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1. An improvement from a rectangular lattice.

In a recent article in this journal [1983: 128], we showed how to use a dissection of a triangle to obtain the following result:

1.1. **THEOREM.** If \( m > 1 \) and \( n > 1 \) are integers, and if \( p, q > 0 \) with \( p + q < 1 \), then

\[
(1 - p^m)^n + (1 - q^n)^m > 1. \tag{1.11}
\]

The proof used only Boolean arguments, such as:

1.2. If two events \( A \) and \( B \) are mutually exclusive, then \( \Pr(A) + \Pr(B) \leq 1 \), with equality if and only if \( A \lor B \) is exhaustive.

Using an idea of the editor, it is possible to sharpen (1.11). Arrange \( mn \) beads in a rectangular array of \( m \) columns and \( n \) rows, with a dogleg of \( \max\{m,n\} \) beads stretching from the top left to the bottom right bead. (Figure 1.3 illustrates the case \( m = 5, n = 7 \).) Color each bead randomly with exactly one of three distinct colors: using red with probability \( p > 0 \), using green with probability \( q > 0 \), or using blue with probability \( r > 0 \). The following three events are mutually exclusive, and their union is exhaustive if and only if \( m = n = 1 \):

1.31. At least one row is entirely red.
1.32. At least one column is entirely green.
1.33. The dogleg is entirely blue.

It is easy to see that \( \Pr(1.31) \) is the complement \( 1 - (1 - p^m)^n \) of the probability \( (1 - p^m)^n \) that none of the rows be entirely red. Similarly, \( \Pr(1.32) = 1 - (1 - q^n)^m \), and it is clear that \( \Pr(1.33) = r^{\max\{m,n\}} \). Assertion 1.2 can be extended to three mutually exclusive events, and gives

\[
\{1 - (1 - p^m)^n\} + \{1 - (1 - q^n)^m\} + r^{\max\{m,n\}} \leq 1.
\]

Since \( r = 1-p-q \), we have the following result:

1.4. **THEOREM.** If \( m \geq 1 \) and \( n \geq 1 \) are integers, and if \( p, q > 0 \) with \( p + q \leq 1 \), then

\[
(1 - p^m)^n + (1 - q^n)^m \geq 1 + (1-p-q)^{\max\{m,n\}},
\]

with equality if and only if \( m = n = 1 \).
Theorem 1.4 can be extended in several ways. One extension, the proof of which requires calculus, is to let \( m \geq 1 \) and \( n \geq 1 \) be real numbers. Another kind of extension results if, for some given integer \( k \geq 2 \), we replace \( m, n \) by \( k \) positive integers \( n_i \), and \( p, q \) by \( k \) positive real numbers \( p_i, q_i \). It will be helpful to first state and give a (Boolean) proof of this result for \( k = 3 \), after which we will state without proof a far-reaching generalization which encompasses both types of extensions.

1.5. **THEOREM.** If \( n_1, n_2, n_3 \geq 1 \) are integers, and if \( p_1, p_2, p_3 > 0 \) are real numbers with \( p_1 + p_2 + p_3 \leq 1 \), then

\[
(1.51) \quad \sum_{i=1}^{3} (1 - p_i l_i)^{n_i} + 2 \geq (1 - p_1 l_1 p_2 l_2 - p_3 l_3)^{\max\{n_1, n_2, n_3\}},
\]

where \( l_1 = n_2 n_3 \), \( l_2 = n_3 n_1 \), \( l_3 = n_1 n_2 \), with equality if and only if \( n_1 = n_2 = n_3 = 1 \).

*Proof.* Arrange \( n_1 n_2 n_3 \) beads in an \( n_1 \times n_2 \times n_3 \) rectangular parallelepiped with \( n_1 \) planes of dimensions \( n_2 \times n_3 \), \( n_2 \) planes of dimensions \( n_3 \times n_1 \), \( n_3 \) planes of dimensions \( n_1 \times n_2 \), and a "fractured" dogleg of \( \max \{n_1, n_2, n_3\} \) beads extending from one corner bead in a direction with equal direction cosines \( 1/\sqrt{3} \) until it reaches a face, proceeding then in that face in a direction with equal direction cosines \( 1/\sqrt{2} \) until it reaches an edge, and then remaining in that edge until it reaches the bead in the corner opposite the starting corner. (Of course, there is no "fracture" if two of the \( n_i \) are equal, and the "leg" is straight if all three of the \( n_i \) are equal.) Color each bead randomly with exactly one of four distinct colors \( 1, 2, 3, 4 \): using color \( i \) with probability \( p_i > 0 \) for \( i = 1, 2, 3 \), and using color \( 4 \) with probability \( r \geq 0 \). The following four events are mutually exclusive, and their union is exhaustive if and only if \( n_1 = n_2 = n_3 = 1 \):

1.5.2. At least one of the \( n_1 \) planes of dimensions \( n_2 \times n_3 \) is entirely of color \( 1 \).

1.5.3. At least one of the \( n_2 \) planes of dimensions \( n_3 \times n_1 \) is entirely of color \( 2 \).

1.5.4. At least one of the \( n_3 \) planes of dimensions \( n_1 \times n_2 \) is entirely of color \( 3 \).

1.5.5. The dogleg is entirely of color \( 4 \).

It is easy to see that \( \Pr(1.52) \) is the complement \( 1 - (1 - p_1 l_1)^{n_1} \) of the probability \( (1 - p_1 l_1)^{n_1} \) that none of the \( n_1 \) planes of dimensions \( n_2 \times n_3 \) be entirely of color \( 1 \), with similar results for \( \Pr(1.53) \) and \( \Pr(1.54) \). Obviously \( \Pr(1.55) = r^{\max\{n_1, n_2, n_3\}} \), so that

\[
\sum_{i=1}^{3} \left( 1 - (1 - p_i l_i)^{n_i} \right) + r^{\max\{n_1, n_2, n_3\}} \leq 1,
\]
and (1.51) follows from the fact that $r = 1 - p_1 - p_2 - p_3$; the inequality degenerates to equality if and only if $n_1 = n_2 = n_3 = 1$. \(\blacksquare\)

Finally, we state the generalization announced earlier.

1.6. **THEOREM.** Let $k \geq 2$ be a given positive integer, and suppose that, for $i = 1, 2, \ldots, k$, we have $k$ real numbers $n_i \geq 1$ and $k$ real numbers $p_i > 0$ with $S = \sum p_i \leq 1$. If $N = \prod n_i$ and $l_i = N/n_i$, then

$$
\sum_{i=1}^{k} (1 - p_i l_i)^{n_i} \geq k - 1 + (1 - S)^{\max\{n_i\}},
$$

with equality if and only if all the $n_i = 1$.

The proof of this theorem in full generality requires calculus.

2. **Inequalities from a polygonal lattice.**

In the Euclidean plane, there are three regular tessellations: the rectangular one $T_1$ used in Section 1, the tessellation $T_2$ with equilateral triangles, and the dual tessellation $T_3$ with regular hexagons. In this section, inequalities are derived by using $T_2$. See Figure 2.1, in which a regular hexagon is subdivided into equilateral triangles by 45 lines, 15 (equally spaced) in each of the three prime directions. A collection of chords adjacent to and symmetrically situated with respect to a given diagonal is called a set of central lines. Thus, a set of central lines consists of an odd number, $2k+1$, of chords, where $0 \leq k \leq 7$.

Color each of the 169 vertices in or on the hexagon at random with exactly one of four distinct colors 1, 2, 3, 4: using colors 1, 2, 3 with probabilities $p, q, r > 0$, respectively, and using color 4 with probability $s > 0$. A line will be said to be of color $i$ if all the vertices on that line are of color $i$. For each $k$, $0 \leq k \leq 7$, the events 2.11k-2.14 given below are mutually exclusive and their union is not exhaustive.

2.11k. At least one of the $2k+1$ central vertical lines is of color 1.

2.12k. At least one of the 15-2k central NE-SW lines is of color 2.

2.13k. At least one of the 15-2k central NW-SE lines is of color 3.

2.14. Two of the main diagonals are of color 4.

For $k = 0$ and $k = 7$, nothing new emerges. For $k = 1$ (the case illustrated in Figure 2.1), the respective probabilities are:

$$
\Pr(2.111) = 1 - (1 - p^{14})^2(1 - p^{15}), \\
\Pr(2.121) = 1 - (1 - q^9)^2(1 - q^{10})^2(1 - q^{14})^2(1 - q^{15}), \\
\Pr(2.131) = 1 - (1 - r^9)^2(1 - r^{10})^2(1 - r^{14})^2(1 - r^{15}), \\
\Pr(2.14) = s^{29}.
$$
Generalizing from $15$ to arbitrary odd integer $l \geq 5$, and from $k=1$ to arbitrary integer $k$, $0 < k < \frac{1}{2}(l-1)$, the usual (Boolean) argument gives the following result:

2.2. THEOREM. Let $l \geq 5$ be an odd integer, and let $p,q,r > 0$, $s \geq 0$, with $p+q+r+s = 1$. Then, if $k$ is an integer such that $0 < k < \frac{1}{2}(l-1)$, the following inequality holds:
\( (1 - p^1)(1 - p^{l-1})^2 \ldots (1 - p^{l-k})^2 \)

\( + (1 - q^l)(1 - q^{l-1})^2 \ldots (1 - q^2(l+1)) \)

\( + (1 - r^l)(1 - r^{l-1})^2 \ldots (1 - r^2(l+1)) \)

> 2 + s^{2l-1}.

In particular,

\( (1 - p^l)(1 - p^{l-1})^2 \)

\( + (1 - q^l)(1 - q^{l-1})^2 \ldots (1 - q^2(l+3)) \)

\( + (1 - r^l)(1 - r^{l-1})^2 \ldots (1 - r^2(l+3)) \)

> 2 + s^{2l-1};

and, if \( l \equiv 1 \pmod{4} \) and \( a = \frac{1}{2}(l-1) \),

\( (2.21a) \)

\( f(p) + f(q) + f(r) > 2 + s^{2l-1} \).

where \( f(p) = (1 - p^l)(1 - p^{l-1})^2 \ldots (1 - p^{\frac{1}{2}(3l+1)}) \).

2.22. COROLLARY. If \( l \equiv 1 \pmod{4} \) and \( p, q, r, s \) are as above, then

\( (2.23) \)

\( (1 - p^l)^{\frac{1}{2}(l+1)} + (1 - q^l)^{\frac{1}{2}(l+1)} + (1 - r^l)^{\frac{1}{2}(l+1)} > 2 + s^{2l-1} \).

Proof. This corollary is just a weaker form of \( (2.21a) \), since

\[ 1 - p^l \geq 1 - p^{l-1} \geq \ldots \geq 1 - p^{\frac{1}{2}(3l+1)}, \]

and the number of factors in each term of \( (2.23) \) is the same as in \( f(p) \). \( \Box \)

The assertion

\( (2.24) \)

\( (1 - p^l)^l + (1 - q^l)^l + (1 - r^l)^l > 2 \)

is valid and is stronger than \( (2.23) \). But to obtain it a new type of diagram must be used. This is discussed in the next section.

3. Special results obtainable from square arrays.

In this section, straightforward applications of the formula for the sum of the probabilities of mutually exclusive events are used to obtain new inequalities.

3.1. THEOREM. If \( p^i > 0 \) for \( i = 1, 2, 3, 4 \) and \( \sum p^i \leq 1 \), then

\( (3.2) \)

\( (1 - p^1)^2 + (1 - p^2)^2 + (1 - p^3)^2 > 2, \)

\( (3.3) \)

\( (1 - p^1)^3 + (1 - p^2)^3 + (1 - p^3)^3 + (1 - p^4)^3 > 3, \)

\( (3.4) \)

\( (1 - p^1)^4 + (1 - p^2)^4 + (1 - p^3)^4 > 2, \)

\( (3.5) \)

\( (1 - p^1)^5 + (1 - p^2)^5 + (1 - p^3)^5 + (1 - p^4)^5 > 3. \)
Proof of (3.2). Take four beads (numbered 1,2,3,4) and color them (with probabilities $p_1, p_2, p_3$) with colors $c_1, c_2, c_3$. The following events are mutually exclusive:

<p>| At least one | At least one | At least one |</p>
<table>
<thead>
<tr>
<th>of the pairs</th>
<th>of the pairs</th>
<th>of the pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2;</td>
<td>1, 3;</td>
<td>1, 4;</td>
</tr>
<tr>
<td>3, 4</td>
<td>2, 4</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

is monochromatic of color $c_1$. is monochromatic of color $c_2$. is monochromatic of color $c_3$.

The respective probabilities are

$$1 - (1 - p_1^2)^2, \quad 1 - (1 - p_2^2)^2, \quad 1 - (1 - p_3^2)^2,$$

and assertion (3.2) follows.

Proof of (3.3). Take 9 beads, four colors, and events summarized by the arrays:

| 1, 2, 3; | 1, 4, 7; | 1, 5, 9; | 1, 6, 8; |
| 4, 5, 6; | 2, 5, 8; | 2, 6, 7; | 2, 4, 9; |
| 7, 8, 9; | 3, 6, 9; | 3, 4, 8; | 3, 5, 7; |

The patterns for (3.4) and (3.5) should now be clear. In fact, the idea is effective for four terms (like (3.5)) when the exponents are odd, and for three terms (like (3.4)) when the exponents are even. To follow the idea, note three things: (i) the rows of the second array are the columns of the first array; (ii) the rows of the third array are the (broken) positive diagonals of the first array; (iii) the rows of the fourth array are the (broken) negative diagonals of the first array. □

The results that this method can deliver are limited. The inequality

$$1 - (1 - p_1^2)^2 + (1 - p_2^2)^2 + (1 - p_3^2)^2 + (1 - p_4^2)^2 > 3$$

is true, but not obtainable from a picture with beads. Here is the impossibility argument. The terms $p_i^2$ imply that the sets being colored must have two beads each. The terms $(1 - p_i^2)^2$ imply that there are two sets of two beads each. But a total of four beads can be partitioned into two such sets in only three ways.

4. Inequalities obtainable from a (nearly) regular octagon.

The next theorem is proved from Figure 4.1.

4.2. THEOREM. If $p_i > 0$ for $i = 1, 2, \ldots, 5$ and $\sum p_i = 1$, then

$$1 - p_1^3 + (1 - p_2^3)^3 + (1 - p_3^3)^2 (1 - p_4^3) + (1 - p_4^3)^2 (1 - p_5^3) > 3 + p_5^3.$$
Proof. There are 21 intersection points marked with open or solid circles. Place beads at the 21 points, and color the beads (at random) with colors \( \sigma_i \) with respective probabilities \( p_i \) (1 \( \leq i \leq 5 \)). The reader is invited to invent five mutually exclusive events and thus derive (4.3). The fifth event, with probability \( p_7 \), is suggested by the seven solidly marked intersection points. \( \square \)

In an analogous fashion, Figure 4.4 leads to the next theorem.

**Theorem 4.5.** If \( p_i > 0 \) for \( i = 1, 2, \ldots, 5 \) and \( \sum p_i = 1 \), then

\[
(1 - p_7^4)(1 - p_9^4) + (1 - p_7^2)(1 - p_9^2) + (1 - p_5^8)(1 - p_3^2)(1 - p_3^2) + (1 - p_5^2)(1 - p_4^2)(1 - p_4^2) > 3 + p_5^{13},
\]

This result (4.6) is not a corollary of either of the inequalities \( \sum (1 - p_5^5) > 3 \), \( \sum (1 - p_5^7) > 3 \) of Section 3.

5. Conclusion.

In this set of two articles (the first appeared in [1983: 128]), Boolean arguments are used to derive some inequalities in which the parameters \( m, n, l_i, n_i \) are integers. No essentially analytical arguments are used. It is interesting that the same inequalities are valid when the parameters are real. This extension will be carried out in a more technical article; in it several other sharp inequalities are obtained. The article will appear in 1985 in *J. Math. Anal. & Appl.*

Note that many corollaries of the theorems of this article can be written down. One of them is:

5.1. **Corollary to (3.2).** If \( a, b, c > 0 \), then

\[
((a+b+c)^2 - a^2)^2 + ((a+b+c)^2 - b^2)^2 + ((a+b+c)^2 - c^2)^2 > 2(a+b+c)^4.
\]

**Proof.** Set \( p_1 = a/(a+b+c) \), \( p_2 = b/(a+b+c) \), \( p_3 = c/(a+b+c) \) in (3.2). \( \square \)

The symbol manipulation required to prove (5.2) directly is rather tedious.

10 Phillips Rd., Palo Alto, California 94303.
FOURTH-ORDER ADDITIVE DIGITAL BRACELETS IN BASE FIVE

CHARLES W. TRIGG

A bracelet is defined [1] as "one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Hence a bracelet may be regenerated by starting at any arbitrary position and applying the generating law." The distance between terms may be measured in degrees or steps. A bracelet may be cut at any arbitrary point for straight-line representation without loss of any properties.

The generating law $u_{n+2} = u_{n+1} + u_n$, where each sum is reduced modulo $b$ (the base of the system of numeration), produces a second-order bracelet. Thus, in base five the starting pair $(1, 3)$ leads to the four-digit bracelet shown in the adjoining figure, which in straight-line representation is

$$B: 1 \ 3 \ 4 \ 2' \ 1 \ 3.$$ 

In what follows, the degree measures of rotations and all numbers clearly identifiable from the context as magnitudes of sets are given in decimal notation (when they are not spelled out). But all sums and products of bracelet elements are reduced modulo five. Two digits with a sum of zero (modulo five) are called complementary.

Fourth-order additive bracelets.

The recursive formula

$$u_{n+4} = u_{n+3} + u_{n+2} + u_{n+1} + u_n$$

produces fourth-order additive digital bracelets when each sum is reduced modulo $b$. In base five, each of the 625 possible digit quartets appears in either the all-zero $A$ or in one of the 312-digit bracelets $H$ and $J$ shown in Table 1. Bracelet $H$ may be called the tetranacci bracelet, since it consists of the units' digits of the tetranacci sequence [2] in base five. The number of digits in this bracelet is $312_{\text{ten}} = 2222_{\text{five}}$.

In both $H$ and $J$, the digits $90^\circ$ apart are either all zeros or they form a cyclic permutation of $B$. It follows that diametrically opposite elements are complementary, and that the sum of the digits in each bracelet is zero. Furthermore, multiplication of either bracelet by 1, 3, 4, or 2 will rotate the bracelet through $0^\circ, 90^\circ, 180^\circ$, or $270^\circ$, respectively.
Table 1: Fourth-order bracelets

\[ A: \quad 0000'0000 \]
\[ H: \quad 0001 \quad 1243 \quad 04133 \]
\[ \quad 1302 \quad 1143 \quad 42321 \]
\[ \quad 3403 \quad 0200 \quad 24124 \]
\[ \quad 1303 \quad 2331 \quad 41404 \]
\[ \quad 4200 \quad 1343 \quad 11440 \]
\[ \quad 4201 \quad 2031 \quad 10023 \]
\[ \quad 0003 \quad 3124 \quad 02344 \]
\[ \quad 3401 \quad 3324 \quad 21413 \]
\[ \quad 4204 \quad 0100 \quad 12312 \]
\[ \quad 3404 \quad 1443 \quad 23202 \]
\[ \quad 2100 \quad 3424 \quad 33220 \]
\[ \quad 2103 \quad 1043 \quad 30014 \]
\[ \quad 0004 \quad 4312 \quad 01422 \]
\[ \quad 4203 \quad 4412 \quad 13234 \]
\[ \quad 2102 \quad 0300 \quad 31431 \]
\[ \quad 4202 \quad 3224 \quad 14101 \]
\[ \quad 1300 \quad 4212 \quad 44110 \]
\[ \quad 1304 \quad 3024 \quad 40032 \]
\[ \quad 0002 \quad 2431 \quad 03211 \]
\[ \quad 2104 \quad 2231 \quad 34142 \]
\[ \quad 1301 \quad 0400 \quad 43243 \]
\[ \quad 2101 \quad 4112 \quad 32303 \]
\[ \quad 3400 \quad 2131 \quad 22330 \]
\[ \quad 3402 \quad 4012 \quad 20041'0001 \]
\[ J: \quad 1111 \quad 4230 \quad 44143 \]
\[ \quad 2044 \quad 0313 \quad 24040 \]
\[ \quad 3244 \quad 3344 \quad 40201 \]
\[ \quad 3100 \quad 4043 \quad 13133 \]
\[ \quad 0233 \quad 3102 \quad 14241 \]
\[ \quad 1344 \quad 2332 \quad 03334 \]
\[ \quad 3333 \quad 2140 \quad 22324 \]
\[ \quad 1022 \quad 0434 \quad 12020 \]
\[ \quad 4122 \quad 4422 \quad 20103 \]
\[ \quad 4300 \quad 2024 \quad 34344 \]
\[ \quad 0144 \quad 4301 \quad 32123 \]
\[ \quad 3422 \quad 1441 \quad 04442 \]
\[ \quad 4444 \quad 1320 \quad 11412 \]
\[ \quad 3011 \quad 0242 \quad 31010 \]
\[ \quad 2311 \quad 2211 \quad 10304 \]
\[ \quad 2400 \quad 1012 \quad 42422 \]
\[ \quad 0322 \quad 2403 \quad 41314 \]
\[ \quad 4211 \quad 3223 \quad 02221 \]
\[ \quad 0002 \quad 2431 \quad 03211 \]
\[ \quad 2222 \quad 3410 \quad 33231 \]
\[ \quad 2104 \quad 2231 \quad 34142 \]
\[ \quad 4033 \quad 0121 \quad 43030 \]
\[ \quad 1433 \quad 1133 \quad 30402 \]
\[ \quad 1200 \quad 3031 \quad 21211 \]
\[ \quad 0411 \quad 1204 \quad 23432 \]
\[ \quad 3402 \quad 4012 \quad 20041'0001 \]
\[ \quad 2133 \quad 4114 \quad 01113'1111 \]

As written, in rows of thirteen elements, the sum of the digits in every column of the two bracelets is zero. In the successive 78-digit quadrants of \( H \), the digit sums are 4, 2, 1, and 3. In \( J \), the successive quadrant sums are 2, 1, 3, and 4. Two more appearances of the ubiquitous \( B \).

In Table 2 are recorded the frequencies of occurrence of the five digits in the successive four quadrants of \( H \) and \( J \). In each case, like frequencies of non-zero digits occur on diagonals slanting down toward the right.

In Table 3, the digits of each of \( H \) and \( J \) are consecutively arranged in thirteen 24-digit rows. There the sum of the digits in each column is zero.
Table 2: Frequency of digits in fourth-order bracelets

<table>
<thead>
<tr>
<th>Quadrants</th>
<th>Digits</th>
<th>Quadrants</th>
<th>Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1 3 4 2</td>
<td></td>
<td>0 1 3 4 2</td>
</tr>
<tr>
<td>$H_1$</td>
<td>18 17 16 15 12</td>
<td>$J_1$</td>
<td>13 14 21 20 10</td>
</tr>
<tr>
<td>$H_2$</td>
<td>18 12 17 16 15</td>
<td>$J_2$</td>
<td>13 10 14 21 20</td>
</tr>
<tr>
<td>$H_3$</td>
<td>18 15 12 17 16</td>
<td>$J_3$</td>
<td>13 20 10 14 21</td>
</tr>
<tr>
<td>$H_4$</td>
<td>18 16 15 12 17</td>
<td>$J_4$</td>
<td>13 21 20 10 14</td>
</tr>
</tbody>
</table>

It follows that the sum of the digits at the vertices of any regular thirteen-gon inscribed in $H$ or $J$ will be zero.

The persistent $B$ appears in the fifth row of $H$, and diagonally downward to the right in alternate columns of $J$ in the form of triads of like digits interwoven with quartets of the same digits. The fifth row of $H$ also contains two consecutive sets of four consecutive digits in increasing order of magnitude; and the twelfth row contains the five consecutive digits in decreasing order of magnitude.

The recordings of the two bracelets in three rows of 104 elements start out as

\[
H: \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 4 \quad 3 \quad 0 \quad 4 \quad 1 \quad 3 \quad 3 \quad \ldots
\]
\[
4 \quad 2 \quad 0 \quad 4 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2 \quad \ldots
\]
\[
1 \quad 3 \quad 0 \quad 0 \quad 4 \quad 2 \quad 1 \quad 2 \quad 4 \quad 4 \quad 1 \quad 1 \quad 0 \quad \ldots
\]

and

\[
J: \quad 1 \quad 1 \quad 1 \quad 1 \quad 4 \quad 2 \quad 3 \quad 0 \quad 4 \quad 4 \quad 1 \quad 4 \quad 3 \quad \ldots
\]
\[
4 \quad 1 \quad 2 \quad 2 \quad 4 \quad 2 \quad 2 \quad 2 \quad 0 \quad 1 \quad 0 \quad 3 \quad \ldots
\]
\[
0 \quad 3 \quad 2 \quad 2 \quad 2 \quad 4 \quad 0 \quad 3 \quad 4 \quad 1 \quad 3 \quad 1 \quad 4 \quad \ldots
\]

The sum of the digits in each of the columns above is zero. It follows that the sum of the digits at the vertices of any equilateral triangle inscribed in $H$ or $J$ will be zero. Indeed, the vertex digit sum will vanish for any inscribed regular $n$-gon where $n$ is any factor of $312 = 2^3 \cdot 3 \cdot 13$.

Since all possible digit quartets appear in $A$, $H$, and $J$, in any clockwise matching of two of these bracelets, the sums of the corresponding digits will form a bracelet wherein the law of formation holds, so it must be a repeated $A$ or some orientation of $H$ or $J$. For example, an $H$ (or $J$) added to another $H$ (or $J$) that has been rotated through $180^\circ$ forms a bracelet of zeros. The operation $H + J$, with the first digit of $J$ matched with the third digit of $H$ (from Table 1), then with the fourth digit, and so on, until it has been matched with every digit of $H$, produces a bracelet of 312 bracelets beginning, in order, with

\[
H \ J \ J \ H \ H \ J \ J \ H \ J \ J \ J \ J \ \ldots
\]
Table 3: Rearranged fourth-order bracelets

| H: | 0 0 0 1 1 2 4 3 0 4 1 3 3 1 3 0 2 1 1 4 3 4 2 3 |
|    | 2 1 3 4 0 3 0 2 0 0 2 4 1 2 4 1 3 0 3 2 3 3 1 4 |
|    | 1 4 0 4 4 2 0 0 1 3 4 3 1 4 4 0 4 2 0 1 2 0 3 |
|    | 1 1 0 0 2 3 0 0 0 3 3 1 2 4 0 2 3 4 3 4 0 1 3 |
|    | 3 2 4 2 1 4 1 3 4 0 4 0 1 0 0 1 2 3 1 2 3 4 0 |
|    | 4 1 4 4 3 2 3 2 0 2 2 1 0 0 3 4 4 3 2 2 0 2 |
|    | 1 0 3 1 0 4 3 3 0 0 1 4 0 0 0 4 4 3 1 2 0 1 4 2 |
|    | 2 4 2 0 3 4 4 1 2 1 3 2 3 4 2 1 0 2 0 3 0 3 1 1 |
|    | 4 3 1 4 2 0 2 3 2 2 4 1 4 1 0 1 1 3 0 0 4 2 1 2 |
|    | 4 4 1 1 0 1 3 0 4 3 0 2 4 4 0 0 3 2 0 0 0 2 2 4 |
|    | 3 1 0 3 2 1 1 2 1 0 4 2 2 3 1 3 4 1 4 2 1 3 0 1 |
|    | 0 4 0 0 4 3 2 4 3 2 1 0 1 4 1 1 2 3 2 3 0 3 3 4 |
|    | 0 0 2 1 3 1 2 2 3 3 0 3 4 0 2 4 0 1 2 2 0 0 4 1 0 0 0 1 |

| J: | 1 1 1 2 0 4 2 3 4 3 2 2 1 3 3 4 1 1 4 0 1 1 1 3 |
|    | 1 1 1 1 4 2 3 0 4 4 1 4 3 2 0 4 4 0 3 1 3 2 4 0 |
|    | 4 0 3 2 4 4 3 3 4 4 4 4 0 2 0 1 3 1 0 0 4 0 4 3 1 |
|    | 3 1 3 3 0 2 3 3 3 1 0 2 1 4 2 4 1 1 3 4 4 2 3 3 |
|    | 2 0 3 3 4 4 3 3 3 2 1 4 0 2 2 3 2 4 1 0 2 2 0 |
|    | 4 3 4 1 2 0 2 0 4 1 2 2 4 4 2 2 2 0 1 0 3 4 3 0 |
|    | 0 2 0 2 4 3 4 3 4 4 0 1 4 4 4 3 0 1 3 2 1 2 3 3 |
|    | 4 2 2 1 4 4 1 0 4 4 4 2 4 4 4 4 1 3 2 0 1 1 4 1 |
|    | 2 3 0 1 1 0 2 4 2 3 1 0 1 0 2 3 1 1 2 2 1 1 1 0 |
|    | 3 0 4 2 4 0 0 1 0 1 2 4 2 4 2 2 0 3 2 2 2 4 0 3 |
|    | 4 1 3 1 4 4 2 1 1 3 2 2 3 0 2 2 2 1 2 2 2 2 3 4 |
|    | 1 0 3 3 2 3 1 4 0 3 3 0 1 2 1 4 3 0 3 0 1 4 3 3 |
|    | 1 1 3 3 3 0 4 0 2 1 2 0 0 3 0 3 1 2 1 2 1 0 4 1 1 1 2 |

In order to compress the sequence of bracelets, the notation $H^m$ has been employed to indicate that $m$ bracelets $H$ have been produced in succession, and $J^n$ to indicate that $n$ bracelets $J$ have been produced in succession. In this notation, the entire bracelet of 312 bracelets becomes

$$H^1J_2H^2J_3H^1J_2H^1J_3H^1J_4H^1J_1H^2J_3H^2J_1H^1J_2H^3J^1H^1$$

$$J_2H^2J_3H^4J_1H^2J_2H^1J_1H^2J_2H^1J_3H^1J_4H^2J_4H^1J_3H^1J_5H^1J_1H^2J_3H^1$$

$$H^1J_2H^3J_3H^1J_2H^1J_4H^2J_1H^3J_2H^1J_2H^1J_1H^1J_2H^4J_2H^1J_2H^1J_2H^1J_2H^1$$

$$J_4H^3J_2H^2J_3H^1J_2H^1J_2H^3J_2H^1J_1H^1J_1H^4J_1H^1J_2H^2J_3H^1J_2H^1$$
The cyclic sequence of the frequencies of $H$ is

```
12111 11122 13124- 22212 21142 13115 34112 11132
11131 14123 22321 11114 22532 31111 56131 11131 lll'l2.
```

This is the reverse of the cyclic sequence of the frequencies of $J$ beginning with the frequency of the second $J$ in the fifth line of the bracelet of bracelets.

The longest palindromes imbedded in the fourth-order bracelets are in $J$. In order of appearance as underscored in Table 1, they are

```
3334333, 4442444, 2221222, and 1113111.
```

The leading digits, as well as the central digits, reproduce bracelet $B$.

Fourth-order bracelets in the decimal system have been discussed previously [3,4,5], and a discussion of second- and third-order bracelets in base five appears in [6].

REFERENCES


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MATHEMATICAL CLERIHEWS

William Rowan Hamilton, A toping and untrammeled son Of the Hibernians, Devised quaternions.

Charles Lutwidge Dodgson (Lewis Carroll) Gave lectures in a clerk's apparel, Was fond of logic, pun and riddle, And also Alice Pleasance Liddell.

ALAN WAYNE, Holiday, Florida
Later on in this column I will give solutions to several problems proposed here earlier. But first I give two new problem sets. The first consists of the problems set at the final round of the 1982 Swedish Olympiad. The second is a set of problems proposed in the March 1983 issue of Középiskolai Matematikai Lapok (Hungarian Mathematical Journal for Secondary Schools). As usual, for all of these problems I solicit elegant solutions, which should be sent directly to me at the address given at the end of this column.

1982 SWEDISH OLYMPIAD

1. Let \( N \) be a positive integer. How many solutions to the equation
   \[ x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2 \]
are there in the interval \( 1 \leq x \leq N \)?

2. Let \( a, b, c \) be positive numbers. Prove that
   \[ abc \geq (b+c-a)(c+a-b)(a+b-c). \]

3. Suppose one can find a point \( P \) in the interior of the quadrilateral \( ABCD \) such that the four triangles \( PAB, PBC, PCD, \) and \( PDA \) have the same area. Show that \( P \) is on one of the diagonals \( AC \) or \( BD \).

4. In the triangle \( ABC \) the sides are \( AB = 33 \) cm, \( AC = 21 \) cm, and \( BC = m \) cm, where \( m \) is an integer. It is possible to find a point \( D \) on \( AB \) and a point \( E \) on \( AC \) such that
   \[ AD = DE = EC = n \text{ cm}, \]
   where \( n \) is an integer. What values can \( m \) take?

5. In an orthonormal coordinate system one considers the points \((x,y)\), where \( x \) and \( y \) are integers with \( 1 \leq x \leq 12, 1 \leq y \leq 12 \). Each of these 144 points is coloured red, white, or blue. Show that there is a rectangle with sides parallel to the axes and having all its vertices the same colour.

6. If \( 0 \leq a \leq 1 \) and \( 0 \leq x \leq \pi \), prove that
   \[ (2a-1)\sin x + (1-a)\sin(1-a)x \geq 0. \]
F. 2410, Solve the system of equations

\[
\begin{align*}
x + y + z &= 5 \\
x^2 + y^2 + z^2 &= 9 \\
xy + u + vx + vy &= 0 \\
yz + u + vy + vz &= 0 \\
zx + u + vz + vx &= 0.
\end{align*}
\]

F. 2411, Show that there is no party of 10 members in which the members have 9, 9, 9, 8, 8, 8, 7, 6, 4, 4 acquaintances, respectively, among themselves. (Acquaintances are supposed to be mutual.)

F. 2412, We have \(d\) cartons and one box. The cartons are numbered from 1 to \(d\) and some of them (maybe all) contain balls. We want to collect the contents of all the cartons in the box. Carton \(i\) may be emptied if it contains exactly \(i\) balls, and this is done in such a way that one ball is placed into the box and the remaining \(i-1\) balls are placed one by one into the cartons 1, 2, ..., \(i-1\), respectively. For which \(n\) is it possible to place \(n\) balls into an appropriate number of cartons so that they all could be collected?

F. 2413, Given is a square ABCD. Find the locus of the points \(P\) for which

\[
PA + PC = \sqrt{2} \max \{PB, PD\}.
\]

F. 2414, We have two regular octagons \(N_1 = A_1B_1...H_1\) and \(N_2 = A_2B_2...H_2\).

The sides \(A_1B_1\) and \(A_2B_2\) lie on the same line, and also the sides \(D_1E_1\) and \(D_2E_2\) lie on the same line. Furthermore, \(G_2\) coincides with \(C_1\), and \(A_2B_2 < A_1B_1\). The regular octagon \(N_{i+1}\) is in the same relation with \(N_i\) as \(N_2\) is with \(N_1\) (\(i = 2,3,\ldots\)). Assuming that all octagons \(N_i\) have been constructed, show that the sum \(R_1 + R_2 + \ldots\) of the radii of their circumcircles converges to the length of the segment \(A_1M\), where \(M = A_1C_1 \cap D_1O_2\) and \(O_2\) is the center of \(N_2\).

F. 2415, Choose 400 different points inside a unit cube. Show that 4 of these points lie inside some sphere of radius 4/23.

P. 375, Does there exist a function \(f: \mathbb{R} \to \mathbb{R}\) such that \(\lim_{x \to \infty} f(x) = \infty\) and

\[
\lim_{x \to \infty} \frac{f(x)}{\ln(\ln(\ldots(\ln x)\ldots))} = 0
\]

holds for all \(n\) (where \(n\) is the number of logarithm functions in the denominator)?
P. 376. Determine the sizes of those circles whose interior points can be painted with two colours in such a way that the endpoints of all line segments of unit length have different colours.

9. [1981: 236; 1982: 45] Let $P$ be a polynomial of degree $n$ satisfying

$$P(k) = \binom{n+1}{k}^{-1}, \quad k = 0, 1, \ldots, n.$$ 

Determine $P(n+1)$.

II. Second solution by Noam D. Elkies, student, Columbia University.

Since

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} P(k)$$

is the $(n+1)$st finite difference of the $n$th-degree polynomial $P$, this sum must vanish. Thus

$$\sum_{k=0}^{n} (-1)^k + (-1)^{n+1} P(n+1) = 0,$$

and it follows that

$$P(n+1) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

1. (Second Round) [1982: 12] The sequence \{a_1, a_2, a_3, \ldots\} is defined as follows: $a_1$ is an arbitrary positive integer and, for $n > 1$, $a_n = [3a_{n-1}/2] + 1$. Is it possible to choose $a_1$ such that $a_{100001}$ is odd and $a_n$ is even for all $n \leq 100000$?

Solution by John Morvay, Dallas, Texas.

The answer is yes. Just choose $a_1 = 2^{100000} - 2$. It then follows easily that

$$a_n = 3^{n-1} \cdot 2^{100001-n} - 2 \text{ for } n \leq 100000$$

and

$$a_{100001} = 3^{100000} - 2.$$ 

2. [1982: 270] Let $(a_1, a_2, \ldots, a_n, \ldots)$ be a sequence of positive real numbers such that $a_n^2 \leq a_n - a_{n+1}$ for all $n$. Show that $a_n < 1/n$ for all $n$.

Solution by K.S. Murray, Brooklyn, N.Y.

The defining relation shows that we must have $a_2 \leq a_1(1-a_1)$. Thus
We now use \( a_{n+1} \leq a_n (1-a_n) \), \( n = 2, 3, 4, \ldots \) and proceed by induction. Assume that \( a_k < 1/k \) for some \( k \geq 2 \). Since \( x(1-x) \) is increasing in the interval \([0, \frac{1}{2}]\), we have
\[
 a_{k+1} \leq a_k (1-a_k) \leq \frac{1}{k} (1-\frac{1}{k}) = \frac{k-1}{k^2} < \frac{1}{k+1},
\]
and the induction is complete.

3. [1982: 270] A round track has \( n \) fueling stations (some possibly empty) containing a combined total of fuel sufficient for a car to travel once around the track. Prove that, irrespective of the initial distribution of fuel among the stations, it is always possible for a car with an empty tank to start from one of the stations and complete a round trip without running out of fuel on the way.

Comment.

See Crux 354 [1979: 57] for a solution and for references to other appearances of this problem by various proposers. However, the version given here, from a 1964 Peking Mathematics Contest, predates all the other references known to us at this time.

1. [1983: 137] Which of \((17091982!)^2\) and \(17091982^{17091982}\) is greater?

Solution by Noam D. Elkies, student, Columbia University.

The result follows by setting \( n = 17091982 \) in the inequality
\[
(n!)^2 > n^n, \quad n > 2,
\]
which itself follows from
\[
(n!)^2 = \prod_{j=1}^{n} j(n+1-j) > \prod_{j=1}^{n} n = n^n.
\]

Comment by M.S.K.

Inequality (1) was also established nicely by Peter Ross (University of Santa Clara) by using the concavity of \( \ln x \), and by Lones Smith (Nepean, Ontario) by means of the A.M.-G.M. inequality.

18-3. [1983: 107, 141] Show how to construct a triangle having its vertices on three given skew lines so that the centroid of the triangle coincides with a given point.

II. Solution by Howard Eves, University of Maine.

Denote the three mutually skew lines by \( l, m, n \), and the given point by \( G \), as shown in the figure. Let \( \mathcal{L}' \) be the map of \( \mathcal{L} \) under the homothety of center \( G \) and
ratio $-\frac{1}{2}$. Let $p$ be the plane midway between the plane through $m$ parallel to $n$ and the plane through $n$ parallel to $m$; this plane is the locus of midpoints of all segments joining points of $m$ to points of $n$. Let $p$ cut $l'$ in $A'$. Let $A'G$ cut $l$ in $A$. Let the line of intersection of the plane determined by $A'$ and $m$ with the plane determined by $A'$ and $n$ cut $m$ in $B$ and $n$ in $C$. Then $ABC$ is the sought triangle.

If $l'$ lies in $p$, any point on $l'$ may serve as point $A'$, and the problem has infinitely many solutions. If $l'$ is parallel to $p$, the problem has no solution.

Comment by M.S.K.

I also received a solution equivalent to the above from Esther Szekeres (New South Wales, Australia). These two solutions were submitted in response to my request for a synthetic solution, following an analytic solution I had given earlier [1983: 141].

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

*  *  *

Hollywood Arithmetic

Item by Leonard Lyons in the September 1976 Reader's Digest, page 90: An actress who was offered a co-starring role with Zero Mostel declined, because Mostel is too overwhelming. "I'd be lost working with him. One plus Zero would still be Zero."
Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

871, Proposed by Allan Wm. Johnson Jr., Washington, D.C.

In each of the following four (independent) cryptarithms, assign each X a different decimal digit to obtain decimal integers that make the cryptarithm arithmetically true.

(a) \(X \cdot X \cdot X \cdot X = XX = X \cdot X\),
(b) \(X \cdot X \cdot X \cdot X = XX\) simultaneously with \(X \cdot X = X\),
(c) \(X \cdot X \cdot X \cdot X = XX\) simultaneously with \(X \cdot X = X + X\),
(d) \(X \cdot X \cdot X \cdot X = XX = (X + X) \cdot (X + X)\).

872, Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \(T\) be a triangle \(ABC\) with sides \(a, b, c\) and circumradius \(R\), and let \(P\) be a point other than a vertex in the plane of \(T\). It is known (M.S. Klamkin, "Triangle inequalities from the triangle inequality", *Elemente der Mathematik*, Vol. 34 (1979), No. 3) that there exists a triangle \(T_0\) with sides \(a \cdot PA, b \cdot PB,\) and \(c \cdot PC\). If \(R_0\) is the circumradius of \(T_0\), prove that

\[PA \cdot PB \cdot PC \leq R \cdot R_0.\]

When does equality occur?

873, Proposed by W.R. Utz, University of Missouri-Columbia.

Show that the sequence \(\{3n^2 + 3n + 1\}, n\) an integer, contains an infinite number of squares but only one cube.

874, Proposed by the COPS of Ottawa.

Let \(S_n\) be the sum of the first \(n\) primes. Show that for every \(n\) there is at least one square between \(S_n\) and \(S_{n+1}\).


Can a square be dissected into three congruent nonrectangular pieces?
Let ABC be a triangle with sides $a, b, c$, and let $K_a, K_b, K_c$ be the circles with centers $A, B, C$, respectively, and radii $\lambda \sqrt{bc}, \lambda \sqrt{ca}, \lambda \sqrt{ab}$, respectively, where $\lambda \geq 0$. Find the locus of the radical center of $K_a, K_b, K_c$ as $\lambda$ ranges over the nonnegative real numbers.

877. Proposed by Charles W. Trigg, San Diego, California.
Prove or disprove the following statement: Every prime which is the reverse of a square integer is congruent to 1 modulo 6.
Examples: $61 = 10 \cdot 6 + 1$ is the reverse of $4^2$, and $12391 = 2065 \cdot 6 + 1$ is the reverse of $139^2$.

878. Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.
Let $A$ be a real $n \times n$ matrix with nonzero entries. If $A$ is singular (i.e., $\det A = 0$), does there always exist a real $n \times n$ matrix $B$ such that $\det(AB + BA) = 0$?

879. Proposed by Leroy F. Meyers, The Ohio State University.
The U.S. Social Security numbers consist of 9 digits (with initial zeros permitted). How many such numbers are there which do not contain any digit three or more times consecutively?

880. Proposed by Clark Kimberling, University of Evansville, Indiana.
For a given triangle ABC, what curve is formed by all the points P in three-dimensional space satisfying
\[ \angle BPC = \angle CPA = \angle APB? \]

INTRODUCING MAYBE THE NEXT EDITOR OF CRUX MATHEMATICORUM

After the second round of crossword competition, Stanley Newman jumped up and said: "The Wicked Wasp of Twickenham! That's Alexander Pope! With a mathematics background, I'm not supposed to know such things, but I do. I picked them up from doing puzzles."

From a news item in the New York Times, Sunday, August 21, 1983, about the Second United States Open Crossword Puzzle Championship, which took place at the Loeb Student Center at New York University on August 20. One of the four puzzles everyone had done by mail to qualify them for the event had a mathematical clue that stumped a lot of people. It was "Kilenc plus kilenc". Paul Erdős will confirm that the correct answer is "Tizennyolc".
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


A propos of the editor's comment following Crux 611 [1982: 30], verify that, with decimal integers,
(a) uniquely, TRIGG is three times WRONG;
(b) independently, but also uniquely, WAYNE is seven times RIGHT.

III. Letter to the editor by Stewart Metchette, Culver City, California.

Per your comment to Problem 721 [1983: 89]:
I hope digitomaniacs continue their efforts, even at the risk of repeating what has been done before. For example, you quote Victor Thébault's work [1] on the 2-factor digital products: if your quote is accurate, his solution list for the 10-digit case is incomplete; if inaccurate, you have done him a disservice.

There are an additional 9 and 6 solutions (for a total of 13 and 9) for

\[ A \cdot BCDE = FGHIJ \quad \text{and} \quad AB \cdot CDE = FGHIJ, \]

respectively. These are listed by Madachy [2] and attributed by him to Charles L. Baker.

You go on to say that "finding all products containing each of the digits exactly once... has already been done." In Madachy's book, he mentions that some 3-factor solutions might be possible, but knew of none.

Inspired by Madachy's comment, I ran a computer check and reported the results in [3]:

\[ A \cdot BC \cdot DE = FGHI, \quad 12 \text{ solutions}; \]
\[ A \cdot BC \cdot DE = FGHIJ, \quad 10 \text{ solutions}; \]
\[ A \cdot BC \cdot DEF = GHIJ, \quad 2 \text{ solutions}. \]

There is still much work for digitomaniacs to perform. All problems requiring solutions using the nine nonzero digits can be redefined in terms of nine distinct digits, which need investigation.

I wish them good hunting!

Editor's comment.

In his earlier comment, the editor meant and should have written "finding all two-factor products... has already been done." The italicized qualifier was unfortunately left out. The editor then went on to give Thébault's solutions, which
were quoted correctly and completely from [1]. However, a more careful reading of [1] shows that Thébault wrote: "In this case [the ten-digit case] the investigation is still more complicated, and the following solutions are submitted...". Thébault then listed the 7 solutions given earlier. But he does not actually claim that these are the only solutions.

The editor, no digitomaniac, thanks reader Metchette for the clarification, and apologizes to Victor Thébault's angry shade.

REFERENCES

1. V. Thébault, Solution to Problem E 13 (proposed by W.F. Cheney, Jr.), American Mathematical Monthly, 41 (1934) 265-266.


Find the locus of points with coordinates

\[(\cos A + \cos B + \cos C, \sin A + \sin B + \sin C)\]

(a) if \(A, B, C\) are real numbers with \(A + B + C = \pi\);
(b) if \(A, B, C\) are the angles of a triangle.

Solution by Jordi Dou, Barcelona, Spain.
(a) Let \(A = 2\alpha\) and \(B = \tau\), so that \(C = \pi - 2\alpha - \tau\). A point \((x, y)\) lies on the locus if and only if

\[x = \cos 2\alpha + \cos \tau + \cos(\pi - 2\alpha - \tau) = \cos 2\alpha + 2\sin \alpha \sin(\alpha + \tau)\]

and

\[y = \sin 2\alpha + \sin \tau + \sin(\pi - 2\alpha - \tau) = \sin 2\alpha + 2\cos \alpha \sin(\alpha + \tau)\]

The resulting

\[\frac{y - \sin 2\alpha}{x - \cos 2\alpha} = \cot \alpha\]

and

\[(x - \cos 2\alpha)^2 + (y - \sin 2\alpha)^2 = 4\sin^2(\alpha + \tau)\]

show that, for fixed \(\alpha\) and variable \(\tau\), the locus of \((x, y)\) is a segment PQ of
length 4 whose midpoint \((\cos 2\omega, \sin 2\omega)\) lies on the unit circle \(\gamma\) with centre at the origin 0. The endpoints of \(PQ\) correspond to \(t = -\omega \pi / 2\); their coordinates are

\[
x_P = \cos 2\omega - 2 \sin \omega, \quad y_P = \sin 2\omega - 2 \cos \omega
\]

and

\[
x_Q = \cos 2\omega + 2 \sin \omega, \quad y_Q = \sin 2\omega + 2 \cos \omega.
\]

Since \(P(\omega + \pi) = Q(\omega)\), it follows that, as \(\omega\) varies, \(P\) and \(Q\) describe the same curve. This curve, shown in the figure, is a 3-cusped hypocycloid, or deltoid, generated by a fixed point on a circle \(\mu\) of radius 1 which rotates without slipping inside a circle \(\phi\) of radius 3 with centre at the origin 0. The interior and boundary of the deltoid constitute the desired locus. Its area is \(2\pi\) and its perimeter is 16.

(b) If \(A, B, C\) are the angles of a triangle (possibly degenerate), then we have the additional restrictions
With these restrictions and for fixed $\omega$, the segment $PQ$ is either vertical or else it has a positive slope $\cot \omega$. The coordinates of the lower endpoint $P$ are obtained by setting $t = 0$. They are

$$x_p = \cos 2\omega + 2 \sin^2 \omega = 1 \quad \text{and} \quad y_p = 2 \sin 2\omega;$$

so, as $\omega$ varies, $P$ describes the segment $RT$ shown in the figure. The coordinates of the higher endpoint $Q$ are obtained by setting $t = -\omega + \frac{\pi}{2}$. They are

$$x_q = \cos 2\omega + 2 \sin \omega \quad \text{and} \quad y_q = \sin 2\omega + 2 \cos \omega.$$

As $\omega$ varies, $Q$ describes the arcs $RS$ and $ST$ of the deltoid. The required locus consists of the interior and boundary of the mixtilinear triangular lamina $RST$ (shaded in the figure). The segment $RT$ contains the images of all degenerate triangles, the arcs $RS$ and $ST$ the images of all isosceles triangles, the point $S$ is the image of the equilateral triangle, and all scalene triangles are mapped into the interior of the lamina.

Also solved by HENRY E. FETTIS, Mountain View, California; J.T. GROENMAN, Arnhem, The Netherlands (partial solution); VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (partial solution); and the proposer.

Editor's comment.

Referring to the figure, suppose the tangent to the deltoid at $T$ meets the arc $RS$ in $U$. The proposer showed that the segment $TU$ contains the images of all right-angled triangles, so $U$ is the image of the isosceles right-angled triangle; and that the parts of the shaded lamina above and below $TU$ contain the images of all acute-angled and all obtuse-angled triangles, respectively.

Suppose all triangles are labelled so that $A \leq B \leq C$, and suppose the tangent to the deltoid at $S$ meets the segment $RT$ in $V$. As Murty showed earlier in this journal [1982: 64-65], the points of segment $VS$ correspond to triangles in which $B = \pi/3$; the points of the shaded lamina above and below $VS$ correspond to the triangles with $B > \pi/3$ and $B < \pi/3$, respectively; and the point $V$ corresponds to the degenerate triangle with angles $0$, $\pi/3$, and $2\pi/3$.

If $TU \cap VS = W$, then $W$ corresponds to the triangle with angles $\pi/6$, $\pi/3$, and $\pi/2$. Finally, $R$ and $T$ correspond to the degenerate triangles with angles $0$, $0$, $\pi$ and $0$, $\pi/2$, $\pi/2$, respectively.

* * *
Proposed by Jordi Dou, Barcelona, Spain.

Given a triangle ABC, construct with ruler and compass, on AB and AC as bases, directly similar isosceles triangles ABX andACY such that BY = CX. Prove that there are exactly two such pairs of isosceles triangles.

Solution by F.G.B. Haskell, Algonquin College, Ottawa.

We introduce a rectangular coordinate system such that the vertices of the triangle have coordinates A(0,2a), B(2b,0), and C(2c,0), as shown in the figure. The midpoints of AB and AC are then M(b,a) and N(c,a). It is easy to verify that ABX and ACY are directly similar isosceles triangles if and only if the coordinates of X and Y are

X(b+2ap, a+2bp)

and

Y(c+2ap, a+2cp),

where p is an arbitrary constant. (As a check, we find that each of the angles BAX and CAY then equals Arccos \( \frac{1}{1+4p^2} \).)

Now BY = CX if and only if

\((-2b+c+2ap)^2 + (a+2cp)^2 = (-2c+b+2ap)^2 + (a+2bp)^2,\)

an equation equivalent to

\(4(b+c)p^2 + 16ap - 3(b+c) = 0.\)

If \(b+c \neq 0\), this is a quadratic equation with a positive discriminant, so there are exactly two distinct satisfactory values of p. Those values are constructible, so are X and Y, and there are exactly two pairs of satisfactory triangles ABX and ACY.

If \(b+c = 0\), the only acceptable value is \(p = 0\). In this case AB = AC, and the only solution to the problem consists of the pair of degenerate triangles ABM and ACN.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. One incorrect solution was received.
76]. Proposed by Allan Wm. Johnson Jr., Washington, D.C.

Solve the following independent base eight alphametics, each of which has a unique solution and is doubly-true in German.

(a) EINS (b) EINS
   EINS EINS
   EINS ZWEI
   EINS VIER
   VIER

I. Solution to part (a) by Kenneth M. Wilke, Topeka, Kansas.

For ease of comprehension, we carry out all our calculations in base ten.

Immediately E = 1; and since $4x \equiv 4$ or 0 (mod 8), N is even and $0 < 4S < 16$. Hence $(S,R) = (2,0)$ or $(3,4)$. If $(S,R) = (2,0)$, then $N = 4$ or 6 and no valid choice of I is possible. Thus $(S,R) = (3,4)$ and $N = 2$ or 6. But only $N = 2$ permits a valid choice for I, which is $I = 5$, and $V = 6$ follows. The unique solution is

1523
1523
1523
1523
6514

II. Solution to part (b) by the proposer.

The alphametic is equivalent to the following system of equations, which are written in base ten:

\[2S + I = R + 8c_1, \quad (1)\]
\[2N + c_1 = 8c_2, \quad (2)\]
\[I + W + c_2 = 8c_3, \quad (3)\]
\[2E + Z + c_3 = V. \quad (4)\]

By (2), $c_1$ is even, so $c_1 = 0$ or 2. Also by (2), $c_2 \leq 2$, so by (3) we have $c_3 = 1$. R and I have the same parity by (1). We tabulate solutions of

\[2E + Z + 1 = V; \quad (4')\]

then for $c_2 = 0, 1, 2$ we enumerate solutions of

\[I + W + c_2 = 8; \quad (3')\]

then we determine the possible R by parity with I; and finally we compute S by

\[S = \frac{R - I}{2} + 4c_1, \quad (1')\]

remembering that, by (2), $(c_2, N, c_1)$ is one of $(0, 0, 0)$, $(1, 4, 0)$, $(1, 3, 2)$, or $(2, 7, 2)$. This work [the details are omitted (Editor)] leads to the unique solution

1043
1043
2710
5016
Also solved by the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; J.A.H. HUNTER, Toronto, Ontario (part (b) only); J.A. McCALLUM, Medicine Hat, Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas (also part (b)); ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer (also part (a)). In addition, one incorrect solution to part (a) and one to part (b) were received.

Editor's comment.

Feder noted that part (a) also has a unique solution in base ten:

\[
\begin{align*}
1329 + 1329 + 1329 + 1329 & = 5316.
\end{align*}
\]


ABC is a triangle with area \( K \) and sides \( a, b, c \) in the usual order. The internal bisectors of angles A, B, C meet the opposite sides in D, E, F, respectively, and the area of triangle DEF is \( K' \).

(a) Prove that

\[
\frac{3abc}{4(a^3+b^3+c^3)} < \frac{K'}{K} < \frac{1}{4}.
\]

(b) If \( a = 5 \) and \( 3abc/4(a^3+b^3+c^3) = 5/24 \), determine \( b \) and \( c \), given that they are integers.

I. Solution to part (a) by M.S. Klamkin, University of Alberta.

We have \( \frac{AF}{FB} = \frac{b}{a} \) and \( \frac{AE}{EC} = \frac{c}{a} \); hence, with brackets denoting area,

\[
\frac{[AFE]}{K} = \frac{AF \cdot AE}{AB \cdot AC} = \frac{bc}{(a+b)(a+c)}.\]

With this and two similar results, we obtain

\[
\frac{K'}{K} = 1 - \sum_{\text{cyclic}} \frac{bc}{(a+b)(a+c)} = \frac{2abc}{(b+c)(c+a)(a+b)},
\]

and we have to show that

\[
\frac{3abc}{4(a^3+b^3+c^3)} \leq \frac{2abc}{(b+c)(c+a)(a+b)} \leq \frac{1}{4}.
\]

The first inequality will follow from

\[
3(b+c)(c+a)(a+b) \leq 8(a^3+b^3+c^3) \tag{1}
\]

and the second from
Now (1) and (2) are both known to hold for all nonnegative \(a, b, c\), with equality in each case if and only if \(a = b = c\). According to Bottema [1], the first is due to A. Padoa (1925) and the second to E. Cesàro (1880). □

Proofs for (1) and (2) are not given in [1]. The proof of (2) is nearly trivial (\(2\sqrt{3}c \leq b+c\), etc.), so for completeness we give here only a proof of (1), for which the Padoa reference is not easily accessible.

From the A.M.-G.M. inequality,

\[
6abc \leq 2(a^3 + b^3 + c^3);
\]

and from Problem 6-3 in this journal [1979: 198],

\[
3(b^2c + c^2a + a^2b) \leq 3(a^3 + b^3 + c^3)
\]

and

\[
3(bc^2 + ca^2 + ab^2) \leq 3(a^3 + b^3 + c^3).
\]

Now adding (3), (4), and (5) yields (1).

The inequality \(K'/K \leq \frac{3}{4}\) of part (a) was already known. It is included in the following more general result: if \(AD, BE,\) and \(CF\) are three concurrent interior cevians of a triangle \(ABC\), then the area of triangle \(DEF\) is a maximum if and only if the point of concurrency is the centroid of \(ABC\). (See the end of solution II of Crux 323 [1978: 256] and the end of the solution of Crux 585 [1981: 304].) So here we have \(K'/K \leq \frac{3}{4}\) with equality if and only if the incenter coincides with the centroid, that is, if and only if \(ABC\) is equilateral.

II. Solution to part (b) by W.J. Blundon, Memorial University of Newfoundland.

The equation we must solve for positive integers \(b\) and \(c\) is equivalent to

\[
18bc = 125 + b^3 + c^3,
\]

which is itself equivalent to

\[
3bc = (b+c)^2 - 6(b+c) + 36 - \frac{91}{b+c+6}.
\]

Since \(b+c+6|91\), we have \(b+c+6 = 13\) or \(91\), so \(b+c = 7\) or \(85\), and the corresponding values of \(bc\) are \(12\) and \(2250\). Only the pair \((b+c, bc) = (7, 12)\) is satisfactory, and it leads to the unique solution \(\{b, c\} = \{3, 4\}\).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland (part (a) also); JORDI DOU, Barcelona, Spain; G.C. GIRI, Midnapore College, West Bengal, India (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, California (part (b) only); ROBERT S. JOHNSON, Montréal, Québec (part (b) only); M.S. KLAMKIN,
Find all positive integer pairs \( (m,n) \) such that

\[
\frac{1}{m} + \frac{1}{n} = \frac{q}{p},
\]

where \( p \) and \( q \) are consecutive primes with \( p > q \).

**Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.**

It is known [1] that, if \( p \) and \( q \) are positive integers, then \( q/p \) can be expressed as \( 1/m + 1/n \) if and only if there exist divisors \( p_1 \) and \( p_2 \) of \( p \) such that \( q|p_1+p_2 \). But if \( p \) is prime, its only divisors are 1 and \( p \). Thus \( q|p+1 \).

Now by Bertrand's postulate, if \( q > 3 \) then \( p+1 < 2q \), and so \( q|p+1 \). Thus there are no solutions with \( q > 3 \). If \( q = 2 \) or 3, it is easily shown that the only solutions are

\[
\frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \quad \frac{1}{2} + \frac{1}{10} = \frac{3}{5},
\]

so the required pairs are \( (m,n) = \{3,3\}, \{2,6\}, \text{ and } \{2,10\} \).

Also solved by the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; ERNEST W. FOX, South Lancaster, Ontario; J.T. GROENMAN, Arnhem, The Netherlands; KELVIN MACBETH, Queensland Institute of Technology, Brisbane, Australia and DAVID R. STONE, Georgia Southern College, Statesboro, Georgia (jointly); BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; LAWRENCE SOMER, Washington, D.C.; DAVID STEINSALTZ, Hewlett Harbor, N.Y.; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer (two solutions).

**Reference**


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**Reference**


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**Reference**

1. Problem E 2875 (proposed by K.P. Shum and R.F. Turner-Smith, The Chinese University of Hong Kong).
If \( n \) is a given positive integer, find all solutions \( \theta \in [0, 2\pi) \) of the equation
\[
\cos^n \theta + \sin^n \theta = 1.
\]
(The trivial case \( n = 2 \) may be omitted.)

Solution by M.S. Klamkin, University of Alberta (revised by the editor).

For \( n = 1 \), we have
\[
\cos \theta + \sin \theta = \sqrt{2} \sin \left( \theta + \frac{\pi}{4} \right) = 1 \iff \theta = 0 \text{ or } \frac{\pi}{2}.
\]

For \( n \geq 3 \), we will use the fact that, for every \( \theta \),
\[
\cos^n \theta + \sin^n \theta \leq \cos^2 \theta + \sin^2 \theta = 1. \tag{1}
\]
Now \( \theta \) is a solution of the given equation if and only if equality holds in (1), that is, if and only if
\[
\cos^n \theta \geq 0 \quad \text{and} \quad \sin^n \theta \geq 0, \tag{2}
\]
and
\[
|\cos \theta| = 1 \quad \text{or} \quad |\sin \theta| = 1. \tag{3}
\]
The values of \( \theta \) satisfying (2) and (3) are easily found to be
\[
\theta = \begin{cases} 
0, \frac{\pi}{2}, & \text{for odd } n; \\
0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, & \text{for even } n \neq 2.
\end{cases} \tag{4}
\]
Combining the cases \( n = 1 \) and \( n \geq 3 \), we find that (4) gives all the solutions for all \( n \neq 2 \).

As a generalization, we will characterize all \( r \)-dimensional \textit{unit} vectors
\[
\mathbf{u} = (u_1, u_2, \ldots, u_r) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \ldots, v_r),
\]
where \( r \geq 2 \), whose components satisfy the equation
\[
\sum_{\ell=1}^{r} u_{m,\ell} v_{n,\ell} = 1, \tag{5}
\]
where \( m \geq 0 \) and \( n \geq 1 \) are integers. (The proposed problem corresponds to the case \( r = 2, m = 0 \).)

Case 1: \( m = 0 \). Here, of course, the pair \((\mathbf{u}, \mathbf{v})\) satisfies (5) if and only if \( \mathbf{v} \) satisfies
We consider several subcases.

Case 1.1: \( m = 0, n = 1 \). It is easy to see that there is an infinite class of vectors \( \mathbf{v} \) satisfying (6) (for which a more precise characterization depends upon the value of \( r \)), except in the case \( r = 2 \), where

\[
\sum_{i=1}^{r} v_i^n = 1. \tag{6}
\]

but only \((1,0)\) and \((0,1)\) satisfy (6). (This suggests an alternative proof for the case \( n = 1 \) in the original problem.)

Case 1.2: \( m = 0, n = 2 \). It is clear that (6) holds for all unit vectors \( \mathbf{v} \).

Case 1.3: \( m = 0, n > 3 \). Here we will use the fact that, for every \( \mathbf{v} \),

\[
\sum_{i=1}^{r} v_i^n \leq \sum_{i=1}^{r} v_i^2 (= 1). \tag{7}
\]

Now \( \mathbf{v} \) satisfies (6) if and only if equality holds in (7), that is, if and only if

\[
v_i^n \geq 0 \text{ for } i = 1,2,\ldots,r \tag{8}
\]

and

\[
|v_{i_0}| = 1 \text{ and } v_i = 0 \text{ for } i \neq i_0 \tag{9}
\]

for some \( i_0 \in \{1,2,\ldots,r\} \). The vectors \( \mathbf{v} \) whose components satisfy (8) and (9) are characterized by

\[
\begin{cases} 
  v_{i_0} = 1 \text{ and } v_i = 0 \text{ for } i \neq i_0, \text{ if } n \text{ is odd;} \\
  v_{i_0} = \pm 1 \text{ and } v_i = 0 \text{ for } i \neq i_0, \text{ if } n \text{ is even.}
\end{cases}
\]

Case 2: \( m = 1, n = 1 \). Cauchy's inequality gives, for all vectors \( \mathbf{u} \) and \( \mathbf{v} \),

\[
\left( \sum_{i=1}^{r} u_i v_i \right)^2 \leq \sum_{i=1}^{r} u_i^2 \cdot \sum_{i=1}^{r} v_i^2 (= 1). \tag{10}
\]

If the pair \((\mathbf{u},\mathbf{v})\) satisfies (5), then equality must hold in (10), and hence

\[ u_i = kv_i \text{ and } u_i v_i = kv_i^2, \quad i = 1,2,\ldots,r, \]

for some constant \( k \). But (5) holds only for \( k = 1 \), so the only solutions are the pairs \((\mathbf{u},\mathbf{v})\) with \( \mathbf{v} = \mathbf{v} \).
Case 3: $m \geq 1, n \geq 2$. Here we will use the fact that, for all vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\sum_{i=1}^{r} u_i^m v_i^n \leq \sum_{i=1}^{r} v_i^2 \quad (= 1). \quad (11)$$

Now the pair $(\mathbf{u}, \mathbf{v})$ satisfies (5) if and only if equality holds in (11), that is, if and only if

$$u_i^m v_i^n \geq 0 \text{ for } i = 1, 2, \ldots, r \quad (12)$$

and

$$|u_{i_0}| = |v_{i_0}| = 1 \text{ and } u_i = v_i = 0 \text{ for } i \neq i_0 \quad (13)$$

for some $i_0 \in \{1, 2, \ldots, r\}$. The vectors $\mathbf{u}$ and $\mathbf{v}$ whose components satisfy (12) and (13) are characterized by

$$\begin{cases} u_i = v_i = 0 \text{ for } i \neq i_0 \text{ and} \\ u_{i_0} = v_{i_0} = \pm 1, \text{ if } m \text{ and } n \text{ are both odd;} \\
 u_{i_0} = 1 \text{ and } v_{i_0} = \pm 1, \text{ if } m \text{ is odd and } n \text{ is even;} \\
 u_{i_0} = \pm 1 \text{ and } v_{i_0} = 1, \text{ if } m \text{ is even and } n \text{ is odd;} \\
 u_{i_0} = \pm 1 \text{ and } v_{i_0} = \pm 1, \text{ if } m \text{ and } n \text{ are both even.} \end{cases}$$

Also solved by HIPPOLYTE CHARLES, Waterloo, Québec; CURTIS COOPER, Central Missouri State University; CLAYTON W. DODGE, university of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; J. WALTER LYNCH, Georgia Southern College; and the proposers. Three incorrect solutions were received.

*    *    *

Merrimack, New Hampshire.

Let $ABC$ be an equilateral triangle with center $O$. Prove that, if $P$ is a variable point on a fixed circle with center $O$, then the triangle whose sides have lengths $PA, PB, PC$ has a constant area.

Solution by George Tsintsifas, Thessaloniki, Greece.

Let $T$ denote triangle $ABC$, and let $\gamma, R, K$ be its circumcircle, circumradius, and area, respectively. Now let $P$ be any point on a circle $\gamma'$ with center $O$ and radius $\rho$, for some $\rho > 0$. Applying the ptolaemaic inequality [1] to the quadruple of noncollinear points $(P, B, C, A)$, we obtain, in magnitude only,

$$PB \cdot CA + PC \cdot AB \geq PA \cdot BC, \quad \text{whence} \quad PB + PC \geq PA;$$
and PC + PA ≥ PB and PA + PB ≥ PC follow likewise from the quadruples \((P, C, A, B)\) and \((P, A, B, C)\). Each inequality is strict if \(P\) does not lie on \(\gamma\) (i.e., \(\rho \neq R\)), and equality holds if \(P\) lies on \(\gamma\) (in one or two of the three cases, depending on the position of \(P\) on \(\gamma\)).

So there is a triangle (possibly degenerate) with sides \(PA, PB, PC\). If this triangle is \(T'\), we must show that its area, \(K'\), is independent of the position of \(P\) on \(\gamma'\). We already know that this is true if \(\rho = R\), when \(T'\) is degenerate and \(K' = 0\) for all positions of \(P\) on \(\gamma\). We will henceforth assume that \(\rho \neq R\).

If \(T_1 = AB_1C_1\) is the pedal triangle of \(P\) with respect to \(T\) (with \(PA_1 \perp BC\), etc.), then it is known \([2, p. 139]\) that its area \(K_1\) satisfies

\[
K_1 = \frac{\sqrt{3}}{2} \rho A, \quad C_1A_1 = \frac{\sqrt{3}}{2} \rho B, \quad A_1B_1 = \frac{\sqrt{3}}{2} \rho C,
\]

from which follows

\[
K_1 = \frac{3}{4} K'.
\]

Finally, from (1) and (2),

\[
K' = \frac{R^2 - \rho^2}{3R^2} \cdot K = \frac{\sqrt{3}}{4} |R^2 - \rho^2|,
\]

which is independent of the position of \(P\) on \(\gamma'\).

Also solved by O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; HENRY E. FETTIS, Mountain View, California; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles (three solutions); and the proposer. A comment was received from DAN PEDOE, University of Minnesota.

Editor's comment.

Several solutions involved lengthy calculations (some with complex numbers) resulting from the use of Heron's formula. And several solvers, trusting souls that they are, blithely set out to calculate \(K'\) without first assuring themselves that there was a triangle \(T'\).

An interesting fact brought out by the above solution is that \(T'\) is similar to the pedal triangle \(T_1\). Eves noted the following special results: if \(\rho = 2R\), then \(K' = K\); and if \(\rho = R\), the inradius of \(T\), then \(K' = \frac{3}{4} K\). Finally, Klamkin showed that \(R'\), the circumradius of \(T'\), is not constant for all positions of \(P\).
on $\gamma'$, but that

$$R'_{\text{max}} = \frac{R^2 - R\rho + \rho^2}{3|R - \rho|} \quad \text{and} \quad R'_{\text{min}} = \frac{R^2 + R\rho + \rho^2}{3(R + \rho)}.$$

REFERENCES


THE PUZZLE CORNER

**Puzzle No. 42:** Rébus (1 4 2 8 8)

$$1 \equiv 2$$

Poor Cinna's slain
In Shakespeare's play.
Alas! It's plain—
COMPLETE, I say.

**Puzzle No. 43:** Phonetic Rebus (3 14)

T

At logarithms, though you're able,
You'll never find THIS in the table.

**Puzzle No. 44:** Enigmatic rebus (*3 *8 *6)

$$\pi$$

By Robert Frost, an essay you must see;
Its title, clearly pictured, is MY KEY.

**Puzzle No. 45:** Rebus (7 4 5)

$$Y = HS$$

The manufacturer has made his will
To put his REBUS that have shown their skill.

**Puzzle No. 46:** Rebus (10)

$$C$$

The THESE are used, you see,
In trigonometry.

ALAN WAYNE, Holiday, Florida