SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Let \(a, b, c\) be the sides, \(A, B, C\) the angles (measured in radians), and \(s\) the semi-perimeter of a triangle.

(i) Prove that

\[
\sum \frac{bc}{A(s-a)} \geq \frac{12s}{\pi},
\]

where the sums here and below are cyclic.

(ii) It follows easily from the proof of Crux 1611 (see [1992: 62] and the correction on [1993: 79]) that also

\[
\sum \frac{b+c}{A} \geq \frac{12s}{\pi}.
\]

Do the two summations above compare in general?

IV. Comment by Waldemar Pompe, student, University of Warsaw, Poland.

Here we give a short demonstration of the equality

\[
\sum \frac{bc}{s-a} = s + \frac{(4R+r)^2}{s},
\]

which has appeared on [1995: 55]. Since

\[
\sum a = 2s, \quad \sum bc = s^2 + r^2 + 4Rr, \quad abc = 4sRr,
\]

we get

\[
\sum (s-b)(s-c) = 3s^2 - 4s^2 + \sum bc = r^2 + 4Rr.
\]

Using the above equality we obtain

\[
\sum \frac{1}{s-a} = \frac{r^2 + 4Rr}{(s-a)(s-b)(s-c)} = \frac{r^2 + 4Rr}{sr^2} = \frac{r + 4R}{sr}.
\]

Therefore

\[
\sum \frac{bc}{s-a} = abc \sum \frac{1}{a(s-a)} = 4Rr \sum \left( \frac{1}{a} + \frac{1}{s-a} \right)
\]

\[
= 4Rr \left( \frac{s^2 + r^2 + 4Rr}{4sRr} + \frac{r + 4R}{sr} \right)
\]

\[
= s + \frac{r^2 + 4Rr}{s} + \frac{4Rr + 16R^2}{s} = s + \frac{(4R + r)^2}{s}.
\]
A (not quite as) short demonstration of (1) has also been sent in by Toshio Seimiya, Kawasaki, Japan.

2006. [1995: 20] Proposed by John Duncan, University of Arkansas, Fayetteville; Dan Velleman, Amherst College, Amherst, Massachusetts; and Stan Wagon, Macalester College, St. Paul, Minnesota.

Suppose we are given \( n \geq 3 \) disks, of radii \( a_1 \geq a_2 \geq \cdots \geq a_n \). We wish to place them in some order around an interior disk so that each given disk touches the interior disk and its two immediate neighbours. If the given disks are of widely different sizes (such as 100, 100, 100, 100, 1), we allow a disk to overlap other given disks that are not immediate neighbours. In what order should the given disks be arranged so as to maximize the radius of the interior disk? [Editor's note. Readers may assume that for any ordering of the given disks the configuration of the problem exists and that the radius of the interior disk is unique, though, as the proposers point out, this requires a proof (which they supply).]

Solution by the proposers.

Let \( r \) be the radius of the central disk. First look at a single tangent configuration made up of the central disk, and two disks of radii \( x \) and \( y \). The three centres form a triangle with sides \( r + x, r + y, \) and \( x + y \); let \( \theta = \theta_r(x, y) \) be the angle at the centre of the disk with radius \( r \). Applying the law of cosines to this angle and simplifying gives

\[
\theta_r(x, y) = \arccos \left( 1 - \frac{2xy}{r^2 + ry + rx + xy} \right). \]

A routine calculation shows that the mixed partial derivative \( \theta_{12} \) is given by

\[
\theta_{12} = \frac{-r^2}{2 \sqrt{xy(2r^2 + rx + ry)^{3/2}}};
\]

therefore \( \theta_{12} > 0 \) (for \( x, y > 0 \)). Integrating from \( a \) to \( a + s \) and \( b \) to \( b + t \) (where \( s, t > 0 \)) yields:

\[
\theta(a + s, b + t) + \theta(a, b) > \theta(a + s, b) + \theta(a, b + t). \tag{1}
\]

Note that equality occurs if and only if \( s = 0 \) or \( t = 0 \).

We may assume \( n \geq 4 \) and \( a_1 \geq a_2 \geq \cdots \geq a_n \); let \( D_i \) denote the disk with radius \( a_i \). And let \( S(r) \) denote \( \sum_{i=1}^{n} \theta_r(a_i, a_{i+1}) \), the total angle made by the configuration; when \( S < 2\pi \), then the disk of radius \( r \) is too large and the ring is not yet closed up.

THEOREM. The largest inner radius \( r \) occurs by placing \( D_2 \) and \( D_3 \) on either side of \( D_1 \), then \( D_4 \) alongside \( D_2, D_5 \) alongside \( D_3 \), and so on around the ring.
Proof. By induction on $n$. We actually prove the stronger assertion that for any radius $r$, $S(r)$ for the configuration of the statement of the theorem is not less than $S(r)$ for any other configuration. This suffices, for if $r$ is such that $S(r) = 2\pi$, then $S(r)$ for any other configuration is not greater than $2\pi$.

For $n = 4$ there are only three arrangements:

$$
\begin{array}{ccc}
D_1 & D_2 & D_3 \\
D_4 & D_5 & D_6 \\
\end{array}
$$

Arrangement I  
Arrangement II  
Arrangement III

for which (suppressing the subscripts $r$ in the $\theta$'s)

$$
S_I(r) = \theta(a_1, a_2) + \theta(a_2, a_4) + \theta(a_3, a_4) + \theta(a_1, a_3),
$$

$$
S_{II}(r) = \theta(a_1, a_2) + \theta(a_2, a_3) + \theta(a_3, a_4) + \theta(a_1, a_4),
$$

$$
S_{III}(r) = \theta(a_1, a_3) + \theta(a_2, a_3) + \theta(a_2, a_4) + \theta(a_1, a_4).
$$

By (1), $S_I(r) \geq S_{II}(r) \geq S_{III}(r)$, which proves our assertion for this case.

For the general case, suppose $E_2, \ldots, E_n$ is an arrangement of $D_2, \ldots, D_n$, with $b_i$ denoting the radius of $E_i$. Then $a_2 \geq b_2$ and we may also assume $a_3 \geq b_3$ (otherwise flip and relabel). Now it is sufficient, by the induction hypothesis, to show that

$$
\theta(a_1, a_2) + \theta(a_1, a_3) - \theta(a_2, a_3) \geq \theta(a_1, b_2) + \theta(a_1, b_3) - \theta(b_2, b_3).
$$

But (1) implies that

$$
\theta(a_1, a_2) + \theta(a_1, a_3) + \theta(b_2, b_3) \geq \theta(a_1, a_3) + \theta(a_1, b_2) + \theta(a_2, b_3),
$$

and this means it is sufficient to prove:

$$
\theta(a_1, a_3) + \theta(a_2, b_3) \geq \theta(a_1, b_3) + \theta(a_2, a_3).
$$

But this too is a consequence of (1). 

Note. A similar argument shows that the smallest inner radius occurs for the configuration: $\ldots D_5 D_{n-3} D_3 D_{n-1} D_1 D_n D_2 D_{n-2} D_4 \ldots$.

There were no other solutions sent in.

The problem was motivated by the special case of three pennies and two nickels, which was in (the late) Joe Konhauser's collection of problems.

Find two primes $p$ and $q$ such that, for all sufficiently large positive real numbers $r$, the interval $[r, 16r/13]$ contains an integer of the form

$$2^n, \quad 2^n p, \quad 2^n q, \quad \text{or} \quad 2^n pq$$

for some nonnegative integer $n$.

Solution by Peter Dukes, student, University of Victoria, B.C.

The prime numbers $p = 3$ and $q = 13$ solve the problem. To prove this, it suffices to find an increasing sequence of positive integers $s_1, s_2, \ldots$ such that each $s_i$ is of one of the forms stated in the problem, and $s_{i+1}/s_i \leq 16/13$ for all $i = 1, 2, \ldots$. Then, given any real number $r \geq s_1$, the integer $s_k$ (where $k = \min\{i \in \mathbb{Z}^+ : r \leq s_i\}$), is no more than $16r/13$. For if $s_k \notin [r, 16r/13]$, then $s_k/s_{k-1} \geq 16/13$, contrary to construction. Of course if $k = 1$ then $r = s_1$; so $s_1 \in [r, 16r/13]$. Now, consider the sequence

$$\{s_i\} : 24, 26, 32, 39, 48, 52, 64, 78, \ldots$$

$$\ldots, 3 \cdot 2^{m+3}, 13 \cdot 2^{m+1}, 2^{m+5}, 3 \cdot 13 \cdot 2^m, \ldots$$

It is a simple matter to verify that the ratio of consecutive terms $s_{i+1}/s_i$ does not exceed $16/13$ for any $i \in \mathbb{Z}^+$. Thus, for all $r \geq 24$, the interval $[r, 16r/13]$ contains an integer of the form $2^n, 2^n p, 2^n q, \text{or} \ 2^n pq$.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, British Columbia; ROBERT B. ISRAEL, University of British Columbia; DAVID E. MANES, State University of New York, Oneonta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One other solution contained a correct pair of primes, but the editor could not decipher the proof that they worked.

Most solvers simply found a single pair of primes, some of which allow the constant $16/13$ to be replaced by a slightly smaller constant. As Engelhaup, Hess and the proposer point out, the best you can do in this direction is that $16/13$ can be replaced by any real number greater than $\sqrt[3]{2}$. Hess says how: find primes "near" $2^{k_1+1/4}$ and $2^{k_2+1/2}$ for integers $k_1$ and $k_2$ (and he gives the example $p = 4871, q = 181$, which gets you within $0.00013$ of $\sqrt[3]{2}$). More precisely, if

$$p = 2^{k_1+1/4} \cdot t_1 \quad \text{and} \quad q = 2^{k_2+1/2} \cdot t_2$$

(1)

where $k_1, k_2$ are positive integers and $t_1, t_2$ are real numbers close to 1, then $2^{k_1+k_2} < p \cdot 2^{k_2} < q \cdot 2^{k_1} < pq$, and the ratios of consecutive terms are

$$\sqrt[3]{2} \cdot t_1, \quad \sqrt[3]{2} \cdot t_2, \quad \sqrt[3]{2} \cdot t_1, \quad \sqrt[3]{2} \cdot \frac{1}{t_1 t_2}.$$
all of which can be made arbitrarily close to $\sqrt{2}$. The reason (1) is possible is because (by the Prime Number Theorem) there is always a prime between $n$ and $nt$ for any given $t > 1$, if $n$ is big enough. By increasing the number of primes allowed, with an analogous change in the kind of integers you want the interval to contain, the proposer can get the interval down to $[r, \mu r]$ for any given $\mu > 1$.

Subsequent to submitting the problem the proposer was able to generalize it. He writes:

For given $\mu > 1$ we claim that there exists squarefree $D$ such that the consecutive integers $n_1, n_2, n_3, \ldots$ of the form $2^a d$ with $d|D$ satisfy $n_{i+1}/n_i \leq \mu$ for all $i$ sufficiently large. It suffices to find an infinite subsequence \{ $m_i$ \}$_{i=1}^\infty$ such that $m_{i+1}/m_i \leq \mu$ for all $i$ sufficiently large. Let $3 = p_1 < p_2 < p_3 < \ldots$ be the sequence of odd consecutive primes. For $n \geq 1$ let $p_{i+n}$ denote the smallest prime exceeding $q^n$ and let $p_{jn}$ denote the greatest prime less than $q^n+1$. Using the Prime Number Theorem it follows that $\lim_{n \to \infty} \frac{p_{i+n}}{p_i} = 1$, so there exists $n$ such that $p_i/p_{i+1} \leq \mu$ for $i = i_m, \ldots, j_n + 1$. Put $q_0 = 2^n$, $q_1 = p_{i+n}$, $q_2 = p_{i+n+1}$, \ldots, $q_s = p_{jn}$ and $D = \prod_{i=1}^s q_i$. Put $m_{i+a(r+1)} = 2^r q_a$, for $0 \leq a \leq s$, $r \geq 0$. Notice that $1 < m_{i+a}/m_i \leq \mu$ for every $i \geq 1$. Thus the claim holds with $D = \prod_{i=1}^s q_i$.

Remark. A similar result holds with 2 replaced by an arbitrary prime.


Let $I$ be the incentre of triangle $ABC$, and suppose there is a circle with centre $I$ which is tangent to each of the excircles of $\Delta ABC$. Prove that $ABC$ is equilateral.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

We solve the problem assuming that the circle with centre $I$ is tangent either externally or internally to all the excircles. Without this assumption the problem is not true; below we give a counterexample. (The triangle $ABC$ in the figure has sides 2, 9, 9, and of course is not equilateral.)

Assume first there is a circle $C$ with centre $I$ tangent externally to all the excircles [i.e., $C$ does not contain the excircles — Ed.]. Then according to Feuerbach's theorem the circle $C$ is the nine-point circle of $ABC$. Therefore $C$ is also tangent to the incircle of $ABC$, but since $I$ is the centre of $C$, the circle $C$ and the incircle of $ABC$ must coincide. It follows that the triangle $ABC$ is equilateral.

Now let $C$ be the circle with centre $I$ and tangent internally to all the excircles. Let $DE$ and $FG$ be the chords of $C$ containing the segments $AB$ and $AC$ respectively. Since $C$ and the incircle of $ABC$ are concentric, the lines $DE$ and $FG$ are symmetric to each other with respect to the line $AI$. Therefore, since the excircles lying opposite to $B$ and $C$ are uniquely determined by the chords $DE$, $FG$ and the circle $C$, they also have to be symmetric.
to each other with respect to the line \( AI \). Thus \( r_B = r_C \). Analogously we show that \( r_C = r_A \), which implies that \( ABC \) has to be equilateral, as we wished to show.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIOSHI SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution received.

Pompe was the only solver to find the counterexample. Everyone else either ignored the possibility, or (in a couple of cases at least) believed that it could not occur! Half the solvers considered both cases of tangency, that is, where the circle is tangent externally to all the excircles or internally to all of them, and the others considered only one.


Sarah got a good grade at school, so I gave her \( N \) two-dollar bills. Then, since Tim got a better grade, I gave him just enough five-dollar bills so that he got more money than Sarah. Finally, since Ursula got the best grade, I gave her just enough ten-dollar bills so that she got more money than Tim. What is the maximum amount of money that Ursula could have received? (This is a variation of problem 11 on the 1994 Alberta High School Mathematics Contest, First Part; see the January 1995 Skoliad Corner [1995: 6].)
Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.
The most Ursula could receive is $2N + 14$ dollars.
Sarah received $2N$ dollars and Tim received $2N + 1, 2N + 2, 2N + 3, 2N + 4$ or $2N + 5$ dollars depending on what residue class $N$ belongs to modulo 5. But since Tim gets $5$ bills his amount is divisible by $5$. Ursula will then receive either $5$ more than Tim (if Tim's amount is not divisible by $10$) or $10$ more than Tim (if Tim's amount is divisible by $10$). Clearly $2N + 5$ is odd, thus not divisible by $10$. So the maximum occurs when Tim receives $2N + 4$ dollars, which means $2N \equiv 1 \mod 5$, i.e. $N \equiv 3 \mod 5$, and that maximum is $2N + 14$. \[ \]

Also solved by HAYO AHLBURG, Benidorm, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; PAUL COLUCCI, student, University of Illinois; PETER DUKES, student, University of Victoria, B.C.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; J. K. FLOYD, Newnan, Georgia; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; RICHARD K. GUY, University of Calgary; DAVID HANKIN, John Dewey High School, Brooklyn, New York; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; J. A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JURGEN SEIFFERT, Berlin, Germany; and the proposer. Three other readers sent in solutions which the editor judges are not precise enough.

Here Ursula gets at most $\frac{1}{14}$ more than Sarah, which is $14/15$ of the obvious maximum difference $5.1/14.10 = 15.1$. How small can this ratio be, if we replace the denominations $2.5.10$ by three other positive integers? (Choosing $N, 2N - 1, 2N$ for large $N$ gets it down arbitrarily close to $3/4$.)

In triangle $ABC$ with $\angle C = 2\angle A$, line $CD$ is the internal angle bisector (with $D$ on $AB$). Let $S$ be the centre of the circle tangent to line $CA$ (produced beyond $A$) and externally to the circumcircles of triangles $ACD$ and $BCD$. Prove that $CS \perp AB$.

Composite of solutions by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore and Roland H. Eddy, Memorial University, St. John's, Newfoundland.

First, note that $\angle BCD = \angle CAD$, so that the line $BC$ is tangent to the circumcircle of $\triangle ACD$. Next, perform an inversion with respect to $C$.

The line $BC$ inverts into the line $B'C$, so that the circumcircle of $\triangle ACD$ inverts into the line $A'D'$, which is parallel to $B'C$. The circumcircle of $\triangle BCD$ inverts into the line $B'D'$. The line $AD$ inverts into the circumcircle of the quadrilateral $CA'D'B'$. Finally, the circle tangent to the
line $CA$, and externally tangent to the circumcircles of $\triangle ACD$ and $\triangle BCD$ inverts into a circle $\Gamma$, which is tangent to the line segments $A'C$, $A'D'$ and $B'D'$ as shown in the diagram.

Now, $\angle BCD = \angle ACD$, so that $\angle B'CD' = \angle A'C'D'$. Thus $B'D' = A'D'$. Since $A'D'$ and $B'C$ are parallel, we have $A'D' = B'D' = A'C$. Hence, the circumcentre of the quadrilateral $A'D'B'C$ lies on the angle bisectors of $\angle CA'D'$ and $\angle A'D'B'$. These two angle bisectors intersect at the centre of $\Gamma$. Thus, quadrilateral $A'D'B'C$ and $\Gamma$ are concentric. Let their common centre be $O$.

Thus, $C$, $O$ and $S'$ (the inverse of $S$) are collinear, and so $CS$ is perpendicular to $AB$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; KEE-WAI LAU, Hong Kong; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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2012. [1995: 52] Proposed by K. R. S. Sastry, Dodballapur, India. Prove that the number of primitive Pythagorean triangles (integer-sided right triangles with relatively prime sides) with fixed inradius is always a power of 2.

Solution by Carl Bosley, student, Topeka, Kansas

The formula for the inradius of a right triangle, $r = (a + b - c)/2$, where $a$, $b$ are the legs and $c$ is the hypotenuse, together with the formula $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$, for the sides of the triangle, where $m$ and $n$ are relatively prime and not both odd, gives $r = n(m - n)$.

Let $p$ be a prime which divides $r$. If $p = 2$, $p$ can divide $n$ but not $m - n$, since then $m$ and $n$ would either be both even or both odd. If $p$ is not 2, then $p$ must divide $n$ or $m - n$, but cannot divide both, since $m$ and $n$ would have a common factor. Since each prime $p$ other than 2 which divides $r$ divides $n$ or $m - n$, but not both, and each combination of choices produces a pair $m$, $n$ which generates a right triangle, so if $r$ has $k$ distinct prime factors greater than 2, there are $2^k$ primitive Pythagorean triangles with inradius $r$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; DAVID DOSTER, Choate Rose-
mary Hall, Wallingford, Connecticut; H. ENGELHAUP, Franz-Ludwig-Gymnasium, Bamberg, Germany; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, British Columbia; JAMSHID KHOIDI, New York, N. Y.; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; HEINZ-JURGEN SEIFFERT, Berlin, Germany; LAWRENCE SOMER, Catholic University of America, Washington, D. C.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There were seven incorrect solutions received, most of which did not handle the case \( p = 2 \) correctly.

Kholdi points out that the problem is solved on page 43 of Sierpiński's Pythagorean Triangles, published by the Graduate School of Science, Yeshiva University, 1962.

\[ 2013. \quad [1995: 52] \quad \text{Proposed by Waldemar Pompe, student, University of Warsaw, Poland.} \]

Given is a convex \( n \)-gon \( A_1A_2 \ldots A_n \) and a point \( P \) in its plane. Assume that the feet of the perpendiculars from \( P \) to the lines \( A_1A_2, A_2A_3, \ldots, A_nA_1 \) all lie on a circle with centre \( O \).

(a) Prove that if \( P \) belongs to the interior of the \( n \)-gon, so does \( O \).

(b) Is the converse to (a) true?

(c) Is (a) still valid for nonconvex \( n \)-gons?

Solution by Jerzy Bednarczuk, Warszawa, Poland.

(a) We show that (a) is true for a polygon that is not necessarily convex (so we will have treated (a) and (c) at the same time). Let us call the given circle \( C \). We first see that when \( P \) is inside the \( n \)-gon, then it must also be inside \( C \). The half-line on \( OP \) starting at \( P \) and going away from \( O \) intersects some side of the \( n \)-gon, say \( A_iA_{i+1} \), at a point \( S \). If \( P \) were exterior to \( C \), then the circle with diameter \( PS \) would lie in the exterior of \( C \). Since the latter circle is the locus of points \( B \) with \( \angle PBS = 90^\circ \), the foot of the perpendicular from \( P \) to \( A_iA_{i+1} \) would then lie outside of \( C \), contrary to the definition of \( C \). Also by definition, \( P \) cannot lie on \( C \) so we conclude that \( P \) must lie inside \( C \) as claimed. By way of contradiction we now assume that \( O \) lies outside of the given \( n \)-gon. Since \( P \) is inside, the segment \( OP \) would intersect some side of the \( n \)-gon, say \( A_jA_{j+1} \), at some point \( T \). Since \( P \) is also inside \( C \), the circle whose diameter is \( PT \) would be contained inside \( C \), which would force the foot of the perpendicular from \( P \) to \( A_jA_{j+1} \) to lie inside \( C \) contrary to its definition. We conclude that \( O \) lies inside the given \( n \)-gon as desired.

(b) No. We can find a counterexample for any \( n > 2 \)! First draw a circle \( C \) and then choose any point \( P \) lying OUTSIDE of this circle. Next take \( n \) points \( B_1, B_2, \ldots, B_n \) on the circle \( C \) and through each \( B_i \) draw the line perpendicular to the line \( PB_i \). Those lines will form an \( n \)-gon which will
contain $O$ if you choose each $B_i$ so that the half-plane determined by the line perpendicular to $PB_i$ will contain all other $B_j$ together with $O$. 

Also solved by the proposer.


(a) Show that the polynomial

$$2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$$

has $x + y + z$ as a factor.

(b) Is the remaining factor irreducible (over the complex numbers)?

1. Solution to (a) by Jayabrata Das, Calcutta, India.

Let $f(x, y, z) = 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$. If we can show that $f(x, y, z) = z$ when $x + y + z = 0$, we are done.

We know, for $x + y + z = 0$, that $x^3 + y^3 + z^3 = 3xyz$. Thus

$$x^7 + y^7 + z^7 = x^3y^4 + x^3z^4 + y^3z^4 + y^3x^4 + z^3y^4 + z^3x^4$$

so that

$$x^7 + y^7 + z^7 = 3xyz(x^4 + y^4 + z^4)$$

$$-x^3y^4 - x^3z^4 - y^3z^4 - y^3x^4 - z^3y^4 - z^3x^4$$

Therefore

$$f(x, y, z) = 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$$

$$= 6xyz(x^4 + y^4 + z^4)$$

$$-2(x^3y^4 + x^3z^4 + y^3z^4 + y^3x^4 + z^3y^4 + z^3x^4)$$

$$-7xyz(x^4 + y^4 + z^4)$$

$$= -xyz(x^4 + y^4 + z^4)$$

$$-2x^3y^3(x + y) - 2y^3z^3(y + z) - 2z^3x^3(z + x)$$

$$= -xyz(x^4 + y^4 + z^4) + 2x^3y^3z + 2xy^3z^2 + 2x^3yz^2$$

$$= -xyz(x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2)$$

$$= -xyz( (x^2 + y^2 + z^2)^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) ) .$$

Since $x^2 + y^2 + z^2 = -2(xy + yz + zx)$, we now have that

$$f(x, y, z) = -xyz(4(xy + yz + zx)^2 - 4(x^2y^2 + y^2z^2 + z^2x^2))$$

$$= -4xyz(2x^2yz + 2xy^2z + 2xyz^2)$$

$$= -8xyz(2xyz(x + y + z)) = 0 .$$


11. Solution to (a) by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.
Consider the sequence $a_n = x^n + y^n + z^n$. The characteristic equation with roots $x, y, z$, is
\[ a^3 - Aa^2 + Ba - C = 0, \]
where $A = x + y + z$, $B = xy + yz + zx$ and $C = xyz$.
The sequence $\{a_n\}$ follows the recurrence relation:
\[ a_{n+3} = Aa_{n+2} - B a_{n+1} + C a_n. \]
Now, we have
\[
\begin{align*}
a_0 & = x^0 + y^0 + z^0 = 3, \\
a_1 & = x^1 + y^1 + z^1 = A, \\
a_2 & = x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = A^2 - 2B
\end{align*}
\]
From the recurrence relation, we see:
\[
\begin{align*}
a_3 & = A a_2 - B a_1 + C a_0 \\
& = A^3 - 2AB - AB + 3C \\
& = A k_3 + 3C, \text{ where } k_3 \text{ is some term in } x, y \text{ and } z
\end{align*}
\]
Similarly
\[
\begin{align*}
a_4 & = A k_4 + 2B^2, \text{ where } k_4 \text{ is some term in } x, y \text{ and } z, \\
a_5 & = A k_5 - 5BC, \text{ where } k_5 \text{ is some term in } x, y \text{ and } z, \\
a_6 & = A k_6 - 2B^3 + 3C^2, \text{ where } k_6 \text{ is some term in } x, y \text{ and } z, \\
a_7 & = A k_7 + 3B^2 C, \text{ where } k_7 \text{ is some term in } x, y \text{ and } z.
\end{align*}
\]
Thus,
\[
\begin{align*}
2 (x^7 + y^7 + z^7) - 7xyz (x^4 + y^4 + z^4) \\
& = 2a_7 - 7C a_4 \\
& = a (A k_7 + 3B^2 C) - 7C (A k_4 + 2B^2) \\
& = A k
\end{align*}
\]
where $k$ is some term in $x, y$ and $z$; that is,
x + y + z divides $2 (x^7 + y^7 + z^7) - 7xyz (x^4 + y^4 + z^4)$. ■

Part (a) was also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, Grade 8 student, Upper Canada College, Toronto, Ontario; TIM CROSS, Wolverley High School, Kidderminster, U. K.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VACLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; J. A. MCCALLUM,
Medicine Hat, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution to part (b) was received.

Other solvers of part (a) made use of computer algebra, properties of roots, the substitution \(x = -y - z\), or direct division. Some solvers pointed out that computer algebra failed to factorize the remaining factor. The editor (Shawyer) tried using DERIVE on a MSDOS 486 66MHz computer. The factorisation stopped with no factors after 155 seconds. MAPLE also failed to find any factors. McCallum commented "An asterisk on a question of Klamkin's is equivalent to a DO NOT ENTER sign!"

2015. [1995: 53 and 129 (Corrected)] Proposed by Shi-Chang Chi and Ji Chen, Ningbo University, China.

Prove that

\[
\left( \sin(A) + \sin(B) + \sin(C) \right) \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) \geq \frac{27\sqrt{3}}{2\pi},
\]

where \(A, B, C\) are the angles (in radians) of a triangle.

1. Solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia, Canada.

If \(A = B = C = \pi/3\), equality obtains. It then suffices to show that each factor has an absolute minimum at the point. Note that \(C = \pi - (A + B)\).

Let \(S = \{(A, B) : A > 0, B > 0, A + B < \pi\}\).

Let \(f(A, B) = \frac{1}{A} + \frac{1}{B} + \frac{1}{\pi - (A + B)}\). Then \(f\) is unbounded on \(S\). So, if there is a unique critical point for \(f\) on \(S\), it must be an absolute minimum.

Now, \(F_A(A, B) = -\frac{1}{A^2} + \frac{1}{(\pi - (A + B))^2} = 0\) implies that \(\pi - (A + B) = A\). Similarly, \(F_B(A, B) = 0\) implies that \(\pi - (A + B) = B\), and so that \(A = B\). Hence \(A = B = \pi - (A + B) = \frac{\pi}{3}\).

Let \(g(A, B) = \sin(A) + \sin(B) + \sin(\pi - (A + B)) = \sin(A) + \sin(B) + \sin(A + B)\). We now obtain that \(0 = g_A(A, B) = \cos(B) + \cos(A + B), 0 = g_B(A, B) = \cos(A) + \cos(A + B)\), so that \(\cos(A) = \cos(B)\). Since no two distinct angles in \((0, \pi)\) have equal cosines, we have that \(A = B\).

Then \(0 = \cos(A) + \cos(2A) = 2\cos^2(A) + \cos(A) - 1 = 2\cos(A) - 1\) \((\cos(A) + 1)\). Since \(\cos(A)\) cannot have the value \(-1\), it must then have value \(\frac{1}{2}\), and so we have \(A = B = C = \frac{\pi}{3}\). \(\blacksquare\)
11. Solution by the proposers.

Let $y(x) = x^{-1/3} \cos(x)$ for $0 < x \leq \frac{\pi}{2}$. Differentiating twice yields

$$x^{7/3} \sec(x), y''(x) = \frac{2x \tan(x)}{3} + \frac{4}{9} - x^2$$

$$> \frac{2x}{3} \left( x + \frac{x^3}{3} \right) + \frac{4}{9} - x^2$$

$$= \frac{2}{9} \left( x^2 - \frac{3}{4} \right)^2 + \frac{23}{72} > 0.$$  

By the AM–GM inequality and the Jensen inequality, we have

$$\sum \sin(A) \times \sum \frac{1}{A} \geq 4 \prod \cos(A/2) \times \sum \frac{1}{A}$$

$$\geq \frac{6 \prod \cos(A/2)}{\prod \left( \frac{A}{2} \right)^{1/3}}$$

$$\geq 6 \left( \cos \left( \frac{\pi}{6} \right) \right)^3 = \frac{27}{2\pi}.$$

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria; and BOB PRIELIPP, University of Wisconsin–Oshkosh. Their solutions were based on known geometric inequalities. Also some readers wrote in after the initial publication of the problem, pointing out that the original result could not be true, and suggesting possible corrections. Thank you.