The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name EUREKA.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name Crux Mathematicorum.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name Crux Mathematicorum with Mathematical Mayhem.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name Crux Mathematicorum.
CRUX MATHEMATICORUM

Vol. 9, No. 7

August - September 1983

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa
Printed at Carleton University

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22 in Canada, US$20 elsewhere. Back issues: $7 each. Bound volumes with index: Vols. 1-2 (combined) and each of Vols. 3-5, $17 in Canada and US$15 elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, Ontario, Canada K1S 1N3.
Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada KIS 0C5.
Typist-compositor: Nghi Chung.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

CONTENTS

The Mystery of the Double Sevens ................. R.C. Lyness 194
Why the Margin Was Too Small .......................... 199
Additive Digital Bracelets in the Quinary System ... Charles W. Trigg 200
More Mathematical Venery .............................. Friend H. Kierstead, Jr. 204
The Olympiad Corner: 47 ............................... M.S. Klamkin 205
Problems - Problèmes .................................. 208
The Puzzle Corner ..................................... 209
Solutions .............................................. 210
What Price Common Logs? ............................ Alan Wayne 222

- 193 -
THE MYSTERY OF THE DOUBLE SEVENS

R.C. LYNES

Prologue.

Sherlock Holmes would have called it a three-pipe problem, and the title is one that Dr. Watson might well have chosen. I usually send a Christmas card with a problem each year to puzzle-minded friends. The reactions of some friends to my Christmas 1982 problem have encouraged me to expand it into the present article. Following the statement of the problem in Section 1, I outline in Section 2 the first proof that I found, one by induction. This is followed in Section 3 by a short direct proof and then, in Section 4, by a combinatorial proof. Finally in Section 5, after three proofs, like Holmes after three pipes, we will together attain full enlightenment with a surprising generalisation.

I apologise to the more sophisticated readers for spelling out some of the processes of double summation, which otherwise the less sophisticated might find difficult. We are not all like Sherlock Holmes in the elucidation of mysteries, and perhaps the Dr. Watsons will be grateful.

1. The problem.

Consider Pascal's triangle of binomial coefficients, with \(^{\binom{a}{b}}\) at the intersection of row \(a\) and column \(b\) (see figure). We assume as usual that \(^{\binom{a}{b}} = 0\) if \(b < 0\) or \(b > a\), so that each row can be extended by zeros as far as necessary in both directions. A \(t\)-size seven with corner \(^{\binom{a}{b}}\) in Pascal's triangle consists of \(^{\binom{a}{b}}\), the \(t\) numbers immediately to the left of \(^{\binom{a}{b}}\), and the \(t\) numbers immediately below \(^{\binom{a}{b}}\). For example, it can be verified from the figure that the following are 2-size sevens with corners \(^{\binom{3}{1}}\), \(^{\binom{4}{2}}\), \(^{\binom{3}{3}}\), and \(^{\binom{7}{3}}\), respectively (the corners are circled in the figure):

\[
\begin{array}{cccccccc}
0 & 1 & 8 & 1 & 4 & 6 & 3 & 1 & 0 & 7 & 21 & 35 \\
9 & 10 & 1 & & & & & & 56 \\
10 & 15 & 5 & & & & & & 84 \\
\end{array}
\]

and the following are 3-size sevens with corners \(^{\binom{5}{1}}\), \(^{\binom{5}{4}}\), and \(^{\binom{7}{6}}\), respectively:

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 7 & 5 & 10 & 10 & 5 & 35 & 35 & 21 & 7 \\
8 & 15 & & & & & & & & 28 \\
9 & 35 & & & & & & & & 84 \\
10 & 70 & & & & & & & & 210 \\
\end{array}
\]

Suppose we take two \(t\)-size sevens, \(S_1\) and \(S_2\). Below the \(t\)th row of Pascal's triangle...
triangle write down the numbers in the row of \( S_1 \) and below that the numbers in the column of \( S_2 \). This gives an array of 3 rows and \( t+1 \) columns. The dot product of \( S_1 \) by \( S_2 \), denoted by \( S_1 \cdot S_2 \), is the sum of the products of the numbers in each of the \( t+1 \) columns of the array. \( S_2 \cdot S_1 \) is defined in the same way by interchanging the roles of \( S_1 \) and \( S_2 \). For example, let \( S_1 \) and \( S_2 \) be the 2-size sevens with corners \((^4_2)\) and \((^7_3)\), respectively. We have

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & 4 & 6 \\
35 & 56 & 84 \\
\end{pmatrix}
\]

\[
\frac{35+448+504}{35+448+504} = 987 = S_1 \cdot S_2
d and
\frac{6+10+15}{42+420+525} = 987 = S_2 \cdot S_1.
\]

And if \( S_1 \) and \( S_2 \) are the 3-size sevens with corners \((^7_1)\) and \((^5_4)\), respectively, then we have

\[
\begin{pmatrix}
1 & 3 & 3 & 1 \\
0 & 0 & 1 & 7 \\
5 & 15 & 35 & 70 \\
\end{pmatrix}
\]

\[
\frac{5+15+35+70}{0+0+105+490} = 595 = S_1 \cdot S_2
d and
\frac{7+8+9+10}{35+240+270+50} = 595 = S_2 \cdot S_1.
\]

It will be observed that in each case \( S_1 \cdot S_2 = S_2 \cdot S_1 \). The Mystery of the Double Sevens is that, if \( S_1 \) and \( S_2 \) are any two sevens of the same size in Pascal's triangle, then \( S_1 \cdot S_2 = S_2 \cdot S_1 \). That is, the dot product of sevens is commutative.

If \( S_1 \) and \( S_2 \) are both of size \( t \) with corners \((^P_{r+t})\) and \((^Q_{s+t})\), respectively, then the row numbers of \( S_1 \) are

\[
(^P_r), (^P_{r+1}), \ldots, (^P_{r+t}), \ldots, (^P_{r+t})
\]

and the column numbers of \( S_2 \) are

\[
(^Q_s), (^Q_{s+1}), \ldots, (^Q_{s+t}), \ldots, (^Q_{s+t}).
\]
Hence \( S_1 S_2 = S_2 S_1 \) if and only if

\[
f(t) = \sum_{u=0}^{t} \binom{t}{u} \binom{p+u}{r+u} = \sum_{u=0}^{t} \binom{t}{u} \binom{q+u}{s+u} \equiv g(t). \tag{1}
\]

2. A proof by induction.

My initial proof of (1) came from noticing that

\[
f(0) = \binom{p}{r} \binom{q}{s},
\]
\[
f(0) + f(1) = \binom{p+1}{r+1} \binom{q+1}{s+1},
\]
\[
f(0) + 2f(1) + f(2) = \binom{p+2}{r+2} \binom{q+2}{s+2},
\]

etc., and that these results remain true if \( f \) is replaced by \( g \), for interchanging the pairs \((p,r)\) and \((q,s)\) transforms \( f(t) \) into \( g(t) \). If one could prove that

\[
\sum_{v=0}^{t} \binom{t}{v} f(v) = \binom{p+t}{r+t} \binom{q+t}{s+t},
\tag{2}
\]

then it would follow that

\[
\sum_{v=0}^{t} \binom{t}{v} \{f(v) - g(v)\} = 0. \tag{3}
\]

A proof of (1) would then come by induction, for the assumption that \( f(v) = g(v) \) is true for \( v = 0,1,...,k \) makes each term except the last vanish in the sum

\[
\sum_{v=0}^{k+1} \binom{k+1}{v} \{f(v) - g(v)\}.
\]

But by (3) with \( t = k+1 \), the entire sum is zero, and so the last term, namely \( f(k+1) - g(k+1) \), is zero. Assuming (2), we have thus proved that if \( f(v) = g(v) \) is true when \( v = 0,1,...,k \), then it is true when \( v = k+1 \). With \( f(0) = \binom{p}{r} \binom{q}{s} = g(0) \), we are done.

I do not give my proof of (2). It is a straightforward summation using two lemmas which, as they are also used in the next section, I state now. Both are identities.

**Lemma 1.** \( \binom{t}{u} \binom{u}{m} = \binom{t-m}{t-u} \binom{t}{m} \).

**Proof.** Each side equals \( \frac{t!}{(t-u)!(u-m)!} \).

**Lemma 2.** \( \binom{q}{w} \binom{u}{0} + \binom{q}{w-1} \binom{u}{1} + \ldots + \binom{q}{w-m} \binom{u}{m} + \ldots + \binom{q}{w-u} \binom{u}{u} = \binom{q+u}{w} \).

**Proof.** This is a Vandermonde identity. Compare the coefficients of \( x^w \) on each side of \((1+x)^q(1+x)^u = (1+x)^{q+u}\).
3. A direct proof of (1).

Using Lemma 2 on \( f(t) \), we have

\[
f(t) = \sum_{u=0}^{t} \binom{t}{u} \binom{p}{r+u} \binom{q}{s+u} + \binom{q}{s+1} \binom{t}{1} + \cdots + \binom{q}{s+m} \binom{u}{m} + \cdots + \binom{q}{t},
\]

and using Lemma 1 on this result gives

\[
f(t) = \sum_{u=0}^{t} \binom{t}{u} \binom{p}{r+u} \binom{q}{s+u} + \binom{q}{s+1} \binom{t}{1} + \cdots + \binom{q}{s+m} \binom{u}{m} + \cdots + \binom{q}{t},
\]

Here the coefficient of \( \binom{q}{s+m} \binom{t}{m} \) is \( \sum_{u=0}^{t} \binom{p}{r+u} \binom{q}{s+u} = \binom{p+t-m}{r+1} \) by Lemma 2. Hence

\[
f(t) = \binom{p+t}{r+t} \binom{q}{0} + \binom{p+t-1}{r+t} \binom{q}{1} + \cdots + \binom{p+t-m}{r+t} \binom{q}{t} + \cdots + \binom{q}{t},
\]

and reversing the order of the terms in this sum gives, since \( \binom{t}{u} = \binom{t}{t-u} \),

\[
f(t) = \sum_{u=0}^{t} \binom{t}{u} \binom{p+u}{r+1} = g(t).
\]

4. A combinatorial proof of (1).

Consider the number of distinct committees of \( r+t \) men and \( s+t \) women that can be formed from \( t \) married couples, \( p \) single men, and \( q \) single women, with the proviso that no married couple is allowed on any committee. This number is \( f(t) \) by one argument and \( g(t) \) by another analogous argument.

On each allowable committee, there must be \( u \) husbands not on the committee, where \( u \) is one of \( 0, 1, 2, \ldots, t \). From their wives and the single women the \( s+t \) women on the committee can be chosen in \( \binom{s+t}{u} \) ways. The \( t-u \) husbands on the committee can be chosen in \( \binom{t}{t-u} \) ways, and the single men can be chosen in \( \binom{p}{r+u} \) ways. Hence the number of committees containing exactly \( t-u \) husbands and none of their wives is \( \binom{t}{u} \binom{p}{r+u} \binom{s+t}{u} \), and the total number of allowable committees is

\[
\sum_{u=0}^{t} \binom{t}{u} \binom{p}{r+u} \binom{s+t}{u} = f(t).
\]

By a similar argument, which interchanges husband and wife, man and woman, \( p \) and \( q \), \( r \) and \( s \), the total number of allowable committees is

\[
\sum_{u=0}^{t} \binom{t}{u} \binom{q}{s+u} \binom{p+u}{r+1} = g(t).
\]
5. A generalisation.

Pascal’s triangle is constructed by using the following well-known property:

\[
\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1}.
\]

But property (4) can be used to construct a seven of any size \(t\) from any given \(t+1\) numbers. For example, a 3-size seven \(S_1\) with row numbers \((a, b, c, d)\) and corner \(d\) can be found as follows:

\[
\begin{array}{cccc}
  a & b & c & d \\
  a+b & b+c & c+d & \\
  a+2b+c & b+2c+d & \\
  a+3b+3c+d & \\
\end{array}
\]

If another 3-size seven \(S_2\) with row numbers \((A, B, C, D)\) and corner \(D\) is formed in the same way, then we have

\[
S_1 \ast S_2 = ad + 3b(C+D) + 3c(B+2C+D) + d(A+3B+3C+D)
\]

and

\[
S_2 \ast S_1 = Ad + 3B(c+d) + 3C(b+2c+d) + D(a+3b+3c+d).
\]

It is easy to verify that \(S_1 \ast S_2 = S_2 \ast S_1\).

In this illustration, we have not assumed that \(u_0 = \binom{r}{s}\). It seems that the double-seven property holds with sevens of any size and arbitrary row numbers provided the column numbers are found by using (4). (This was pointed out to me by Mr. C.P. Ormell, School of Education, University of East Anglia.)

To show that this is true generally, let \(S_1\) and \(S_2\) be \(t\)-size sevens with row numbers \(u_r\) and \(w_r\), respectively, \(r = 0, 1, \ldots, t\). We must show that \(S_1 \ast S_2 = S_2 \ast S_1\), where

\[
S_1 \ast S_2 = f(u, w) = u_0 w_0 t + t u_1 (w_{t-1} + w_t) + \binom{t}{2} u_2 (w_{t-2} + 2w_{t-1} + w_t) + \ldots + u_t w_0,
\]

and \(S_2 \ast S_1 = f(w, u)\). The coefficient of \(u_p w_q\) in this sum is \(\binom{t}{p+s}\) where \(s = p+q-t\); and since

\[
\binom{t}{p+s} = \frac{t!}{(t-p)! (p+q-t)! (t-q)!}
\]

which is symmetrical in \(p\) and \(q\), the coefficients of \(u_p w_q\) and \(u_q w_p\) are equal, and the general result is proved.
"You consider that to be important?" he [Inspector Gregory] asked.
"Exceedingly so."
"Is there any other point to which you would wish to draw my attention?"
"To the curious incident of Pascal's triangle in the proof."
"Pascal's triangle did nothing in the proof."
"That was the curious incident," remarked Sherlock Holmes.

2 Godyll Road, Southwold, Suffolk, England IP18 6AJ.

WHY THE MARGIN WAS TOO SMALL

As every baby knows by now, Fermat's Last "Theorem" states that the equation \( a^n + b^n = c^n \) has no solution in positive integers if \( n > 2 \). Fermat announced this in the "small" margin of his copy of Bachet's edition of Diophantos's *Arithmetica* as follows (as quoted in [1]):

> Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere; cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non oaperet.

As reported in [2], Samuel S. Wagstaff, Jr. proved in 1978 that Fermat's Last Theorem is true for all exponents \( n \leq 125000 \), but a complete proof has eluded number theorists for more than 300 years. Now *Newsweek* (August 1, 1983) reports that a giant step towards a complete solution has just been taken. "Gerd Faltings of Wuppertal University near Dusseldorf has written a 40-page proof of a theorem so sweeping that it even sheds light on Fermat's Last Theorem... It claims that certain equations, including those considered by Fermat, have at most a finite number of rational solutions... Faltings found it using nothing more than pencil, paper and 18 solid months in which he thought of almost nothing else."

Forty pages for a partial solution! No wonder the margin was too small.

REFERENCES

A *bracelet* is defined [1] as "one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Hence a bracelet may be regenerated by starting at any arbitrary position and applying the generating law." The distance between terms may be measured in degrees or steps. A bracelet may be cut at any arbitrary point for straight-line representation without loss of any properties.

A second-order digital bracelet may be constructed by starting with a pair of digits, affixing the units' digit of their sum, again affixing the units' digit of the sum of the last two digits, and continuing the process. The generating law employed here is

\[ u_{n+2} = u_{n+1} + u_n, \]

where each sum is reduced modulo *b* when the operation is conducted in base *b*. To be consistent, the results of all operations, such as addition and multiplication, performed on the digits of a bracelet are reduced modulo *b*. Thus the computations are in a modular arithmetic dealing with individual digits.

In the figure, the twelve-digit bracelet in the decimal system results from applying the generating law to the digit pair (1, 8). The terms are 30° or one step apart [2]. The straight-line representation of the bracelet is

1 8 9 7 6 3 9 2 1 3 4 7'1 8.

Second-order quinary additive bracelets.

In base five, three distinct second-order additive digital bracelets—the all-zero *A*, the four-digit *B*, and the twenty-digit *C*—together contain every one of the twenty-five two-digit permutations of the five digits. In Table 1, two different arrangements, *C*₁ and *C*₂, of *C* are shown. *C* may be called the *Fibonacci bracelet*, since it consists of the units' digits of the Fibonacci sequence in base five. Similarly, *B* may be called the *Lucas bracelet*, since it consists of the units' digits of the Lucas sequence.
Table 1: Second-order bracelets

\[
\begin{align*}
A &: 0 \ 0' \ 0 \ 0 \\
B &: 1 \ 3 \ 4 \ 2' \ 1 \ 3 \\
C_1 &: 0 \ 1 \ 1 \ 3 \ 1 \ 4 \\
& \quad \quad \quad 0 \ 4 \ 4 \ 3 \ 2 \\
& \quad \quad \quad 0 \ 2 \ 2 \ 4 \ 1' \ 0 \ 1 \\
C_2 &: 0 \ 1 \ 1 \ 2 \\
& \quad \quad \quad 3 \ 0 \ 3 \ 3 \\
& \quad \quad \quad 1 \ 4 \ 0 \ 4 \\
& \quad \quad \quad 4 \ 3 \ 2 \ 0 \\
& \quad \quad \quad 2 \ 2 \ 4 \ 1' \ 0 \ 1 \\
\end{align*}
\]

Multiplication of \( B \) by 1, 3, 4, or 2 rotates the bracelet counterclockwise (the original bracelet being written clockwise) through 0°, 90°, 180°, or 270°, respectively. Like operations on \( C \) produce like rotations of \( C \).

Successive multiplications of \( B \) by 3 rotate \( B \) in 90° jumps. That is, \( B \) is a cyclic geometric progression \([3]\) with \( r = 3 \). Like multiplications of \( C \) produce like rotations. This could be expected, since the last four columns of \( C_1 \) are cyclic permutations of \( B \), whereas the first column is repetitive of \( A \).

The sums of the digits in the four rows of \( C_1 \) in order are 2, 1, 3, and 4, a rotation of \( B \).

Two digits may be said to be complementary if their sum is 0. In each of the three bracelets, elements 180° apart are complementary. Thus, the sum of the digits in each of the three bracelets is 0.

The sum of the digits in each row of \( C_2 \) is 4. Each column of \( C_2 \) contains the five distinct digits. Alternate columns contain reverse cyclic permutations of the digits. Each column sums to 0.

It follows that the digits at the vertices of any square or regular pentagon inscribed in bracelet \( C \) will have a sum of 0.

Both \( A \) and \( B \) can be repeated to form twenty-digit sequences, \( A' \) and \( B' \). Each of \( A' \), \( B' \), and \( C \) is produced by the same recurrence formula. Consequently, proceeding clockwise to handle them as bracelets, when any two of these sequences, duplication permitted, are matched in any orientation and added, the resulting bracelet must obey the same law of formation. But the three bracelets, \( A \), \( B \), and \( C \), exhaust the twenty-five pair field, so the result of the addition must be \( A' \), \( B' \), or \( C \).

Any matching of \( A' \) and \( B' \) produces \( B' \), and of \( A' \) and \( C \) produces \( C \). The four matchings of \( B' \) and \( C \) produce \( C \). If a particular digit of one \( C \) is matched with successive digits of another \( C \), proceeding clockwise, the twenty sequences produced upon addition, in order, are

\[
C \ C \ B' \ C \ C \ C \ B' \ C \ C \ C \ A' \ C \ C \ C \ B' \ C \ C \ C \ B' \ C.
\]
This is a cyclic palindrome about \( A' \) and the initial (underlined) \( C \).

The longest palindromes imbedded in \( C \), in order of appearance, are 101, 303, 404, and 202. Their leading digits reproduce \( B \).

**Third-order quinary additive bracelets.**

The recursive formula

\[
u_{n+3} = u_{n+2} + u_{n+1} + u_n
\]

produces third-order bracelets when each sum is reduced modulo \( b \). In the quinary scale of notation, each of the one hundred twenty-five possible digit triads appears in either the all-zero \( A \) or one of the four thirty-one-digit bracelets \( D, E, F, \) and \( G \) shown in Table 2. Bracelet \( D \) may be called the tribonacci bracelet, since it consists of the units' digits of the tribonacci sequence \([4]\) in base five. In that bracelet, the number of digits is \( 31 \text{ten} = 111 \text{five}. \)

**Table 2: Third-order bracelets**

\[
\begin{array}{cccc}
\text{A:} & 000'000 \\
D: & 0011 2423 4414 4420 1343 0202 4122 041'001 \\
E: & 0033 1214 2232 2210 3424 0101 2311 023'003 \\
F: & 0044 3132 1141 1130 4212 0303 1433 014'004 \\
G: & 0022 4341 3323 3340 2131 0404 3244 032'002 \\
\end{array}
\]

As arranged in Table 2, each column consists of zeros or is a cyclic permutation of bracelet \( B \), so it is not unexpected that each of \( D, E, F, \) and \( G \) is a multiple of each of the other three. Bracelets \( D \) and \( F \) are said to be complementary, as are \( E \) and \( G \), since their corresponding digits are complementary.

The digits of each bracelet sum to 0.

There are six zeros in each of the \( D, E, F, \) and \( G \) bracelets. Tabulation of the frequency of occurrence of the nonzero digits in the separate bracelets shows an interesting appearance of like frequencies (in base ten) along the diagonals slanting downward toward the right in Table 3.

**Table 3: Frequency of digits in third-order bracelets**

\[
\begin{array}{cccc}
\text{Digits} & 1 & 3 & 4 & 2 \\
\hline
\text{Bracelets} & D & 6 & 3 & 9 & 7 \\
& E & 7 & 6 & 3 & 9 \\
& F & 9 & 7 & 6 & 3 \\
& G & 3 & 9 & 7 & 6 \\
\end{array}
\]
As with the second-order bracelets, when the elements of two third-order bracelets having the same clockwise orientation are matched and added, a member of the five-bracelet set is generated. For example, in the operation $D + F$, when the first digit of $F$ is matched with the first digit of $D$, the addition of the complementary digit pairs produces thirty-one zeros, a reiteration of bracelet $A$. When the first digit of $F$ is matched with the second digit of $D$, the addition

\[
\begin{array}{cccccccccccccccc}
0011 & 2423 & 4414 & 4420 & 1343 & 0202 & 4122 & 041 \\
4004 & 4313 & 2114 & 1113 & 0421 & 2030 & 3143 & 301 \\
4010 & 1231 & 1023 & 0033 & 1214 & 2232 & 2210 & 342
\end{array}
\]

generates a rotation of bracelet $E$. Upon continuing the matching of the first digit of $F$ with successive digits of $D$, the additions generate the sequence of thirty-one bracelets shown in the first row of Table 4.

Table 4: Derived bracelets

<table>
<thead>
<tr>
<th>Generating pairs</th>
<th>Sequences of thirty-one derived bracelets</th>
<th>Frequency of bracelets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D,F$</td>
<td>AEPEDECDGPPDGEPDEEDEFGDFEEFGDFGG</td>
<td>1 6 9 6 9</td>
</tr>
<tr>
<td>$E,G$</td>
<td>APFPEDEDDGGEFDGDFDEEFGDFGGDGEDD</td>
<td>1 9 6 9 6</td>
</tr>
<tr>
<td>$D,E$</td>
<td>FPEDEEEDEDDGDDDGFEEDGDDGDFDEEFGDF</td>
<td>- 9 9 7 6</td>
</tr>
<tr>
<td>$E,F$</td>
<td>GGPEFGDFEEDEEEEDGPPDGFDGDGGGDF</td>
<td>- 6 9 9 7</td>
</tr>
<tr>
<td>$F,G$</td>
<td>DDFGFGDGFPPFPEGFFEDGGEEDGDFGGDF</td>
<td>- 7 6 9 9</td>
</tr>
<tr>
<td>$G,D$</td>
<td>EEDGDFDDGDGPGGGEFDDDFFPDGEDG</td>
<td>- 9 7 6 9</td>
</tr>
<tr>
<td>$E,E$</td>
<td>DFGGDFDFFDEDGDDGGEEDGDDFDFDDG</td>
<td>- 7 6 6 12</td>
</tr>
<tr>
<td>$F,F$</td>
<td>EGDDEGGEFFEDDFDDPFDGDGGGEDDD</td>
<td>- 12 7 6 6</td>
</tr>
<tr>
<td>$G,G$</td>
<td>FDDEPFFDGFPEEEEGGEEFEGDFDDFEED</td>
<td>- 6 12 7 6</td>
</tr>
<tr>
<td>$D,D$</td>
<td>GEEFPGEEDDPFGFEDFFFDGDGEDGFG</td>
<td>- 6 6 12 7</td>
</tr>
</tbody>
</table>

The thirty-one bracelet sequences derived from other pairings are reported in three groups in Table 4. The frequencies of the bracelets in the sequences are expressed in the decimal system. In the generating pairs, the first bracelet listed is stationary and the second bracelet rotates clockwise in one-step moves.

In each of the two sequences derived from the complementary pairs $D,F$ and $E,G$, each member of the generating pair occurs six times. In each sequence, the numbers of like adjacent bracelets are

\[
1 1 1 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 1
\]

a twenty-seven-term cyclic palindrome centered on the first term.

In each of the four sequences derived from the cyclically adjacent thirty-one-digit bracelet pairs in Table 2, each member of the generating pair occurs nine
times. In each sequence, the numbers of like adjacent bracelets are

\[ 2 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 1 \ 1 \ 1 \ 3 \ 1 \ 1 \ 1 \ 2 \ 3 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1. \]

In each of the four sequences derived from a bracelet operating on itself, the generating bracelet appears six times, whereas its complement appears twelve times. In each sequence, the numbers of like adjacent bracelets are

\[ 1 \ 1 \ 2 \ 1 \ 2 \ 1 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2, \]

a twenty-two-term cyclic palindrome symmetrical to each of the underlined terms.

The longest palindromes imbedded in \(D, E, F, G\) are \(44144\), \(22322\), \(11411\), and \(33233\), respectively. Their leading terms, middle terms, and digit sums, all reproduce \(B\).

REFERENCES


2404 Loring Street, San Diego, California 92109.

MORE MATHEMATICAL VENERY [1983: 136]

A welter of primes
An occam of razors
A russell of paradoxes
A complex of roots
A series of Taylors
A surd of irrational numbers
A neumann of programmers
A class of residues
An induction of proofs

A hamming of self-correcting codes
A cardan of cubic equations
An interpolation of Newtons
A napier of logarithms
A dover of reprints
A biaise of Pascal triangles
A clone of repunits
A score of icosahedral faces
A dozen of dodecahedral faces

FRIEND H. KIERSTEAD, JR.
Cuyahoga Falls, Ohio.
The Twenty-Fourth International Mathematical Olympiad (I.M.O.) was held from July 1 to 11, 1983, in Sèvres and Paris, France. Teams from thirty-two countries took part in the competition. This was a record number of participating countries, up from last year's record of thirty countries. As the team size was six students for each country, there was also a record number of 192 participating students (including Jeremy Kahn, a 13-year old American), up from the previous record of 185 students set in 1981. The countries participating for the first time were Morocco and Spain. Italy was back again, and Outer Mongolia abstained.

Continuing with the last two years' break with tradition (see the last two I.M.O. reports in [1981: 220] and [1982: 223]), the six problems of the competition were assigned equal weights of 7 points each, for a maximum possible score of 42. The number of very low scores apparently shows that the level of difficulty for this year's competition was higher than last year's. However, there were four perfect papers, one more than last year. These were achieved by

Bernhard Leeb, West Germany;
Michael Stoll, West Germany;
Frank Wagner, West Germany;
Leonide Parnovski, The Soviet Union.

The results of the competition are announced officially only for individual team members. However, team standings are always compiled unofficially by adding up the scores of individual team members. In this unofficial team standing (see listing on following page), the West German team repeated their last year's exploit of heading the list, and this time with a quite substantial margin over the runner-up.

The West German team was coached by Arthur Engel, and the very high standings the team achieved in the last three competitions (2nd, 1st, 1st) can certainly be attributed, at least in part, to his enthusiasm, knowledge, and training methods. I hope that in a later column in this journal I can get him to describe his training program. This will provide valuable information for those countries which have not done too well.

The problems of this year's competition are given below. Solutions to these problems, along with those of the Twelfth U.S.A. Mathematical Olympiad [1983: 173] will appear later this year in a booklet, Olympiads for 1983, obtainable from

Dr. Walter E. Mientka, Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska 68588.
<table>
<thead>
<tr>
<th>Rank</th>
<th>Country</th>
<th>Score (max 252)</th>
<th>Prizes</th>
<th>Total prizes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>West Germany</td>
<td>212</td>
<td>4 1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>U.S.A.</td>
<td>171</td>
<td>1 3 2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Hungary</td>
<td>170</td>
<td>- 4 2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>U.S.S.R.</td>
<td>169</td>
<td>1 3 2</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>Romania</td>
<td>161</td>
<td>1 2 3</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>Vietnam</td>
<td>148</td>
<td>- 3 3</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>Netherlands</td>
<td>143</td>
<td>1 3 -</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>Czechoslovakia</td>
<td>142</td>
<td>1 1 3</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>Bulgaria</td>
<td>137</td>
<td>- 1 4</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>France</td>
<td>123</td>
<td>- 2 3</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>Great Britain</td>
<td>121</td>
<td>- 3 1</td>
<td>4</td>
</tr>
<tr>
<td>12</td>
<td>East Germany</td>
<td>117</td>
<td>- 5</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>Finland</td>
<td>103</td>
<td>- 3</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>Canada</td>
<td>102</td>
<td>- 4</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>Poland</td>
<td>101</td>
<td>- 3</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>Israel</td>
<td>97</td>
<td>- 5</td>
<td>5</td>
</tr>
<tr>
<td>17</td>
<td>Greece</td>
<td>96</td>
<td>- 3</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>Yugoslavia</td>
<td>89</td>
<td>- 5</td>
<td>5</td>
</tr>
<tr>
<td>19</td>
<td>Australia</td>
<td>86</td>
<td>- 1 2</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>Brazil</td>
<td>77</td>
<td>- 3</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>Sweden</td>
<td>47</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>22</td>
<td>Austria</td>
<td>45</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>23</td>
<td>Spain</td>
<td>37</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>24</td>
<td>Cuba</td>
<td>36</td>
<td>- 1</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>Morocco</td>
<td>32</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>26</td>
<td>Belgium</td>
<td>31</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>27</td>
<td>Tunisia</td>
<td>26</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>28</td>
<td>Colombia</td>
<td>21</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>29</td>
<td>Luxemburg</td>
<td>13</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>30</td>
<td>Algeria</td>
<td>6</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>31</td>
<td>Kuwait</td>
<td>4</td>
<td>- -</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>Italy</td>
<td>2</td>
<td>- -</td>
<td>-</td>
</tr>
</tbody>
</table>
Finally, although it is not yet completely official, the 1984 and 1985 Olympiads are expected to be held in Czechoslovakia and Finland, respectively.

XXIVth INTERNATIONAL MATHEMATICAL OLYMPIAD

First day: Wednesday, July 6, 1983. Time: 4½ hours

1. Find all functions \( f \) defined on the set of positive real numbers which take positive real values and satisfy the conditions:
   \( f(xf(y)) = yf(x) \) for all positive \( x, y \);
   \( f(x) \to 0 \) as \( x \to +\infty \).

2. Let \( A \) be one of the two distinct points of intersection of two unequal coplanar circles \( C_1 \) and \( C_2 \) with centres \( O_1 \) and \( O_2 \), respectively. One of the common tangents to the circles touches \( C_1 \) at \( P_1 \) and \( C_2 \) at \( P_2 \), while the other touches \( C_1 \) at \( Q_1 \) and \( C_2 \) at \( Q_2 \). Let \( M_1 \) be the midpoint of \( P_1Q_1 \) and \( M_2 \) the midpoint of \( P_2Q_2 \). Prove that the angles \( O_1A_2 \) and \( M_1A_2 \) are equal.

3. Let \( a, b, c \) be positive integers, no two of which have a common divisor greater than 1. Show that
   \[
   2abc - bc - ca - ab
   \]
   is the largest integer which cannot be expressed in the form
   \[
   xbc + yca + zab,
   \]
   where \( x, y, z \) are nonnegative integers.

Second day: Thursday, July 7, 1983. Time: 4½ hours

4. Let \( ABC \) be an equilateral triangle, and \( E \) be the set of all points contained in the three segments \( BC, CA, \) and \( AB \) (including \( A, B, \) and \( C \)). Determine whether, for every partition of \( E \) into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

5. Is it possible to choose 1983 distinct positive integers, all less than or equal to \( 10^5 \), no three of which are consecutive terms of an arithmetic progression? Justify your answer.

6. Let \( a, b, \) and \( c \) be the lengths of the sides of a triangle. Prove that
   \[
   b^2c(b-c) + c^2a(c-a) + a^2b(a-b) \geq 0.
   \]
   Determine when equality occurs.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.
Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.


SAM'S IDEA MADE SENSE

Of course, it would not be an odd IDEA, but what makes SENSE?

862. Proposed by George Tsintsifas, Thessaloniki, Greece.

P is an interior point of a triangle ABC.

Lines through P parallel to the sides of the triangle meet those sides in the points A1, A2, B1, B2, C1, C2, as shown in the figure. Prove that

(a) \[ [A_1B_1C_1] \leq \frac{1}{3}[ABC], \]

(b) \[ [A_1C_2B_1A_2C_1B_2] \geq \frac{2}{3}[ABC], \]

where the brackets denote area.


If A is an \( n \times j \) matrix with rank \( j \), and B is an \( n \times k \) matrix with rank \( k \), then the equation \( Ax = By \) has a solution other than \( x = y = 0 \) if and only if

\[
B^+B - B^+A(A^+A)^{-1}A^+B
\]

is singular.

(The \( (r,s) \) element of \( M^+ \) is the complex conjugate of the \( (s,r) \) element of \( M \).)


Find all \( x \) between 0 and \( 2\pi \) such that

\[
2 \cos^2 3x - 14 \cos^2 2x - 2 \cos 5x + 24 \cos 3x - 89 \cos 2x + 50 \cos x > 43.
\]
Proposed by Clark Kimberling, University of Evansville, Indiana.

Let $x, y, z$ be the distances from the sides $a = BC, b = CA, c = AB$, respectively, to a variable point inside a triangle $ABC$. Prove that, for $0 \neq t \neq 1$, the critical point of $x^t + y^t + z^t$ satisfies

$$x : y : z = a^p : b^p : c^p,$$

where $p = 1/(t-1)$. Discuss limiting cases.

Proposed by Jordi Dou, Barcelona, Spain.

Given a triangle $ABC$ with sides $a, b, c$, find the minimum value of

$$a \cdot XA + b \cdot XB + c \cdot XC,$$

where $X$ ranges over all the points of the plane of the triangle.

Proposed by Charles W. Trigg, San Diego, California.

A strip of four equilateral triangles can be folded along the common edges to form a regular tetrahedron with 3 open edges. (An open edge is one through which there is direct access to the centroid of the polyhedron.) How many triangles must be in the strip to form a tetrahedron with no open edges?


The graph of $x^3 + y^3 = 3axy$ is known as the folium of Descartes. Prove that the area of the loop of the folium is equal to the area of the region bounded by the folium and its asymptote $x + y + a = 0$.

Proposed by W.J. Blundon, Memorial University of Newfoundland.

(a) Prove that every integer $n > 6$ can be expressed as the sum of two relatively prime integers both of which exceed 1.

(b) Prove that every integer $n > 11$ can be expressed as the sum of two composite positive integers.

Proposed by Sidney Kravitz, Dover, New Jersey.

Of all the simple closed curves which are inscribed in a unit square (touching all four sides), find the one which has the minimum ratio of perimeter to enclosed area.

THE PUZZLE CORNER

Answer to Puzzle No. 38 [1983: 169]: Fermat as (the symbols are known as firmatas).


Answer to Puzzle No. 40 [1983: 169]: Largando (L-Argand-o).

Answer to Puzzle No. 41 [1983: 169]: Zero.
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


A triangle has sides $a, b, c$, and the medians of this triangle are used as sides of a new triangle. If $r_m$ is the inradius of this new triangle, prove or disprove that

$$r_m \leq \frac{3abc}{4(a^2+b^2+c^2)},$$

with equality just when the original triangle is equilateral.

IV. Comment by Vedula N. Marty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.

In solution III [1983: 149], the solver very neatly established the inequality

$$2R(m_a + m_b + m_c) \geq a^2 + b^2 + c^2,$$  (1)

which is equivalent to that in the proposal, from the inequalities

$$m_a/h_a \geq \frac{b^2+c^2}{2bc}, \quad m_b/h_b \geq \frac{c^2+a^2}{2ca}, \quad m_c/h_c \geq \frac{a^2+b^2}{2ab},$$  (2)

for which he gave a reference ([1] on page 149). The solver claimed that, in the first inequality in (2), for example, equality holds just when $b = c$. This is incorrect: equality holds just when $b = c$ or angle $A = \pi/2$, and this is clearly established in the reference. However, the solver's conclusion that equality holds in (1) if and only if $a = b = c$ is still valid, for in a nonequilateral triangle at least two of the inequalities (2) are strict.


Prove that, if the incentre $I$ of a triangle is equidistant from the circumcentre $O$ and the orthocentre $H$, then one angle of the triangle is $60^\circ$.

II. Comment by Bob Prielipp, University of Wisconsin-Oshkosh.

The following stronger result is known [1]:

For any triangle (other than equilateral) with circumcenter $O$, incenter $I$, and orthocenter $H$, let the angles have (degree) measures $\alpha \leq \beta \leq \gamma$. Then

$$\beta > 60^\circ \Rightarrow 0 < \frac{HI}{IO} < 1, \quad \beta = 60^\circ \Rightarrow HI = IO, \quad \beta < 60^\circ \Rightarrow 1 < \frac{HI}{IO} < 2.$$
REFERENCE

1. Anders Bager, solution to Problem E 2282 (proposed by W.J. Blundon), 

751. [1982: 173] Proposed by Alan Wayne, Pasco-Hernando Community College, 
New Port Richey, Florida.

Who utters this sound?

CALM
BAA
BAA

The sum of this decimal cryptarithmetic addition will answer the question.

I. Solution by Clayton W. Dodge, University of Maine at Orono.

This is one of those problems where the answer is suspected long before a 
solution is deduced. Clearly M is odd and, for each M, there are two values for 
A, and then L is determined. We find that

(M,A,L) = (1,2,5), (1,7,4), (3,1,7), (5,0,9), (7,4,0), (9,3,2), or (9,8,1),

with the triples (3,6,6), (5,5,8), and (7,9,9) eliminated because of duplications.
Now only (M,A,L) = (9,3,2) permits satisfactory values for B and C, B = 5 and C = 1.
The unique answer is the expected one:

2395 = LAMB.

II. "Solution" by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

It is obvious that the animal who utters BAA is LAMB, so we can avoid the 
laborious calculations so dear to the hearts of the alphametic experts and make 
the obvious substitutions L = 2, A = 3, M = 9, B = 5, and C = 1 follows immediately.

See? Alphametics are not really very difficult if you tackle them in the 
proper manner.

III. Comment by Edith Orr, Ottawa, Ontario.

Question:

Little LAMB, who made thee?
Dost thou know who made thee?

Answer:

Mary had a little sheep
And with the sheep she went to sleep.
The sheep turned out to be a ram,
Mary had a little LAMB.
Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; MEIR FEDER, Haifa, Israel; DONALD C. FULLER, Gainesville Junior College, Gainesville, Georgia; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; D.J. SMEENK, Zaltbommel, The Netherlands; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

The author of comment III probably expects that every reader will recognize that her "Question" was taken from William Blake's (1757-1827) Songs of Innocence (1789), but she may be trying to pass off the "Answer" as her own. In the interests of scholarship, we have made a search and found that her answering quatrain appears, word for word, on page 152 of a scrofulous anthology entitled The Purple Book of Locker-Room Humor, published in 1980 by Peek-A-Boo Press, Toronto, Ontario. The author is unidentified, so it might well be Edith Orr.

* * *


For \( i = 1,2,3 \), let \( A_i \) be the vertices of a triangle with angles \( \alpha_i \), sides \( a_i \), circumcenter \( O \), and inscribed circle \( \gamma \). The lines \( A_i O \) intersect \( \gamma \) in \( P_i \) and \( Q_i \).

(a) Prove that

\[
P_1Q_1 : P_2Q_2 : P_3Q_3 = f(\cos \alpha_1) : f(\cos \alpha_2) : f(\cos \alpha_3),
\]

where \( f(x) \) is a function to be determined.

(b) Prove or disprove that \( \alpha_2 = \alpha_3 \) if and only if \( P_2Q_2 = P_3Q_3 \).

Solution by the COPS of Ottawa.

(a) To ensure the determinateness of the ratios in (1), we assume that \( P_i \neq Q_i \), \( i = 1,2,3 \). This implies that \( O \) is an interior point of the triangle, which is therefore acute-angled.

Let \( r \) be the inradius, \( I \) the incentre, and let \( IN \perp OA_1 \), as shown in the figure. Since

\[
\frac{OA_1I}{\frac{1}{2} \alpha_1 - \left( \frac{\pi}{2} - \alpha_3 \right)} = \frac{1}{2} |\alpha_2 - \alpha_3|,
\]

we have

\[
P_1Q_1 = 2 \sqrt{P_1^2 - IN^2} = 2 \sqrt{r^2 - IA_1^2 \sin^2 \frac{\gamma}{2} (\alpha_2 - \alpha_3)} = 2r \sqrt{1 - \frac{\sin^2 \frac{\gamma}{2} (\alpha_2 - \alpha_3)}{r^2/IA_1^2}} = 2r \sqrt{1 - \frac{\sin^2 \frac{\gamma}{2} (\alpha_2 - \alpha_3)}{\sin^2 \frac{\gamma}{2} \alpha_1}}
\]

\[ \begin{align*}
\frac{1}{\cos \alpha_1(1 - \cos \alpha_1)} & = 2r \frac{\sqrt{1 - \frac{1 - \cos(\alpha_2 - \alpha_3)}{1 - \cos \alpha_1}}}{\sqrt{\cos \alpha_2 \cos \alpha_3}} \\
& = 2r \frac{\sqrt{\cos(\alpha_2 + \alpha_3) + \cos(\alpha_2 - \alpha_3)}}{1 - \cos \alpha_1} \\
& = 2r \frac{\sqrt{2 \cos \alpha_2 \cos \alpha_3}}{1 - \cos \alpha_1} \\
& = 2r \frac{1}{\sqrt{\cos \alpha_1(1 - \cos \alpha_1)}}.
\end{align*} \]

With similar results for \( P_2Q_2 \) and \( P_3Q_3 \), it follows that (1) holds with \( f(x) = \frac{1}{\sqrt{x(1-x)}} \).

(b) With \( f(x) \) as found above, it follows from (1) that
\[ P_2Q_2 = P_3Q_3 \iff f(\cos \alpha_2) = f(\cos \alpha_3) \]
\[ \iff \cos \alpha_2(1 - \cos \alpha_2) = \cos \alpha_3(1 - \cos \alpha_3) \]
\[ \iff (\cos \alpha_2 - \cos \alpha_3)(\cos \alpha_2 + \cos \alpha_3 - 1) = 0 \]
\[ \iff \alpha_2 = \alpha_3 \text{ or } \cos \alpha_2 + \cos \alpha_3 = 1. \]

So any triangle in which \( \alpha_2 \neq \alpha_3 \) but \( \cos \alpha_2 + \cos \alpha_3 = 1 \) provides a counterexample that disproves part (b).

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

When an \( n \times n \) matrix \( A \) is factorized as \( \Omega \Omega^{-1} \), with \( \Omega \) unitary and \( T \) upper triangular, the diagonal elements \( d_1, d_2, \ldots, d_n \) of \( T \) are such that
Can the numbers $d_i$ be permuted arbitrarily, given $A$, by suitably choosing $\Omega$?

I. Solution by the proposer.

They can: If $(t-\mu_1)(t-\mu_2)(t-\mu_n) = \det(tI-A)$, then $\Omega$ can be chosen unitary so that $\Omega^{-1}A\Omega$ is upper triangular and its diagonal elements in order are $\mu_1$, $\mu_2$, ..., $\mu_n$. The proof is by induction and depends on the following Lemma:

Suppose $W$ is an $n \times k$ matrix whose columns are orthonormal and $AW = WT$, where $T$ is upper triangular and its diagonal elements in order are $\mu_1$ to $\mu_k$; then there is a unit vector $\omega$ orthogonal to all columns of $W$ such that $A\omega = \omega + \mu_{k+1}\omega$.

Assuming this proved, we have

$$A(W,\omega) = (WT, W\omega+\mu_{k+1}\omega) = (W,\omega) \begin{pmatrix} T & x \\ 0 & \mu_{k+1} \end{pmatrix},$$

and so if $W^k = (W,\omega)$, then $W^k$ is an $n \times (k+1)$ matrix with orthonormal columns such that $AW^k = \mu_{k+1} W^k$, where $T^k$ is upper triangular and its diagonal elements in order are $\mu_1$ to $\mu_{k+1}$. Where $k = 1$, we take for $W$ any column vector which is an eigenvector of $A$ to eigenvalue $\mu_1$; by repeating the argument above we eventually reach $k = n$.

To prove the lemma, choose $V$, an $n \times (n-k)$ matrix such that $(W,V)$ is unitary. Then

$$A(W,V) = (W,V) \begin{pmatrix} T & P \\ 0 & Q \end{pmatrix}$$

for suitable $P$ and $Q$, $Q$ square. Since $\det(tI-A) = \det(tI-T) \det(tI-Q)$, it follows that $\mu_{k+1}$ is an eigenvalue of $Q$. Suppose $Qz = \mu_{k+1} z$ ($z \neq 0$), and set $\omega = Vz$ (having chosen $z$ so that $||\omega|| = 1$). Since $AV = WP + VQ$, we have $A\omega = W(Pz) + \mu_{k+1}\omega$, and with $z = Ps$ this completes the proof.

II. Comment by Kent D. Boklan, student, Massachusetts Institute of Technology.

The fact that the $n \times n$ matrix $A$ can be factored as $\Omega T \Omega^{-1}$ with $\Omega$ unitary and $T$ upper triangular (the diagonal elements being the eigenvalues of $A$) is just a restatement of Schur's Theorem. In his proof of this theorem, Strang [1] proves that the $d_i$ can be permuted arbitrarily by suitably choosing $\Omega$.

REFERENCE


*   *   *


Let $\sigma_n = A_0A_1...A_n$ be an $n$-simplex in $\mathbb{R}^n$ and $P$ an interior point of $\sigma_n$. For $i = 0, 1, ..., n$, let $A_i'$ be the intersection of the line $A_iP$ with the $(n-1)$-simplex $\sigma_{n-1} = A_0A_1...A_{i-1}A_{i+1}...A_n$.

Show that

$$\sum_{i=0}^{n} \frac{\lambda_i A_i'}{\lambda_i} = \vec{0}$$  \hspace{1cm} (1)

if and only if $P$ is the centroid of $\sigma_n$.

**Solution by the proposer.**

We assume that $n \geq 2$. If $P$ is the centroid of $\sigma_n$, then

$$nA_i' A_i = \sum_{j=0}^{n} A_i^\lambda A_j, \quad i = 0, 1, ..., n;$$

hence

$$n \sum_{i=0}^{n} A_i' A_i = \sum_{i=0}^{n} (\sum_{j=0}^{n} A_i^\lambda A_j) = \vec{0},$$

and (1) follows.

Conversely, suppose (1) holds, and let $(\lambda_0, \lambda_1, ..., \lambda_n)$ be the barycentric coordinates of $P$. We will show that $\lambda_0 = \lambda_1 = ... = \lambda_n$, from which it will follow that $P$ is the centroid of $\sigma_n$. The origin of all vectors being some point $0$ outside the $n$-space of $\sigma_n$, we have

$$\vec{P} = \sum_{i=0}^{n} \lambda_i \vec{A}_i, \quad \lambda_i > 0, \quad \sum_{i=0}^{n} \lambda_i = 1.$$

From

$$\vec{P} = \lambda_i \vec{A}_i' + (1-\lambda_i) \vec{A}_i = \lambda_i \vec{A}_i' + (1-\lambda_i) \sum_{j=0}^{n} \lambda_j \vec{A}_j / (1-\lambda_i),$$

we obtain

$$\vec{A}_i' = \sum_{j=0}^{n} \frac{\lambda_j \vec{A}_j}{1-\lambda_i}, \quad i = 0, 1, ..., n.$$
\[ \sum_{i=0}^{n} \hat{A}_i = \sum_{i=0}^{n} \hat{A}_i. \]

we have

\[ \sum_{i=0}^{n} \left( \sum_{j=0}^{n} \frac{\lambda_i \hat{A}_j}{1-\lambda_i} \right) = \sum_{i=0}^{n} \hat{A}_i. \]

Since the \( \hat{A}_i \) are linearly independent, we therefore have

\[ \sum_{i=0}^{n} \frac{\lambda_i}{1-\lambda_i} = 1, \]

from which follows

\[ \lambda_j^n = \lambda_j \sum_{i=0}^{n} \frac{1}{1-\lambda_i} = \frac{1}{1-\lambda_j}, \quad j = 0,1,\ldots,n. \]

So, for \( j = 0,1,\ldots,n \), each \( \lambda_j(1-\lambda_j) \) equals the positive constant \( 1/k \). Now let \( u \) and \( v \) be distinct integers such that \( 0 < u,v < n \). From

\[ \lambda_u(1-\lambda_u) = \lambda_v(1-\lambda_v), \]

we obtain

\[ (\lambda_u - \lambda_v)(1 - \lambda_u - \lambda_v) = 0. \]

As \( n \geq 2 \) implies that \( \lambda_u + \lambda_v < 1 \), we have \( \lambda_u = \lambda_v \), and so

\[ \lambda_0 = \lambda_1 = \ldots = \lambda_n, \]

as required.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands (partial solution); and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

For other characterizations of the centroid of a simplex, see [1] and [2].

REFERENCES


Proposed by Yang Lu and Zhang Jingzhong, China University
of Science and Technology, Hefei, Anhui, People's Republic of China.

Given three vertices $A$, $B$, and $C$ of a parallelogram, find the fourth vertex $D$, using only a rusty compass.


We write $R = P \vee Q$ if $PQR$ is an equilateral triangle with vertices labelled in counterclockwise order. It is known [1] that $P \vee Q$ can be constructed with a rusty compass when distinct points $P$ and $Q$ are given.

Given three consecutive vertices $A$, $B$, and $C$ of a parallelogram (labelled in counterclockwise order), we can construct

$$X = B \vee A, \quad Y = C \vee B, \quad \text{and} \quad D = X \vee Y.$$  \hspace{1cm} (1)

We claim that $D$ is the required fourth vertex of the parallelogram, or, equivalently, that

$$\vec{BD} = \vec{BA} + \vec{BC}. \hspace{1cm} (2)$$

To prove this, we use the well-known fact [2] that if $PQR$ is an equilateral triangle, with vertices labelled in counterclockwise order, imbedded in the complex plane, then the affixes $p$, $q$, $r$ of its vertices satisfy

$$r + pw + qw^2 = 0,$$

where $\omega = e^{2\pi i/3}$.

The affixes of the points $A, B, C, X, Y, D$ in the complex plane being represented by the corresponding lower-case italic letters, we obtain from (1)

$$x + bw + aw^2 = 0, \quad y + cw + bw^2 = 0, \quad d + xw + yw^2 = 0.$$ 

Substituting the values of $x$ and $y$ from the first two equations into the third, and noting that $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$, we obtain an equation equivalent to

$$d - b = (a - b) + (c - b),$$

from which (2) follows.

Also solved by DAN PEDOE, University of Minnesota; and the proposers.

REFERENCES


Given only two distinct points A and B, prove or disprove that the midpoint of the segment AB can be found, using only a rusty compass.


In a brief chapter entitled "Constructions using only compasses with constant opening of legs", Kostovskii [1] makes the following statement:

"Geometric constructions with compasses with a constant opening of the legs, with which it is possible to describe only circles of fixed radius R, were investigated by many scholars. A large part of the work The Book of Geometrical Constructions of the Arab mathematician Abu Yaf is devoted to this subject. Leonardo da Vinci, Cardano, Tartaglia, Ferrari and others have occupied themselves with solving construction problems using only compasses with a constant opening. ... We cannot, however, divide segments and arcs into equal parts, find proportional segments and so on, with these compasses."

However, Kostovskii does not come right out and say whether the constructions mentioned in the last sentence quoted above have been proved to be impossible, or whether the fact is simply that no construction method has yet been discovered.

Editor's comment.

Even with only a well-oiled (not rusty) compass, the problem of finding the midpoint of a segment when only its two endpoints are known is not trivial. Martin Gardner [2] says that this is problem No. 66 in Mascheroni's Geometria del compasso (1797), and he gives the simplest of Mascheroni's five solutions for this problem.

REFERENCES


Find a necessary and sufficient condition on $p, q, r$ so that the roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are the vertices of an equilateral triangle in the complex plane.
Solution by M.S. Klamkin, University of Alberta.

More generally, let \( P \) be the \( n \)-gon (possibly degenerate) the affixes of whose vertices in counterclockwise order are the roots \( x_1, x_2, \ldots, x_n \) of the equation

\[
x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0, \tag{1}
\]

where \( n \geq 3 \). For any given \( a_1 \), the centroid of \( P \) has affix \( (x_1 + x_2 + \ldots + x_n)/n = -a_1/n \). Then \( P \) is nondegenerate and regular if and only if

\[
x_i + a_1/n = \rho \omega^i, \quad i = 1, 2, \ldots, n,
\]

where \( \omega = e^{2\pi i/n} \) and \( \rho \) is a nonzero constant (whose modulus is the circumradius of \( P \)). Equivalently, \( P \) is nondegenerate and regular if and only if the \( x_i \) are the roots of

\[
(x + a_1/n)^n = \rho^n \tag{2}
\]

for some constant \( \rho \neq 0 \). Rewriting (2) in the equivalent form

\[
x^n + a_1 x^{n-1} + \binom{n}{2}(a_1/n)^2 x^{n-2} + \ldots + \binom{n}{n-1}(a_1/n)^{n-1} x + (a_1/n)^n - \rho^n = 0
\]

and comparing this with (1) yields

\[
a_i = \binom{n}{i}(a_1/n)^i, \quad i = 2, 3, \ldots, n-1 \quad \text{and} \quad a_n = (a_1/n)^n - \rho^n.
\]

These are the required necessary and sufficient conditions. Since the only constraint on \( \rho \) is \( \rho \neq 0 \), the conditions can be rewritten in the form

\[
a_1 = \frac{n \cdot a_i}{\binom{n}{i}}, \quad i = 2, 3, \ldots, n-1 \quad \text{and} \quad a_1 \neq n \cdot a_n.
\]

When \( n = 3 \) as in our problem, we obtain, in the notation of the proposal, the necessary and sufficient conditions

\[
p^2 = 3q \quad \text{and} \quad p^3 \neq 27r. \tag{3}
\]

For another kind of generalization, we find necessary and sufficient conditions on the coefficients of the equation

\[
x^3 + px^2 + qx + r = 0 \tag{4}
\]

so that the triangle \( T \) whose vertices in counterclockwise order have the roots \( x_1, x_2, x_3 \) of (4) as affixes, be directly similar to a given nondegenerate triangle \( T' \). We assume that \( T' \) is not equilateral, since otherwise the required conditions
are given by (3). We further assume without loss of generality that \( T' \) is positioned so that the affixes \( \omega_1, \omega_2, \omega_3 \) of its vertices in counterclockwise order satisfy
\[
\omega_1 + \omega_2 + \omega_3 = 0
\]
(centroid at the origin) and
\[
\omega_2 \omega_3 + \omega_3 \omega_1 + \omega_1 \omega_2 = 1
\]
(this is possible for every nonequilateral triangle).

For any given \( p \), the centroid of \( T \) has affix \( (x_1 + x_2 + x_3)/3 = -p/3 \). Hence \( T \) is directly similar to \( T' \) if and only if
\[
x_i = -\frac{p}{3} + \rho \omega_i, \quad i = 1, 2, 3,
\]
for some \( \rho \neq 0 \) (whose modulus is the ratio of similarity). We now obtain from (6)
\[
q = x_2 x_3 + x_3 x_1 + x_1 x_2 = \frac{p^2}{3} + \rho^2
\]
and
\[
-r = x_1 x_2 x_3 = -\frac{p^3}{27} - \frac{p^2}{3} + \rho^3 \omega_1 \omega_2 \omega_3.
\]
Eliminating \( \rho \) from (7) and (8) gives
\[
(2p^3 - 9pq + 27r)^2 + 27(p^2 - 3q)^3 \omega_1^2 \omega_2^2 \omega_3^2 = 0.
\]
Now \( \rho^2 = -(p^2 - 3q)/3 \) from (7), and since the only constraint on \( \rho \) is \( \rho \neq 0 \), the required necessary and sufficient conditions are \( p^2 = 3q \) and (9).

Rewriting the left member of (9) in the form
\[
F(p, q, r) = \{3p(p^2 - 3q) - (p^3 - 27r)\}^2 + 27(p^2 - 3q)^3 \omega_1^2 \omega_2^2 \omega_3^2
\]
will enable us to make a general statement valid for any given nondegenerate triangle \( T' \):

If, as before, \( \omega_1, \omega_2, \omega_3 \) are the affixes of the vertices in counterclockwise order of a given nondegenerate triangle \( T' \) with centroid at the origin, and satisfying (5) if \( T' \) is not equilateral; and if \( T \) is the triangle whose vertices in counterclockwise order have the roots \( x_1, x_2, x_3 \) of (4) as affixes, then \( T \) is directly similar to \( T' \) if and only if
\[
p^2 = 3q \quad \text{and} \quad F(p, q, r) \neq 0, \quad \text{if} \ T' \text{ is equilateral;}
\]
and
\[
p^2 \neq 3q \quad \text{and} \quad F(p, q, r) = 0, \quad \text{if} \ T' \text{ is not equilateral.}
The above method can be used to find necessary and sufficient conditions for
direct similarity with a given nondegenerate \( n \)-gon, but the algebra involved quickly
becomes prohibitive for \( n > 3 \).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; DONALD C.
FULLER, Gainesville Junior College, Gainesville, Georgia; J.T. GROENMAN, Arnhem,
The Netherlands; M.S. KLAMKIN, University of Alberta (second solution); the late
regretted VIKTORS LINIS, University of Ottawa; and the proposer.


Given are four congruent circles intersecting in a point \( O \), and a
quadrilateral \( ABCD \) circumscribing these circles with each side of the quadrilateral
tangent to two circles. Prove that quadrilateral \( ABCD \) is cyclic.

I. Solution by Leon Bankoff, Los Angeles, California.

With \( O \) as center, describe a circle congruent to the other four, the centers
of which now become the vertices of a cyclic quadrilateral \( A'B'C'D' \), as shown in
the figure. Each side of \( A'B'C'D' \) is the perpendicular bisector of the common
chord of two of the original circles, which in turn is perpendicular to their common
tangent. Hence ABCD is cyclic, since its sides are parallel to those of A'B'C'D',
with the result that ABCD has supplementary opposite angles. □

Note that ABCD and A'B'C'D' need not be homothetic, that is, they need not be
similar, even though they have four pairs of parallel sides.

II. Comment by Leroy F. Meyers, The Ohio State University.

The corresponding result does not hold for polygons with more than four sides,
in particular for pentagons. To see this, let the pentagon ABCDE be circumscribed
about the five circles of radius $\sqrt{2}$ whose centers are at

$$(1, 1), (-1, 1), (-1, -1), (1, -1), \text{ and } (\sqrt{2}, 0),$$

respectively, and which are concurrent at the origin O. The vertices of the cir­
cumscribed pentagon are the points

$A=(\sqrt{2}+\sqrt{2}-\sqrt{2}+1, \sqrt{2}+1), \quad B=(-\sqrt{2}-1, \sqrt{2}+1), \quad C=(\sqrt{2}-1, -\sqrt{2}-1), \quad D=(\sqrt{2}+\sqrt{2}+\sqrt{2}-1, -\sqrt{2}-1), \quad \text{and}$

$$E = (\sqrt{2}+\sqrt{2}-\sqrt{2}, 0).$$

The first four vertices, being the vertices of a rectangle, lie on a circle with
center $P = (\sqrt{2}-\sqrt{2}-1, 0)$, and

$$PA^2 = PB^2 = PC^2 = PD^2 = (\sqrt{2}+\sqrt{2}-\sqrt{2})^2 + (\sqrt{2}+1)^2 = 7 + \sqrt{2} + 2\sqrt{4-2\sqrt{2}},$$

while

$$PE^2 = (\sqrt{2} + \sqrt{2}-\sqrt{2} + 1)^2 = 5 + \sqrt{2} + 2\sqrt{4-2\sqrt{2}} + 2\sqrt{2}-\sqrt{2} \neq PA^2.$$

Also solved by the COPS of Ottawa; JORDI DOU, Barcelona, Spain; J.T. GROENMAN,
Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; STANLEY
RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KESIRAJU SATYANARAYANA,
Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands;
and the proposer.

* * *

WHAT PRICE COMMON LOGS?

Seen last summer, in the window of a store in Amagansett, Long Island, N.Y.,
beside a display of Royal Oak brix Instant Charcoal, a sign reading:

NATURAL
LOGS
49¢ each.

ALAN WAYNE
Holiday, Florida

* * *