI$, we have from (1)

$$r = 2PI \cos(\beta \Leftrightarrow x) = 2IE \sin y \cos(\beta \Leftrightarrow x). \tag{3}$$

From (2) and (3) we have (cancelling IE),

$$\begin{aligned} \sin(\alpha + \beta) = 2 \sin y \cos(\beta \Leftrightarrow x) &= \sin(\beta + y \Leftrightarrow x) + \sin(x + y \Leftrightarrow \beta) \\ &= \sin(\beta + y \Leftrightarrow x) + \sin(\alpha \Leftrightarrow \beta). \end{aligned}$$

Therefore $\sin(\beta + y \Leftrightarrow x) = \sin(\alpha + \beta) \Leftrightarrow \sin(\alpha \Leftrightarrow \beta) = 2 \cos \alpha \sin \beta$, thus we obtain

$$\sin \beta \cos(y \Leftrightarrow x) + \cos \beta \sin(y \Leftrightarrow x) = 2 \cos \alpha \sin \beta. \tag{4}$$

Similarly we have (exchanging β for γ, and x for y, y for x, simultaneously) $\sin \gamma \cos(x \Leftrightarrow y) + \cos \gamma \sin(x \Leftrightarrow y) = 2 \cos \alpha \sin \gamma$, or

$$\sin \gamma \cos(y \Leftrightarrow x) \Leftrightarrow \cos \gamma \sin(y \Leftrightarrow x) = 2 \cos \alpha \sin \gamma. \tag{5}$$

Multiplying (4) by $\sin \gamma$ and (5) by $\sin \beta$, we get

$$(\sin \gamma \cos \beta + \cos \gamma \sin \beta) \sin(y \Leftrightarrow x) = 0$$

or $\sin(\beta + \gamma) \sin(y \Leftrightarrow x) = 0$, that is $\sin \alpha \sin(y \Leftrightarrow x) = 0$.

Since $\sin \alpha > 0$, we have $\sin(y \Leftrightarrow x) = 0$, therefore $y = x$, and $\cos(y \Leftrightarrow x) = 1$. Hence we get from (4) $\sin \beta = 2 \cos \alpha \sin \beta$. Because $\sin \beta > 0$ we get $\cos \alpha = \frac{1}{2}$. Thus we have $\alpha = 60^\circ$. As $\alpha = 90^\circ \Leftrightarrow \frac{1}{2} \angle A$, we obtain $\angle A = 60^\circ$. ■

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UKand KEE-WAI LAU, Hong Kong. There was one incorrect solution.

2016. [1995: 53] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Recall that $0.\overline{19}$ stands for the repeating decimal $0.19191919\ldots$, for example, and that the period of a repeating decimal is the number of digits in the repeating part. What is the period of

$$\text{(a) } 0.\overline{19} + 0.\overline{199}\,, \qquad \text{(b) } 0.\overline{19} \times 0.\overline{199}\ ?$$

Solution by Christopher J. Bradley, Clifton College, Bristol, UK

(a) We have

$$0.\overline{19} + 0.\overline{199} = \frac{19}{99} + \frac{199}{999} = \frac{19 \times 10101 + 199 \times 1001}{999999} = \frac{391118}{999999}$$

$$= 0.\overline{391118},$$

so the period is **6**.

(b) Since

$$0.\overline{19} \times 0.\overline{199} = \frac{19 \times 199}{99 \times 999}\,,$$

we look for a number with all nines which is a multiple of 999×99. In order to be a multiple of 999 it must have $3k$ nines for some integer k. When this number is divided by 999, the quotient has k ones interspersed with pairs of zeros: $1001001\ldots1001$. In order that this quotient be divisible by 99 as well, k must be divisible by 9 *and* must be even (using the well known rules for divisibility by 9 and 11). Hence the required number will have $3 \times 9 \times 2 = 54$ nines, so the period is **54**. That the number has the full period of 54 results from the fact that 19 and 199 are both coprime to 99 and 999 ensuring no fortuitous cancellations. ■

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TIM CROSS, Wolverley High School, Kidderminster, UK; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JURGËN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Part (a) *only was solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; and JOHN S. VLACHAKIS, Athens, Greece. Part* (b) *only was solved by KEE-WAI LAU, Hong Kong.*

Konečný and Perz had solutions which were comparable to Bradley's in their simplicity and minimal dependence on calculation.

Cross notes that the product in (b) *equals*

$$\frac{3781}{98901} = \frac{3780}{98901} + \frac{1}{98901},$$

where the first fraction is $140/3663 = 0.\overline{038220}$, *and the second fraction is the "rather interesting" recurring decimal*

$$0.\overline{000010\ 111121\ 222232\ 333343\ 444454\ 555565\ 666676\ 777787\ 888899}\ !$$

Bellot and López mention the following general result of J. E. Oliver which is quoted (for an arbitrary number of fractions) on page 166 *of Dickson's* History of the Theory of Numbers, Volume 1*: if* x'/x *and* z'/z *are periodic fractions, with periods* a *and* b *respectively, then* $(x'/x)(z'/z)$ *has a period of*

$$\frac{xz}{[x,z]} \cdot [a,b],$$

where $[a,b]$ *is the least common multiple of* a *and* b*. This formula, which can be more simply written as* $(x,z)[a,b]$ *where* (x,z) *is the greatest common divisor of* x *and* z*, gives precisely* 54 *in our case. However, perhaps "period" here was intended to mean* **any** *period rather than just the smallest period, because the formula fails in many cases. For example, for the fractions* $11/3$ *and* $1/11$*, which have periods* 1 *and* 2 *respectively, the formula gives* $(3,11)[1,2] = 1 \cdot 2 = 2$ *as the period for the product* $(11/3)(1/11) = 1/3$*, which has period only* 1 *of course. Unfortunately, the reference given for Oliver's result is page* 295 *of* Math. Monthly, *Vol. 1, 1859; this cannot be the familiar* American Math. Monthly, *which only started publishing in 1894. Can any reader supply us with more information?*

2017. [1995: 53] *Proposed by D. J. Smeenk, Zaltbommel, the Netherlands.*

We are given a fixed circle κ and two fixed points A and B not lying on κ. A variable circle through A and B intersects κ in C and D. Show that the ratio

$$\frac{AC \cdot AD}{BC \cdot BD}$$

is constant. [This is not a new problem. A reference will be given when the solution is published.]

Solution by Toshio Seimiya, Kawasaki, Japan.

Let γ be a fixed circle passing through A, B and intersecting κ at X and Y. Then XY, CD, and AB are all parallel or concurrent at the radical centre of the three circles.

Case 1. $XY \| AB$.

If $XY \| AB$ then $CD \| AB$, so $AC = BD$ and $AD = BC$. Hence

$$\frac{AC \cdot AD}{BC \cdot BD} = 1 \text{ (which is a constant).}$$

Case 2. $XY \nparallel AB$.

Let P be the intersection of XY with AB, then P is a fixed point (the radical centre) and CD passes through P.

Since either $\angle CAD = \angle CBD$, or $\angle CAD + \angle CBD = 180°$, we get

$$\frac{[ACD]}{[BCD]} = \frac{AC \cdot AD}{BC \cdot BD}, \tag{1}$$

where $[UVW]$ denotes the area of triangle UVW.

Let A', B' be the feet of perpendiculars from A, B to CD, then $AA' \| BB'$, and

$$\frac{AA'}{BB'} = \frac{AP}{BP}. \tag{2}$$

Because $\dfrac{[ACD]}{[BCD]} = \dfrac{AA'}{BB'}$, we have from (1) and (2)

$$\frac{AC \cdot AD}{BC \cdot BD} = \frac{AP}{BP} = \text{constant.} \quad \blacksquare$$

The proposer tells us that the problem comes from a 1939 book by Dr. P. Molenbroek.

(*Note that $\angle CAD$ and $\angle DBC$ might be supplementary rather than equal, so $\sin\angle CAD = \sin\angle DBC$ still holds, but the triangles CAD and DBC are not necessarily similar.*)

Also solved by: CLAUDIO ARCONCHER, Jundiaí, Brazil; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, Minneapolis, Minnesota, USA; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; there were four incorrect solutions.

2018. [1995: 53] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

How many permutations $(x_1, \dots, x_n)$ of $\{1, \dots, 2\}$ are there such that the cyclic sum $\sum_{i=1}^{n} |x_1 \Leftrightarrow x_{i+1}|$ (with $x_{n+1} = x_1$) is (a) a minimum, (b) a maximum?

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

(a) Let j and k be such that $x_j = 1$ and $k_k = n$. For all m, define $x_{n+m} = x_m$. Then we have

$$\begin{aligned} \sum_{i=1}^{n} |x_i \Leftrightarrow x_{i+1}| &= \sum_{i=j}^{k-1} |x_i \Leftrightarrow x_{i+1}| + \sum_{i=k}^{j-1} |x_i \Leftrightarrow x_{i+1}| \\ &\geq |x_k \Leftrightarrow x_j| + |x_j \Leftrightarrow x_k| \quad \text{by the triangle inequality} \\ &= 2n \Leftrightarrow 2. \end{aligned}$$

Equality holds if and only if both sequences

$$\begin{gathered} \{x_k = n,\ x_{k+1},\ \dots,\ x_{j-1},\ x_j = 1\} \\ \{x_k,\ x_{k-1},\ \dots,\ x_{j+1},\ x_j = 1\} \end{gathered}$$

are monotonic decreasing. Each element of $\{n \Leftrightarrow 1,\ n \Leftrightarrow 2,\ \dots,\ 3,\ 2\}$ may be in either the first or second sequence, but not both. Choosing which elements are in these sequences determines the positions of all of $\{n \Leftrightarrow 1, n \Leftrightarrow 2, \dots, 2, 1\}$ uniquely. Thus there are $n2^{n-2}$ permutations with minimal sum $2n \Leftrightarrow 2$. ■

(b) Consider the effect of a single element x_i on the sum. If $x_i > x_{i+1}$ and $x_i > x_{i-1}$, then the two terms in the sum involving x_i are $(x_i \Leftrightarrow x_{i+1})$ and $(x_i \Leftrightarrow x_{i-1})$. Thus the term x_i contributes $2x_i$ to the sum. Similarly, if $x_i < x_{i+1}$ and $x_i < x_{i-1}$, then the term x_i contributes $\Leftrightarrow 2x_i$ to the sum. Further, if x_i is less than one of x_{i-1}, x_{i+1}, and greater than the other, then x_i has no contribution to the sum.

Suppose that j elements have no contribution to the sum. There are n pairs of elements x_i, x_{i+1}. In each pair, one number is greater and one is less than the other. Thus there are $\frac{n-j}{2}$ numbers x_i which contribute $2x_i$ to the sum, and $\frac{n-j}{2}$ numbers x_i which contribute $\Leftrightarrow 2x_i$ to the sum. If we can arrange the numbers x_i which contribute $2x_i$ to the sum to be the largest $\frac{n-j}{2}$ numbers, and the numbers x_i which contribute $\Leftrightarrow 2x_i$ to the sum to be the smallest $\frac{n-j}{2}$ numbers, then a maximum value is clearly attained with j as small as possible.

This is indeed possible. If n is even ($n = 2k$) and $j = 0$, we must have that $\{x_i,\ x_{i+2},\ \dots,\ x_{i-4},\ x_{i-2}\}$ is a permutation of $\{k+1,\ k+2,\ \dots,\ 2k\}$ for $i = 0$ or $i = 1$, and the other k numbers must be a permutation of $\{1,\ 2,\ \dots,\ k\}$. Otherwise, for some x_i, we would have $x_i < x_{i+1} < x_{i+2}$ which is a contradiction. This gives a same maximal sum of $2(k + 1 + k + 2 + \dots + 2k) \Leftrightarrow 2(1 + 2 + \dots + k) = 2(k^2) = n^2/2$, for any permutation of $\{k+1, k+2 \ldots , 2k\}$ and $\{1, 2, \ldots , k\}$. So there are $2(k!)^2$ possible permutations with maximal sum if $n = 2k$.

If n is odd ($n = 2k + 1$), then j must be odd, so j is at least 1. Placing the middle element $k+1$ in one of the $2k+1$ possible positions gives permutations of $\{1, 2, \ldots , k\}$ and $\{k+2, k+3, \ldots , 2k+1\}$ in alternating positions. This gives the same maximal sum, $(n^2 \Leftrightarrow 1)/2$, for every such permutation. Hence there are $2(2k + 1)(k!)^2$ possible permutations with maximal sum of $n = 2k + 1$. ■

Also solved by NEILS BEJLEGAARD, Stavanger, Norway; TOBY GEE, student, The John of Gaunt School, Trowbridge, England, part (a) only; P. PENNING, Delft, the Netherlands; and the proposer.

2019. [1995: 53] *Proposed by P. Penning, Delft, the Netherlands.*

In a plane are given a circle C with diameter ℓ, and a point P within C but not on ℓ. Construct the equilateral triangles that have one vertex at P, one on C, and one on ℓ.

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Assume that we have a solution triangle, and consider the effect of rotating the circle C by $60°$ about P. We can do this in two directions, clockwise or counterclockwise. The vertex of the equilateral triangle on C is rotated to the vertex on the original diameter. Since each rotated circle intersects the diameter only once (or twice, if ℓ is to be interpreted as the line containing the given diameter), this vertex is now known and we can easily find the vertex on C of the equilateral triangle. We see that there are two such equilateral triangles (four if the extension of the diameter is allowed). To construct the pair of rotated circles, find their centres as the third vertices of the equilateral triangles that have PO as base, where O is the centre C; their radius is half the length of the given diameter. ■

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PETER HURTHIG, Columbia College, Burnaby, BC; D. J. SMEENK, Zaltbommel, the Netherlands; and the proposer. One incorrect solution was received.

Most of the solvers used an argument similar to the featured solution. Bradley remarked that he has seen the rotation technique used to solve the analogous problem in which a line parallel to ℓ is given instead of the circle C. Bejlegaard went even further, pointing out that the given triangle could have any prescribed shape (instead of equilateral) and that C and ℓ could be any two curves for which the points of intersection of ℓ with the rotated image of C could be constructed.

2020. [1995: 53] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Let a, b, c, d be **distinct** real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4 \quad\text{and}\quad ac=bd.$$

Find the maximum value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x=\frac{a}{b}$ and $y=\frac{b}{c}$. Then by $ac=bd$ we have $\frac{c}{d}=\frac{b}{a}=\frac{1}{x}$, $\frac{d}{a}=\frac{c}{b}=\frac{1}{y}$, $\frac{a}{c}=xy$ and $\frac{b}{d}=\frac{b}{c}\cdot\frac{c}{d}=\frac{y}{x}$. Thus we are to find the maximum value of $xy+\frac{y}{x}+\frac{1}{xy}+\frac{x}{y}$ subject to the conditions that a, b, c, d are distinct and

$$(*)\quad x+y+\frac{1}{x}+\frac{1}{y}=4.$$

Let $e=x+\frac{1}{x}$ and $f=y+\frac{1}{y}$. Then $xy+\frac{z}{x}+\frac{1}{xy}+\frac{x}{y}=ef$.

By the Arithmetic Mean-Geometric Mean Inequality, we have $t+\frac{1}{t}\geq 2$ if $t>0$ and $t+\frac{1}{t}\leq \Leftrightarrow 2$ if $t<0$.

By $(*)$, we have that x and y cannot both be negative. If both are positive, then $(*)$ implies $x=y=1$ or $a=b=c$, a contradiction. Hence exactly one of x and y is negative.

Assume, without loss of generality, that $x>0$, $y<0$. Then we get $f\leq \Leftrightarrow 2$, $e=4\Leftrightarrow f\geq 6$ and so $ef\leq \Leftrightarrow 12$. Equality holds, for example, when $a=3+2\sqrt{2}$, $b=1$, $c=\Leftrightarrow 1$ and $d=\Leftrightarrow(3+2\sqrt{2})$. ■

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; NIELS BEJLEGAARD, Stavanger, Norway; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, New York, USA; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, the Netherlands; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; JOHN VLACHAKIS, Athens, Greece; and the proposer. Ten incorrect or incomplete solutions were also received. (*Is this a record?*)

Many of them showed that $\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b} \leq 4$ *and then claimed erroneously that* 4 *is actually the maximum value.*

From Bosley's solutions above, it is not difficult to show that the upper bound $\Leftrightarrow 12$ *is attained if and only if* (a, b, c, d) *equals* $(k, (3 \pm 2\sqrt{2})k, \Leftrightarrow(3 \pm 2\sqrt{2})k, \Leftrightarrow k)$ *or* $(k, \Leftrightarrow k, \Leftrightarrow(3 \pm 2\sqrt{2})k, (3 \pm 2\sqrt{2})k)$ *for some* $k \neq 0$*. This was shown by Flanigan and Mane.*

Arslanagić conjectured that the minimum value of the given sum is $\Leftrightarrow 27\frac{661}{900}$*!*

2021. [1995: 89] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

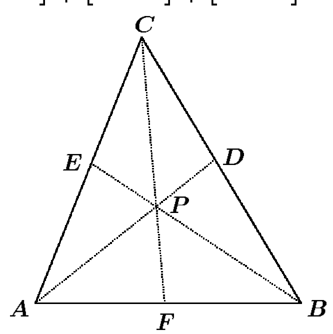
P is a variable interior point of a triangle ABC, and AP, BP, CP meet BC, CA, AB at D, E, F respectively. Find the locus of P so that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

I. *Solution by P. Penning, Delft, the Netherlands.*

First, note that $[PAE] + [PFB] + [PDC] = \frac{1}{2}[ABC]$.



Let $AF/AB = \frac{1}{2} \Leftrightarrow x$, $BD/BC = \frac{1}{2} \Leftrightarrow y$, $CE/CA = \frac{1}{2} \Leftrightarrow z$.
From Ceva's Theorem, we get:

$$\left(\tfrac{1}{2} \Leftrightarrow x\right)\left(\tfrac{1}{2} \Leftrightarrow y\right)\left(\tfrac{1}{2} \Leftrightarrow z\right) = \left(\tfrac{1}{2} + x\right)\left(\tfrac{1}{2} + y\right)\left(\tfrac{1}{2} + z\right)$$

or

$$x + y + z = 4xyz. \qquad (1)$$

From the figure, we can see that

$$\left(\tfrac{1}{2} \Leftrightarrow x\right)[ABC] = [PAF]+[PAE]+[PEC] = [PAE]+\tfrac{1}{2}[ABC]\Leftrightarrow[PBD].$$

Thus

$$x[ABC] = [PBD] \Leftrightarrow [PAE]. \qquad (2)$$

Similarly, we have

$$y[ABC] \;=\; [PCE] \Leftrightarrow [PFB], \qquad (3)$$
$$z[ABC] \;=\; [PAF] \Leftrightarrow [PDC]. \qquad (4)$$

Adding (2), (3), and (4) gives

$$(x+y+z)[ABC] \;=\; \tfrac{1}{2}[ABC] \Leftrightarrow \tfrac{1}{2}[ABC],$$

so that $x+y+z=0$. This, together with (1), gives $xyz=0$, so that $x=0$ or $y=0$ or $z=0$.
Thus the locus consists of the three medians of the triangle (excluding the end points). ∎

II. *Solution by the Austin Academy Problem Solvers, Austin, Texas, USA.*

Assign masses a, b, c to the vertices A, B, C, respectively, so that P is the centre of mass and $a+b+c=2$.
Then $\dfrac{[PAF]}{[ABC]} = \dfrac{FP}{CF}\cdot\dfrac{AF}{AB} = \dfrac{c}{(a+b+c)}\cdot\dfrac{b}{(a+b)}$.
There are similar expression for PBD and PCE.
So we need to find the locus of P such that

$$\frac{bc}{(a+b)}+\frac{ac}{(b+c)}+\frac{ab}{(a+c)} \;=\; \frac{(a+b+c)}{2} \quad (=\;1).$$

Cross multiplying and simplifying leads to

$$(a \Leftrightarrow b)(b \Leftrightarrow c)(c \Leftrightarrow a)(a+b+c)=0.$$

But $a+b+c=2$, so we must have at least one of $a=b$, $b=c$ and $c=a$.
Now, this is exactly when P lies on a median of triangle ABC (excluding the end points). ∎

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallora, Spain; PETER HURTHIG, Columbia College, Burnaby, BC; KEE-WAI LAU, Hong Kong; ASHISH KR. SINGH, Student, Kanpur, India; D. J. SMEENK, Zaltbommel, the Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2022. [1995: 89] *Proposed by K. R. S. Sastry, Dodballapur, India.*
Find the smallest integer of the form

$$\frac{A \star B}{B},$$

where A and B are three-digit integers and $A{\star}B$ denotes the six-digit integer formed by placing A and B side by side.

Solution by M. Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland.

I claim that the answer to the problem is 121.
Note that when $A = 114$ and $B = 950$, then $\frac{A \star B}{B} = 121$, so this number is actually obtained. We must therefore show that in any other case when $D = \frac{A \star B}{B}$ is an integer, then $D \geq 121$.
Observe that $\frac{A \star B}{B} = \frac{(1000A + B)}{B} = \frac{1000A}{B} + 1$, so we must now show that whenever $E = \frac{1000A}{B}$ is an integer, then $E \geq 120$.
Let $B = GX$ where $G = \gcd(B, 1000)$. We will show that whenever $\frac{A}{X}$ is an integer, then $F = \left(\frac{1000}{G}\right)\left(\frac{A}{X}\right) \geq 120$. Note that since $B < 1000$, we have $X < \frac{1000}{G}$. Also, recall that we are only interested in cases when $\frac{A}{X}$ is an integer.

If $G = 1$, 2, 4, 5 or 8, then $\frac{1000}{G} > 120$, and we are done.

If $G = 10$, then $X < 100$, so $\frac{A}{X} \geq 2$ (since $A \geq 100$). In this case, $F \geq 200$, and we are done.

If $G = 20$, then $X < 50$ and $\frac{A}{X} \geq 3$. Thus $F \geq 150$, and we are done.

If $G = 25$, then $X < 40$ and $\frac{A}{X} \geq 3$. Thus $F \geq 120$, and we are done.

If $G = 40$, then $X < 25$ and $\frac{A}{X} \geq 5$. Thus $F \geq 125$, and we are done.

If $G = 50$, then $X < 20$ and $\frac{A}{X} \geq 6$. Thus $F \geq 120$, and we are done.

If $G = 100$, then $X < 10$ and $\frac{A}{X} \geq 12$. (Note that $x \leq 9$). Thus $F \geq 120$, and we are done.

If $G = 125$, then $X < 8$ and $\frac{A}{X} \geq 15$. Thus $F \geq 120$, and we are done.

If $G = 200$, then $X < 5$ and $\frac{A}{X} \geq 25$. (Note that $x \leq 4$). Thus $F \geq 125$, and we are done.

If $G = 250$, then $X < 4$ and $\frac{A}{X} \geq 34$. Thus $F \geq 136$, and we are done.

If $G = 500$, then $B = 500$ and $\frac{1000A}{B} = 2A \geq 200$, and we are now all done. ■

Note that I am assuming that A is an "honest" three-digit integer, that is $100 \leq A$. Otherwise, there is a trivial solution: $A = 001$, $B = 500$, and $\frac{A \star B}{B} = 3$.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JEFFREY R. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California, USA; P. PENNING, Delft, the Netherlands; the SCIENCE ACADEMY PROBLEM SOLVERS; Austin, Texas, USA; DAVID STONE and BILL MEISEL, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. Some solvers made use of computers to search for all possible solutions. One incorrect solution was submitted. Janous suggested an extension to:

Let $\lambda \in \mathbb{N}$. Determine the integer-minimum, λ_n of $\frac{\lambda A}{B}$, where A and B are n-digit numbers.

2023. [1995: 89] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let a, b, c, d, e be positive numbers with $abcde = 1$.

(a) Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bdc}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\geq\frac{10}{3}.$$

(b) Find a generalization!

Solution to (a) by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x_1 = a$, $x_2 = ab$, $x_3 = abc$, $x_4 = abcd$ and $x_5 = abcde = 1$. Multiply the second, the third, the fourth and the fifth fractions on the left by $\frac{a}{a}$, $\frac{ab}{ab}$, $\frac{abc}{abc}$ and $\frac{abcd}{abcd}$ respectively. Then the expression on the left becomes:

$$\frac{x_1+x_3}{x_5+x_2+x_4}+\frac{x_2+x_4}{x_1+x_3+x_5}+\frac{x_3+x_5}{x_2+x_4+x_1}+\frac{x_4+x_1}{x_3+x_5+x_2}+\frac{x_5+x_2}{x_4+x_1+x_3}.$$

Add 1 to each of the five fractions. Then the desired inequality is equivalent to:

$$\left(\sum_{i=1}^{5} x_i\right)\left(\frac{1}{x_5+x_2+x_4}+\frac{1}{x_1+x_3+x_5}+\frac{1}{x_2+x_4+x_1}+\frac{1}{x_3+x_5+x_2}+\frac{1}{x_4+x_1+x_3}\right)\geq\frac{25}{3},$$

which follows by applying the Arithmetic Mean – Harmonic Mean inequality to the second factor on the left. Equality holds if and only if the five denominators are all equal, which happens if and only if $x_1 = x_2 = x_3 = x_4 = x_5$, which is true if and only if $a = b = c = d = e = 1$. ∎

Solution to (b) by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

Suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\ldots a_n = 1$, where $n > 1$. For each $i = 1, 2, \ldots, n$, define $a_{i+n} = a_i$, and for $i, j = 1, 2, \ldots, n$, let $A_{i,j} = \prod_{r=i}^{i+j-1} a_r$. Further, for each $i = 1, 2, \ldots, n$, define $A_{i,j+n} = A_{i,j}$.

Let I be any non-empty subset of $S = \{0, 1, 2, \ldots, n \Leftrightarrow 1\}$. Then the generalized inequality is:

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S-I} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \geq \frac{n\,(n \Leftrightarrow |I|)}{|I|}.$$

The proof is as follows: let $u = \sum\limits_{r=0}^{n-1} A_{1,r}$. Then

$$\frac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{i,k}} = \frac{\sum\limits_{j\in I} A_{1,i-1}A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{1,i-1}A_{i,k}} = \frac{\sum\limits_{j\in I} A_{1,j+i-1}}{\sum\limits_{k=0}^{n-1} A_{1,k+i-1}}$$

$$= \frac{1}{u}\sum_{j\in I} A_{1,j+i-1},$$

so that

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{i,k}}\right) = \sum_{i=1}^{n}\frac{1}{u}\sum_{j\in I} A_{1,j+i-1} = \frac{1}{u}\sum_{j\in I}\sum_{i=1}^{n} A_{1,j+i-1}$$

$$= \frac{1}{u}\sum_{j\in I} u = |I|.$$

By the Arithmetic Mean – Harmonic Mean inequality, we have:

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \geq \frac{n^2}{\sum\limits_{i=1}^{n}\frac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k\in S} A_{i,k}}} = \frac{n^2}{|I|},$$

and thus

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S-I} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \geq \frac{n^2}{|I|} \Leftrightarrow n = \frac{n\,(n \Leftrightarrow |I|)}{|I|}.$$

Note that part (a) is the special case when $n = 5$ and $I = \{0,\ 2,\ 4\}$. ■

Besides Bosley and Wee, both parts were also solved by VEDULA N. MURTY, Andhra University, Visakhapatnam, India; and the proposer. Part (a) only was solved by SABIN CAUTIS, Earl Haig Secondary School, North York, Ontario.

2024. [1995: 90] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is a known result that if P is any point on the circumcircle of a given triangle ABC with orthocentre H, then $(PA)^2 + (PB)^2 + (PC)^2 \Leftrightarrow (PH)^2$ is a constant. Generalize this result to an n-dimensional simplex.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let O be the centre of a circumhypersphere, let R be the radius, and let A_1, A_2, ... , A_n be the vertices. Define H by the vector expression:

$$\overrightarrow{OH} = \sum_{i=1}^{n} \overrightarrow{OA_i}\,.$$

Also, let P be any point on the surface of the circumhypersphere such that $|\overrightarrow{OP}|^2 = R^2$.

Then we claim that $\sum_{i=1}^{n}(PA_i)^2 \Leftrightarrow (PH)^2 = \text{constant}$.

Let $\overrightarrow{OP} = \vec{x}$. Then

$$\begin{aligned}
\sum_{i=1}^{n}(PA_i)^2 \Leftrightarrow (PH)^2 &= \sum_{i=1}^{n}(\vec{x} \Leftrightarrow \overrightarrow{OA_i})\,.\,(\vec{x} \Leftrightarrow \overrightarrow{OA_i}) \\
&\quad \Leftrightarrow \left(\vec{x} \Leftrightarrow \sum_{i=1}^{n}\overrightarrow{OA_i}\right).\left(\vec{x} \Leftrightarrow \sum_{i=1}^{n}\overrightarrow{OA_i}\right) \\
&= n\,|\vec{x}|^2 \Leftrightarrow 2\vec{x}.\sum_{i=i}^{n}\overrightarrow{OA_i} + \sum_{i=1}^{n} OA_i^2 \\
&\quad \Leftrightarrow |\vec{x}|^2 + 2\vec{x}.\sum_{i=1}^{n}\overrightarrow{OA_i} \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\overrightarrow{OA_i}.\overrightarrow{OA_j} \\
&= (n \Leftrightarrow 1)R^2 \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\overrightarrow{OA_i}.\overrightarrow{OA_j} \\
&= \text{constant.} \quad \blacksquare
\end{aligned}$$

CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA, WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, and the proposer had essentially the same solution.

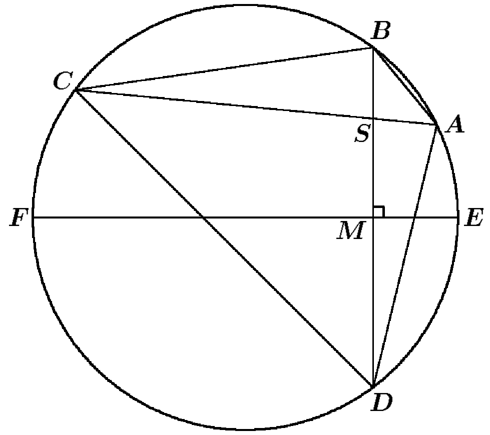
2027. [1995: 90] *Proposed by D. J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is inscribed in a circle Γ, and has an incircle as well. EF is a diameter of Γ with $EF \perp BD$. BD intersects EF in M and AC in S. Show that $AS : SC = EM : MF$.

Solution by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

Note: It is necessary that A and E lie on the same side of the line BD, otherwise the result is false.

Since $ABCD$ has an incircle, we have $AB + CD = BC + AD$, or, equivalently, $BC \Leftrightarrow CD = AB \Leftrightarrow AD$. Thus



$$\begin{aligned}
\frac{AS}{AC} &= \frac{\text{Area}\Delta ABD}{\text{Area}\Delta BCD} = \frac{AB\ AD \sin(\angle BAD)}{BC\ CD \sin(\angle BCD)} \\
&= \frac{AB\ AD}{BC\ CD} \qquad (\text{since } \angle BAD + \angle BCD = 180^\circ) \\
&= \frac{2AB\ AD\ \left((BC \Leftrightarrow CD)^2 \Leftrightarrow BD^2\right)}{2BC\ CD\ \left((AB \Leftrightarrow AD)^2 \Leftrightarrow BD^2\right)} \\
&= \frac{AB\ AD\ \left(BC^2 + CD^2 \Leftrightarrow 2BC\ CD \Leftrightarrow BD^2\right)}{2BC\ CD\ \left(AB^2 + AD^2 \Leftrightarrow 2AB\ AD \Leftrightarrow BD^2\right)} \\
&= \frac{\dfrac{BC^2 + CD^2 \Leftrightarrow BD^2 \Leftrightarrow 2BC\ CD}{2BC\ CD}}{\dfrac{AB^2 + AD^2 \Leftrightarrow BD^2 \Leftrightarrow 2AB\ AD}{2AB\ AD}} \\
&= \frac{\cos(\angle BCD) \Leftrightarrow 1}{\cos(\angle BAD) \Leftrightarrow 1},
\end{aligned}$$

using the cosine rule, applied to triangles ABD and BCD.

Since EF is a diameter, we have $BE + DF = DE + BF$, and so, by an argument similar to the above, we get

$$\frac{EM}{MF} = \frac{\cos(\angle BFD) \Leftrightarrow 1}{\cos(\angle BED) \Leftrightarrow 1}.$$

Since $ABCD$ is a cyclic quadrilateral, we have $\angle BED = \angle BAD$ and $\angle BFD = \angle BCD$, and hence that $AS : SC = EM : MF$. ∎

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; SCIENCE ACADEMY PROBLEM SOLVERS, Austin, Texas, USA; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, student, Kanpur, India; and the proposer. These solvers assumed implicitly that A and E lay on the same side of the line BD.

2028. [1995: 90] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

If $n \geq m \geq k \geq 0$ are integers such that $n + m \Leftrightarrow k + 1$ is a power of 2, prove that the sum $\binom{n}{k} + \binom{m}{k}$ is even.

Solution by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

For any non-negative integer t, let the binary representation of t be $(a_r a_{r-1} \cdots a_0)_2$, where $a_i \in \{0, 1\}$, $i = 0, 1, \ldots, r$. Let $f(t)$ be the largest integer α such that 2^α divides t. Let $g(t)$ be the sum of the digits in the binary representation of t. Then

$$\begin{aligned} f(t!) &= \sum_{j=1}^{\infty} \left[\frac{t}{2^j}\right] = \sum_{j=1}^{r} (a_r a_{r-1} \cdots a_j)_2 \\ &= \sum_{j=1}^{r} \sum_{s=0}^{j-1} 2^s = \sum_{j=1}^{r} a_j (2^j \Leftrightarrow 1) = t \Leftrightarrow g(t). \end{aligned}$$

From this we get

$$f\left(\binom{n}{k}\right) = g(n \Leftrightarrow k) + g(k) \Leftrightarrow g(n).$$

Similarly, $f\left(\binom{m}{k}\right) = g(m \Leftrightarrow k) + g(k) \Leftrightarrow g(m)$. Let $n + m \Leftrightarrow k + 1 = 2^\beta$, that is, $n + m \Leftrightarrow k = 2^\beta \Leftrightarrow 1$. Then, by considering the binary representations of n, $m \Leftrightarrow k$, $2^\beta \Leftrightarrow 1$, m, and $n \Leftrightarrow k$, it is clear that

$$g(n) + g(m \Leftrightarrow k) = \beta = g(n \Leftrightarrow k) + g(m).$$

Thus $g(n \Leftrightarrow k) + g(k) \Leftrightarrow g(n) = g(m \Leftrightarrow k) + g(k) \Leftrightarrow g(m)$, that is, $f\left(\binom{n}{k}\right) = f\left(\binom{m}{k}\right)$. Hence $2|\binom{n}{k} \Leftrightarrow 2|\binom{m}{k}$, so $\binom{n}{k} + \binom{m}{k}$ is even. ∎

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SOFIYA VASINA, student, University of Arizona, Tucson, USA; and the proposer. Kuczma notes that the problem can be solved without any computation as follows:

> Look at Pascal's triangle, rows 0 through 2^{β}, modulo 2, and visualize it geometrically as an equilateral triangle with black and white spots (zeros and ones). It has three (geometric) symmetry axes, and each one of these symmetries preserves the spot design. This fact can be considered as known. And, if not, it suffices to draw the "row 0 through 7 triangle" and notice that it consists of three copies of the (twice) smaller triangle formed by rows 0 through 3, the remaining quarter in the centre of the figure being filled with zeros. The base row is all ones. These observations provide a scheme for an induction proof of the claim for rows 0 through $2^{\beta} \Leftrightarrow 1$. The symmetry with respect to one of the oblique axes is precisely the contents of the problem, expressed in algebraic terms.

Comment by the Editor-in-Chief

Choosing a solution to print is not an easy task. Each collection of submitted solutions is assigned to a member of the Editorial Board, and these editors work independently of one another. The choice of which solution to highlight is left entirely to that editor. Sometimes it happens, as in this issue, that several solutions are chosen from the same solver. I would like to assure all subscribers that every submission is very important to *CRUX*, and that we encourage everyone to submit solutions, as well as proposals for problems, articles for publication, and contributions to the other corners. Every subscriber is very important to us and we really value all contributions.

And while I am on the subject of contributions, please continue to send in proposals for problems. We publish 100 per year, and we do not have too many in reserve at this time. Without your contributions, there would be no *CRUX*.