THE OLYMPIAD CORNER

No. 172

R.E. Woodrow

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Because we are now publishing eight numbers of the Corner rather than ten, I am giving two Olympiad Sets for your pleasure. Besides, here in Canada it is winter, and quite cold, so having a stock of problems to contemplate in a warm spot is a good idea. Both of the sets we give this number were collected by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, when he was Canadian Team Leader to the IMO at Istanbul. Many thanks to him for gathering a wide sample of contests. We begin with the Telecom 1993 Australian Mathematical Olympiad.

TELECOM 1993 AUSTRALIAN MATHEMATICAL OLYMPIAD
Paper 1

Tuesday, 9th February, 1993

(Time: 4 hours)

1. In triangle ABC, the angle ACB is greater than $90°$. Point D is the foot of the perpendicular from C to AB; M is the midpoint of AB; E is the point on AC extended such that $EM = BM$; F is the point of intersection of BC and DE; moreover $BE = BF$. Prove that $\angle CBE = 2\angle ABC$.

2. For each function f which is defined for all real numbers and satisfies

$$f(x,y) = x \cdot f(y) + f(x) \cdot y \qquad (1)$$

and

$$f(x+y) = f(x^{1993}) + f(y^{1993}) \qquad (2)$$

determine the value $f(\sqrt{5753})$.

3. Determine all triples (a_1, a_2, a_3), $a_1 \geq a_2 \geq a_3$, of positive integers in which each number divides the sum of the other two numbers.

4. For each positive integer n, let

$$f(n) = [2\sqrt{n}] - [\sqrt{n-1} + \sqrt{n+1}].$$

Determine all values n for which $f(n) = 1$.
Note: If x is a real number, then $[x]$ is the largest integer not exceeding x.

Paper 2

Wednesday, 10th February, 1993

(Time: 4 hours)

5. Determine all integers x and y that satisfy

$$(x+2)^4 - x^4 = y^3.$$

6. In the acute-angled triangle ABC, let D, E, F be the feet of altitudes through A, B, C, respectively, and H the orthocentre. Prove that

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

7. Let n be a positive integer, $a_1, a_2, \ldots, a_n$ positive real numbers and $s = a_1 + a_2 + \cdots + a_n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{s-a_i} \geq \frac{n}{n-1} \quad \text{and} \quad \sum_{i=1}^{n} \frac{s-a_i}{a_i} \geq n(n-1).$$

8. The vertices of triangle ABC in the $x-y$ plane have integer coordinates, and its sides do not contain any other points having integer coordinates. The interior of ABC contains only one point, G, that has integer coordinates. Prove that G is the centroid of ABC.

Next we give the Final Round of the Japan Mathematical Olympiad.

JAPAN MATHEMATICAL OLYMPIAD
Final Round — 11 February, 1993

(Time: 4.5 hours)

1. Suppose that two words A and B have the same length $n > 1$ and that the first letters of them are different while the others are the same. Prove that A or B is not periodic.

2. Let $d(n)$ be the largest odd number which divides a given number n. Suppose that $D(n)$ and $T(n)$ are defined by

$$D(n) = d(1) + d(2) + \cdots + d(n),$$

$$T(n) = 1 + 2 + \cdots + n.$$

Prove that there exist infinitely many positive numbers n such that $3D(n) = 2T(n)$.

3. In a contest, x students took part, and y problems were posed. Each student solved $y/2$ problems. For every problem, the number of students who solved it was the same. For each pair of students, just three problems were solved by both of them. Determine all possible pairs (x, y). Moreover, for each (x, y), give an example of the matrix (a_{ij}) defined by $a_{ij} = 1$ if the ith student solved the jth problem and $a_{ij} = 0$ if not.

4. Five radii of a sphere are given so that no three of them are in a common plane. Among the 32 possible choices of an end point from each segment, find out the number of choices for which the 5 points are in a hemisphere.

5. Prove that there exists a positive constant C (independent of n, a_j) which satisfies the inequality

$$\max_{0\le x\le 2}\prod_{j=1}^{n}|x-a_j| \le C^n \max_{0\le x\le 1}\prod_{j=1}^{n}|x-a_j|$$

for any positive integer n and any real numbers $a_1, \ldots, a_n$.

Last issue we gave a set of six Klamkin Quickies. Here are his "quick" solutions. Many thanks go to Murray Klamkin, the University of Alberta, for sending them to me.

SIX KLAMKIN QUICKIES

1. Which is larger

$$(\sqrt[3]{2}-1)^{1/3} \qquad \text{or} \qquad \sqrt[3]{1/9}-\sqrt[3]{2/9}+\sqrt[3]{4/9}?$$

Solution. That they are equal is an identity of Ramanujan.
Letting $x = \sqrt[3]{1/3}$ and $y = \sqrt[3]{2/3}$, it suffices to show that

$$(x+y)(\sqrt[3]{2}-1)^{1/3} = x^3+y^3 = 1,$$

or equivalently that

$$(\sqrt[3]{2}+1)^3(\sqrt[3]{2}-1) = 3,$$

which follows by expanding out the left hand side.

For other related radical identities of Ramanujan, see Susan Landau, *How to tangle with a nested radical*, Math. Intelligencer, 16 (1994), pp. 49–54.

2. Prove that

$$3\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

where a, b, c are sides of a triangle.

Solution. Each of the inequalities

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

$$3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

follow from their equivalent forms (which follow by expansion):

$$(b+a-c)(c-a)^2+(c+b-a)(a-b)^2+(a+c-b)(b-c)^2 \geq 0,$$

$$(b+c-a)(c-a)^2+(c+a-b)(a-b)^2+(a+b-c)(b-c)^2 \geq 0.$$

3. Let $\omega = e^{i\pi/13}$. Express $\frac{1}{1-\omega}$ as a polynomial in ω with integral coefficients.

Solution. We have

$$\frac{2}{(1-\omega)} = \frac{(1-\omega^{13})}{(1-\omega)} = 1+\omega+\omega^2+\cdots+\omega^{12},$$

$$0 = \frac{(1+\omega^{13})}{(1+\omega)} = 1-\omega+\omega^2-\cdots+\omega^{12}.$$

Adding or subtracting, we get

$$\frac{1}{(1-\omega)} = 1+\omega^2+\omega^4+\cdots+\omega^{12}$$

$$= \omega+\omega^3+\cdots+\omega^{11}.$$

More generally, if $\omega = e^{i\pi/(2n+1)}$,

$$\frac{1}{(1-\omega)} = 1+\omega^2+\omega^4+\cdots+\omega^{2n}.$$

4. Determine all integral solutions of the simultaneous Diophantine equations $x^2+y^2+z^2 = 2w^2$ and $x^4+y^4+z^4 = 2w^4$.

Solution. Eliminating w we get

$$2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4 = 0$$

or

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z) = 0,$$

so that in general we can take $z = x+y$. Note that if (x, y, z, w) is a solution, so is $(\pm x, \pm y, \pm z, \pm w)$ and permutations of the x, y, z. Substituting back, we get

$$x^2+xy+y^2 = w^2.$$

Since $(x, y, w) = (1, -1, 1)$ is one solution, the general solution is obtained by the method of Desboves, that is, we set $x = r + p$, $y = -r + q$ and $w = r$. This gives $r = \frac{(p^2+pq+q^2)}{(q-p)}$. On rationalizing the solutions (since the equation is homogeneous), we get

$$\begin{aligned} x &= p^2 + pq + q^2 + p(q-p) = q^2 + 2pq, \\ -y &= p^2 + pq + q^2 - q(q-p) = p^2 + 2pq, \\ w &= p^2 + pq + q^2, \\ z &= q^2 - p^2. \end{aligned}$$

5. Prove that if the line joining the incentre to the centroid of a triangle is parallel to one of the sides of the triangle, then the sides are in arithmetic progression and, conversely, if the sides of a triangle are in arithmetic progression then the line joining the incentre to the centroid is parallel to one of the sides of the triangle.

Solution. Let **A**, **B**, **C** denote vectors to the respective vertices A, B, C of the triangle from a point outside the plane of the triangle. Then the incentre I and the centroid G have the respective vector representations **I** and **G**, where

$$\mathbf{I} = \frac{(a\mathbf{A} + b\mathbf{B} + c\mathbf{C})}{(a+b+c)}, \qquad \mathbf{G} = \frac{(\mathbf{A} + \mathbf{B} + \mathbf{C})}{3},$$

(where a, b, c are sides of the triangle). If $\mathbf{G} - \mathbf{I} = k(\mathbf{A} - \mathbf{B})$, then by expanding out

$$(b + c - 2a - k')\mathbf{A} + (a + c - 2b + k')\mathbf{B} + (a + b - 2c)\mathbf{C} = 0,$$

where $k' = 3k(a + b + c)$. Since A, B, C are linearly independent, the coefficient of **C** must vanish so that the sides are in arithmetic progression. Also then $k' = b + c - 2a = 2b - a - c$.

Conversely, if $2c = a + b$, then $\mathbf{G} - \mathbf{I} = \frac{3(\mathbf{A}-\mathbf{B})(b-a)}{6(a+b+c)}$, so that GI is parallel to the side AB.

6. Determine integral solutions of the Diophantine equation

$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-w}{z+w} + \frac{w-x}{w+x} = 0$$

(joint problem with Emeric Deutsch, Polytechnic University of Brooklyn).

Solution. It follows by inspection that $x = z$ and $y = w$ are two solutions. To find the remaining solution(s), we multiply the given equation by the least common denominator to give

$$P(x, y, z, w) = 0,$$

where P is the 4th degree polynomial in x, y, z, w which is skew symmetric in x and z and also in y and w. Hence,

$$P(x,y,z,w) = (x-z)(y-w)Q(x,y,z,w),$$

where Q is a quadratic polynomial. On calculating the coefficient of x^2 in P, we get $2z(y-w)$. Similarly the coefficient of y^2 is $-2w(x-z)$, so that

$$P(x,y,z,w) = 2(x-z)(y-w)(xz-yw).$$

Hence, the third and remaining solution is given by $xz = yw$.

Next we turn to the readers' solutions of problems from earlier numbers of the *Corner*. Let me begin by thanking Beatriz Margolis, Paris France; Bob Prielipp, University of Wisconsin-Oshkosh, USA; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zaltbommel, the Netherlands and Chris Wildhagen, Rotterdam, the Netherlands, for sending in nice solutions to some of the problems of the 1994 Canadian and 1994 U.S.A. Mathematical Olympiads. As we publish "official" solutions to the former, and refer readers to an MAA publication for the latter, I normally do not publish these solutions.

Next, correspondence from Murray Klamkin, the University of Alberta, filed under the September number of the *Corner*, contains comments about problems from several numbers.

3. [1993: 5, 1994: 69] 1991 *British Mathematical Olympiad*.

$ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E. Prove that if AC is perpendicular to BD then

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. \qquad (*)$$

Is it true that, if $(*)$ holds, then AC is perpendicular to BD? Give a reason for your answer.

Klamkin's Comment. A simpler solution than that published, plus a generalization, is given in *Crux*, 1989, p. 243, #1.

3. [1994: 184] 1*st Mathematical Olympiad of the Republic of China* (*Taiwan*).

If $x_1, x_2, \dots, x_n$ are n non-negative numbers, $n \geq 3$ and $x_1 + x_2 + \cdots + x_n = 1$ prove that $x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1 \leq 4/27$.

Klamkin's Comment. This problem appeared as problem 1292 in *The Math. Magazine*, April 1988.

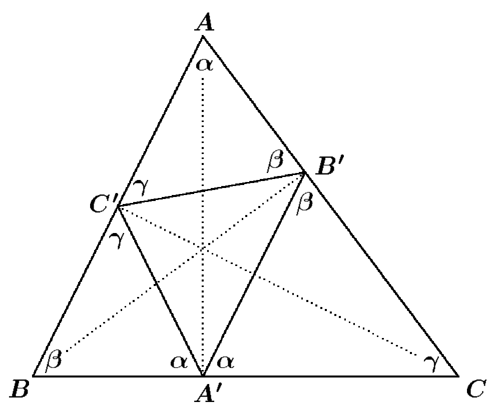
Now we turn to readers' solutions of problems proposed to the Jury, but not used, at the 34th International Mathematical Olympiad at Istanbul. A booklet of "official solutions" was issued by the organizers. Because I do not have formal permission to reproduce these solutions, I will only discuss readers' solutions that are different.

2. [1994: 216] *Proposed by Canada.*

Let triangle ABC be such that its circumradius $R = 1$. Let r be the inradius of ABC and let p be the inradius of the orthic triangle $A'B'C'$ of triangle ABC. Prove that $p \leq 1 - 1/3(1+r)^2$.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

In the first instance I misread the problem: I read $\dfrac{1}{3(1+r)^2}$ instead of $\frac{1}{3}(1+r)^2$!



Now $r_{\triangle ABC} = R(\cos\alpha + \cos\beta + \cos\gamma - 1)$. From $R = 1$, we obtain $r = \cos\alpha + \cos\beta + \cos\gamma - 1$. Similarly,

$$\begin{aligned} p &= \frac{1}{2}[\cos(\pi - 2\alpha) + \cos(\pi - 2\beta) + \cos(\pi - 2\gamma) - 1] \\ &= 2\cos\alpha\cos\beta\cos\gamma. \end{aligned}$$

We are to show that $p + \frac{1}{3}(1+r)^2 \leq 1$. (1)
Equivalently,

$$2\cos\alpha\cos\beta\cos\gamma + \frac{1}{3}(\cos\alpha + \cos\beta + \cos\gamma)^2 \leq 1. \qquad (2)$$

Now

$$\left.\begin{aligned} -1 < \cos\alpha\cos\beta\cos\gamma \leq \tfrac{1}{8}, \\ 1 < \cos\alpha + \cos\beta + \cos\gamma \leq \tfrac{3}{2}. \end{aligned}\right\} \qquad (3)$$

[Bothema, *Geom. Inequalities*, 2.24, resp. 2.16]
Now $2 \cdot \frac{1}{8} + \frac{1}{3}(\frac{3}{2})^2 = 1$, so it is clear that (1) holds. ∎

3. [194: 216] *Proposed by Spain.*

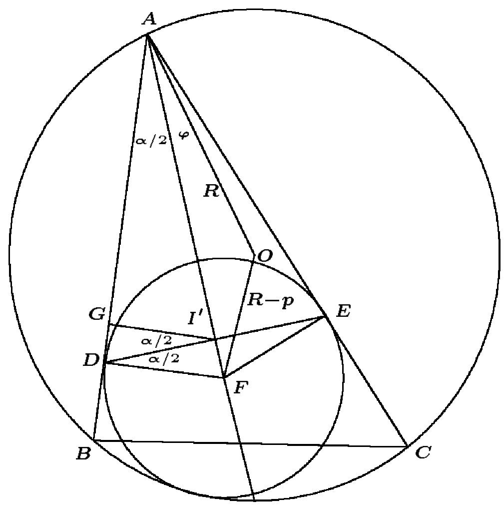
Consider the triangle ABC, its circumcircle k of centre O and radius R, and its incircle of centre I and radius r. Another circle K_c is tangent to the sides CA, CB at D, E, respectively, and it is internally tangent to k. Show that I is the midpoint of DE.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Assume (see figure) that $\beta > \gamma$. Let F be the centre of K_c and ρ its radius. Then F lies on the production of AI. Now $FD = \rho$, so

$$AF = \frac{\rho}{\sin\frac{\alpha}{2}}.$$

Also $AO = R$, $OF = R - \rho$, and $\angle FAO = \varphi = \frac{1}{2}(\beta - \gamma)$.



Apply the Law of Cosines to ΔAFO:

$$OF^2 = AF^2 + AO^2 - 2AF \cdot AO\cos\varphi,$$

or

$$(R-\rho)^2 = \frac{\rho^2}{\sin^2\frac{\alpha}{2}} + R^2 - \frac{2R\rho\cos\left(\frac{\beta-\gamma}{2}\right)}{2\sin\frac{\alpha}{2}}; \quad \rho \neq 0.$$

From this we obtain

$$-2R + \rho = \frac{\rho}{\sin^2\frac{\alpha}{2}} - \frac{2R\cos\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\beta+\gamma}{2}\right)},$$

or

$$\frac{\rho\cos^2\frac{\alpha}{2}}{\sin^2\alpha/2} = \frac{4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\alpha/2}.$$

Thus

$$\rho = \frac{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos^2\frac{\alpha}{2}} = \frac{r}{\cos^2\frac{\alpha}{2}} = FD.$$

Now let DE intersect AF at I'. Then $DI' = DF\cos\frac{\alpha}{2} = \frac{r}{\cos\frac{\alpha}{2}}$.
Let G be the foot of the perpendicular from I' to AB.
Then $I'G = I'D\cos\frac{\alpha}{2} = \frac{r}{\cos\frac{\alpha}{2}}\cos\frac{\alpha}{2} = r$. So I' coincides with I. ■

7. [1994: 217] *Proposed by Israel.*

The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides and S is the area of ABC, prove that

$$DE \geq 2\sqrt{2}\,S\cdot\{a^2+b^2+c^2+4\sqrt{3}\,S\}^{-1/2}.$$

Comment by Murray S. Klamkin, The University of Alberta.

This problem is equivalent to problem #624 of *Crux* [1982: 109–110].

11. [1994: 241] *Proposed by Spain.*

Given the triangle ABC, let D, E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are respectively, the points of tangency with BC of the incircles of the triangles ABD and ACE, show that

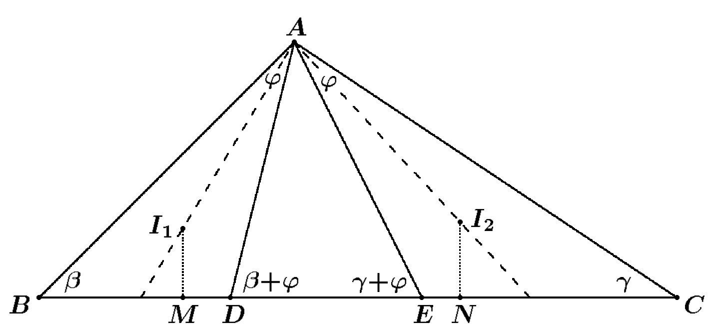
$$\frac{1}{BM}+\frac{1}{MD}=\frac{1}{NC}+\frac{1}{NE}.$$

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

We are to show $\frac{1}{MB}+\frac{1}{MD}=\frac{1}{NC}+\frac{1}{NE}$, or equivalently

$$BD\cdot NC\cdot NE = CE\cdot MB\cdot MD. \tag{1}$$

We denote $\angle BAD = \angle CAE = \varphi$.



Applying the law of sines to ΔABD we obtain

$$BD = \frac{c\sin\varphi}{\sin(\beta+\varphi)},\qquad AD = \frac{c\sin\beta}{\sin(\beta+\varphi)}. \tag{2}$$

The sine law for ΔACE gives

$$CE = \frac{b\sin\varphi}{\sin(\gamma+\varphi)},\qquad AE = \frac{b\sin\gamma}{\sin(\gamma+\varphi)}. \tag{2'}$$

From ΔABD:

$$2MB = AB + BD - AD$$
$$2MD = -AB + BD + AD,$$

and from ΔACE:

$$2NC = AC + CE - AE$$
$$2NE = -AC + CE + AE.$$

From this we see that to show (1) we must show

$$\begin{aligned} & BD(AC + CE - AE)(-AC + CE + AE) \\ = \; & CE(AB + BD - AD)(-AB + BD + AD), \end{aligned}$$

or equivalently

$$\begin{aligned} & BD(CE^2 - AC^2 - AE^2 + 2AC \cdot AE) \\ = \; & CE(BD^2 - AB^2 - AD^2 + 2AB \cdot AD). \end{aligned} \tag{3}$$

Now use the law of cosines in ΔACE, and in ΔABD to obtain

$$\left.\begin{aligned} CE^2 - AC^2 - AE^2 &= -2ACAE\cos\varphi \\ BD^2 - AB^2 - AD^2 &= -2ABAD\cos\varphi. \end{aligned}\right\} \tag{4}$$

Combining (3) and (4) we see that we must show

$$BD \cdot AC \cdot AE(1 - \cos\varphi) = CE \cdot AB \cdot AD(1 - \cos\varphi).$$

As $1 - \cos\varphi \neq 0$, we find with (2) and (2′) that we must verify that

$$\frac{c\sin\varphi}{\sin(\beta+\gamma)} \cdot b \cdot \frac{b\sin\gamma}{\sin(\gamma+\varphi} = \frac{b\sin\varphi}{\sin(\gamma+\varphi)} \cdot c \cdot \frac{c\sin\beta}{\sin(\beta+\varphi)},$$

and as $b\sin\gamma = c\sin\beta$, this holds. ∎

13. [1994: 241] *Proposed by India.*

A natural number n is said to have the property P if, whenver n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$.

(a) Show that every prime number has property P.

(b) Show there are infinitely many composite numbers n that possess property P.

Solution by E.T.H. Wang, Sir Wilfrid Laurier University, Waterloo, Ontario.

(a) Suppose $n = p$ is a prime such that $a^p \equiv 1 \bmod p$. Since $a^p \equiv a \bmod p$ by Fermat's Little Theorem, we have $a \equiv 1 \bmod p$. Thus $a = kp+1$ for some integer k. Hence, by the Binomial Theorem $a^p - 1 = (kp+1)^p - 1 = (kp)^p + \binom{p}{1}(kp)^{p-1} + \cdots + \binom{p}{p-1}kp$. Since $\binom{p}{p-1} = p$ and $p \geq 2$, it follows that $p^2 \mid a^p - 1$.

(b) We show that all composite numbers of the form $n = 2p$, where p is an odd prime have property P. Suppose $2p \mid a^{2p} - 1$. Then $p \mid (a^2)^p - 1$

which, in view of (a), implies $p^2 \mid a^{2p}-1$. On the other hand, $2 \mid a^{2p}-1$ implies $2 \mid (a^p-1)(a^p+1)$. Since a^p-1 and a^p+1 have the same parity, they must both be even, and hence $4 \mid (a^p-1)(a^p+1)$. Since $\gcd(4,p^2)=1$, $4p^2 \mid a^{2p}-1$ follows.

Remarks. (1) The fact that all prime numbers satisfy property P is well known. In fact, using exactly the same argument as in the proof of (a) above, one can show easily that if $a^p \equiv b^p \bmod p$, then $a^p \equiv b^p \bmod p^2$. (See e.g. Ex. 13 on page 190 of *Elementary Number Theory and its Applications* by Kenneth H. Rosen, 3rd Edition).

(2) $n=4$ also has property P. For suppose that $4 \mid a^4-1$. Then $4 \mid (a^2-1)(a^2+1)$, which implies that a is odd. Since any odd square is congruent to 1 modulo 8, we have $8 \mid a^2-1$, which, together with $2 \mid a^2+1$, yields $16 \mid (a^2-1)(a^2+1)$.

(3) In view of (2) and our proof above, $n=8$ is the first natural number which need not possess property P. Indeed, it does not since $3^8-1=6560$ is divisible by 8 but not by 64.

(4) It might be of interest to characterize *all* natural numbers with property P.

We finish this number of the Corner with a solution to one of the IMO problems from the 35th IMO in Hong Kong.

2. [1994: 244]

ABC is an isosceles triangle with $AB=AC$. Suppose that

(i) M is the midoint of BC and D is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE=QF$.

Solution by D. J. Smeenk, Zaltbommel, the Netherlands.

($\Longrightarrow$). See Figure 1. Assume $OQ \perp EF$. We want to show $QE=QF$.

Note that quadrilateral $OBEQ$ is inscribed on the circle with diameter OE. Thus $\angle OEQ=\angle OBQ=\angle OQB=\frac{\alpha}{2}$. Also $OFCQ$ is cyclic and thus $\angle OFQ=\angle OCQ=\angle OAC=\frac{\alpha}{2}$. Together these give $\angle OEQ=\angle OFQ$ and $OQ \perp EF$, so $QE=QF$.

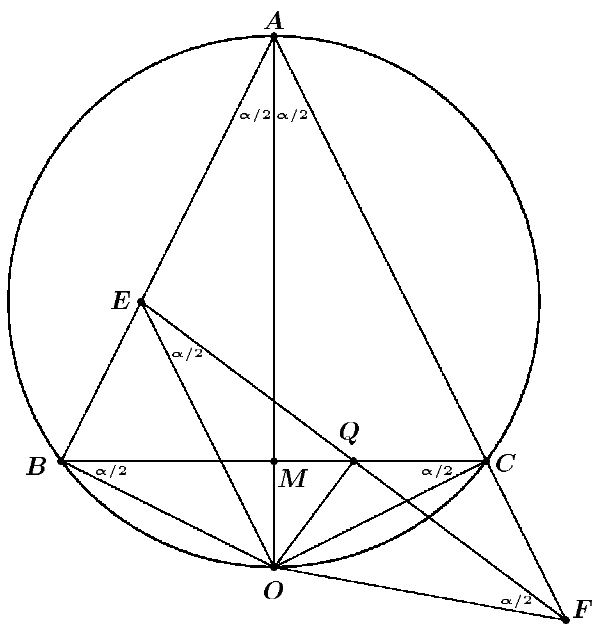


Figure 1.

($\Longleftarrow$). See Figure 2. Assume that $QE = QF$. We want to show that $OQ \perp EF$.

Let D lie on AC with QD parallel to AB. As $QE = QF$ we see that D is the midpoint of AF. As $AB = AC$ we have $DQ = DC$. (1)
Also,

$$QE = 2DQ = 2CD \tag{2}$$

$$BE = AB - AE. \tag{3}$$

From (2) and (3)

$$\begin{aligned} BE &= AC - 2CD = (AC - CD) - CD \\ &= AD - CD = DF - CD. \end{aligned}$$

Thus $BE = CF$. (4)

Draw OB, OE, OC, and OF.

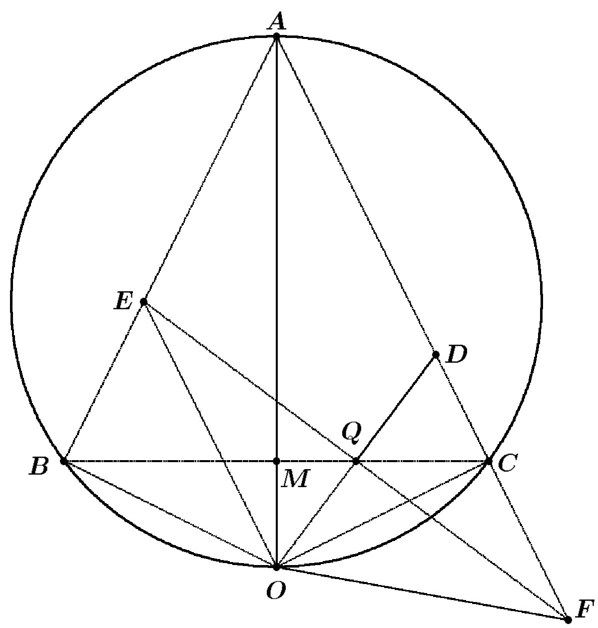


Figure 2.

Then

$$\Delta OBE \cong \Delta COF[BE = CF,\ OB = OC,\ \angle OBE = \angle OCF = n/2]. \quad (5)$$

Now this implies that $OE = OF$, $QE = QF$ giving $OQ \perp EF$.
Remark. This direction could be shortened by using the law of sines in triangles BQF and CQF, but the given solution is, I think, more elementary, and therefore more elegant.

That completes the material we have available for this number. The Olympiad Season is fast approaching. Please collect your contests and send them to me. Also send me your nice solutions to problems posed in the *Corner*.

THE ACADEMY CORNER

No. 2

This will appear in a future issue.