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- 291 -
A SIMPLE PROOF OF THE BUTTERFLY PROBLEM
KESIRAJU SATYANARAYANA

The Butterfly Problem has been attracting attention at least since 1815, and the number of known proofs is quite large (see the extensive list of references in [1], also [2] and [3]). Assuming that mathematical lepidopterists will always be happy to meet a new specimen, we present here a simple elementary analytical proof which we believe to be new.

In its simplest form, the problem may be stated as follows (see Figure 1):

THE BUTTERFLY PROBLEM. Through the midpoint M of a chord AB of a circle, two other chords, CD and EF, are drawn. ED and CF intersect AB in P and Q, respectively. Prove that PM = MQ.

Figure 1

Proof. Let \( \Gamma_1 \) be the given circle. We introduce a rectangular coordinate system with origin M, x-axis AB, and y-axis MO, where O(0,d) is the centre of the circle. If the circle has radius r, its equation is

\[
\Sigma_1 = x^2 + (y-d)^2 - r^2 = 0.
\]

As the lines CD and EF pass through the origin, they form a degenerate conic \( \Gamma_2 \) whose equation is of the form

\[
\Sigma_2 = ax^2 + 2hxy + by^2 = 0.
\]
Now, for any \( k, \ell \),
\[
\Sigma \equiv k\Sigma_1 + \ell\Sigma_2 = 0
\]
represents a conic \( \Gamma \) through the points common to \( \Gamma_1 \) and \( \Gamma_2 \), that is, through \( C, D, E, F \); and every conic through \( C, D, E, F \) is representable in this form.

Suppose the conic \( \Sigma = 0 \) intersects \( AB \) in \( V \) and \( W \). The equation of \( AB \) is \( y = 0 \), and
\[
\Sigma_1(x,0) = x^2 + d^2 - r^2, \quad \Sigma_2(x,0) = \alpha x^2;
\]
hence the abscissas of \( V \) and \( W \) are the roots of \( \Sigma(x,0) = 0 \), that is, of
\[
k(x^2 + d^2 - r^2) + \alpha x^2 = 0.
\]
Since this equation has no first-degree term, the sum of its roots is zero, so
\[
\overline{VM} + \overline{MW} = 0, \quad \text{and}
\]
\[
\overline{VM} = \overline{MW}. \quad (1)
\]
Now (1) holds for all conics through \( C, D, E, F \), and the pair of lines \( ED, CF \) is such a conic, so \( PM = MQ \) follows from (1). \( \square \)

The pair of lines \( CE, DF \) is also a conic through \( C, D, E, F \). If these lines intersect \( AB \) in \( P' \) and \( Q' \), as shown in Figure 1, then \( P'M = MQ' \) also follows from (1).

Suppose that \( \Gamma_1 \) is, instead of a circle, an arbitrary proper conic of equation
\[
\Sigma_1 \equiv Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.
\]
With the rest of the notation as before, we have
\[
\Sigma(x,0) = k(Ax^2 + 2Hxy + C) + \alpha x^2.
\]
If the coordinates of \( A \) and \( B \) are \((-\alpha,0)\) and \((\alpha,0)\), respectively, then \( \Sigma_1(-\alpha,0) = \Gamma_1(\alpha,0) = 0 \) implies that \( C = 0 \); so the equation \( \Sigma(x,0) = 0 \) has no first-degree term, and the rest of the proof is as before. We have thus proved

**THE GENERALIZED BUTTERFLY PROBLEM.**

Through the midpoint \( M \) of a chord \( AB \) of a proper conic \( \Gamma_1 \), two other chords, \( CD \) and \( EF \), are drawn. A conic \( \Gamma \) through \( C, D, E, F \) intersects \( AB \) in \( V \) and \( W \). Prove that \( VM = MW \).

This problem is illustrated in Figure 2. A proof by projective geometry can be found in Eves [4].
REFERENCES


* * *

BIODICAL NOTE

Professor Satyanarayana, who is 84 years young, has been since 1978 one of the most prolific contributors of solutions to Crux problems, mostly, but not exclusively, in the field of geometry. He has contributed problems and solutions to other journals as well; one of his problems, in fact, was recently published in the American Mathematical Monthly (Problem E 2873 in the March 1981 issue). He has also had several articles published in The Mathematics Student.

He was born on October 6, 1897 in Malakapalli, West Godavari District, India. He holds First Class Honours B.A. and M.A. Degrees in Mathematics from Madras University. He lectured for 34 years to B.A., B.Sc., and B.Ed. classes of the Academic and Teachers' Training Colleges at Rajahmundry, and retired in 1953 as Principal of the Teachers' Training College. Since his retirement, he has published three research-oriented books on geometry:

1. Angles and In- and Ex-Elements of Triangles and Tetrahedra (1962). Rs 10/-.  
3. Dihedral Angles and In- and Ex-Elements of n-Space Simplexes (1979). Rs 10/-.

These books are available from Visalaandhra Publishing House, Vijayawada 520 004, Andhra Pradesh, India.

* * *

THE PUZZLE CORNER

Puzzle No. 5: Alphametic

FOUR is a square and FIVE is a prime,  
SIX has been perfect for quite a long time.  
Can you confirm when you've finished this poem  
That SEVEN + NINE is a palindrome?

HANS HAVERMANN, Weston, Ontario

* * *
MORE NINE-DIGIT PATTERNS PALINDROMIC PRIMES

CHARLES W. TRIGG

Delving once more (see [1]) into the set of 5172 nine-digit palindromic primes, a list of which was prepared by Jacques Sauvé on a PDP-11/45 at the University of Waterloo, we find a number of subsets of these primes with the same characteristics or patterns. Some selected subsets are given below. But first we do a bit of juggling with the digits of the cardinal number of the set, 5172.

According to some points of view, all the digits of 5172 are primes, and -(5+1)+7+2 supplies the missing prime digit, 3. Furthermore,

\[
5 - 1 + 7 + 2 = 5 - 1 \cdot 7 + 2
\]

and

\[
-5 + 1 + 7 + 2 = 5, \\
5 + 1 - 7 + 2 = 1, \\
5 - 1 + \sqrt{7} + 2 = 7, \\
5 - \sqrt{1 \cdot 7 + 2} = 2.
\]

Now to our self-appointed task. Three of the primes have eight like digits, namely: 111181111, 111191111, and 777767777.

If a nine-digit palindrome contains seven like digits, the other two digits must be like. The twenty-two such palindromic primes having nine digits fall into four different patterns. They are listed separately below according to the positions of the two like digits.

| 188888881 | 121111121 | 110111011 | 111010111 |
| 199999991 | 131111131 | 112111211 | 111515111 |
| 322222223 | 181111181 | 113111311 | 111616111 |
| 355555553 | 323333323 | 115115111 | 333434333 |
| 722222227 | 331333133 | 335335333 | 335353533 |
| 335335333 | 338338333 | 991999199 |

If a nine-digit palindrome contains six like digits and the other three digits are like, then the palindrome is divisible by 3. So there are no palindromic primes of this type.
The thirty-three palindromic primes with four like digits and five like digits include the seven smoothly undulating primes \[2\]

323232323, 383838383, 727272727, 919191919, 929292929, 979797979, 989898989.

The other twenty-six primes fall into the five patterns separately exhibited below:

\[
\begin{align*}
331111133 & \quad 100111001 & \quad 112212211 & \quad 181888181 & \quad 322323223 \\
772222277 & \quad 133111331 & \quad 118818811 & \quad 181888181 & \quad 355355355 \\
779999977 & \quad 377333773 & \quad 338838833 & \quad 322323233 & \quad 722727227 \\
997222299 & \quad 766777667 & \quad 994494949 & \quad 383888383 & \quad 911919119 \\
995555599 & \quad 944999449 & \quad 998898899 & \quad 959555959 \quad 977999779 \\
& \quad & \quad & \quad & \quad 988999889
\end{align*}
\]

Thus all nine-digit palindromic primes composed of just two distinct digits are accounted for.

There are thirty-seven palindromic primes which are permutations of five consecutive digits. Each digit except the central one appears twice in the prime. These primes are assembled below into columns according to their digit sets.

\[
\begin{align*}
102343201 & \quad 125343201 & \quad 345262543 & \quad 345676543 & \quad 745686547 & \quad 759686957 \\
312040213 & \quad 134525431 & \quad 346525643 & \quad 354767453 & \quad 746858647 & \quad 957686759 \\
320141023 & \quad 142353241 & \quad 356474653 & \quad 756848657 & \quad 967585769 \\
321404123 & \quad 153424351 & \quad 357646753 & \quad 786545687 & \quad 976858679 \\
324101423 & \quad 312545213 & \quad 736545637 & \quad 978656879 \\
& \quad \quad 315424513 & \quad \quad 745363547 & \quad \quad 986757689 \\
& \quad \quad 351242153 & \quad \quad 745636547 \quad \quad \quad \quad \\
& \quad \quad 352141253 & \quad \quad 753646357 \quad \quad \quad \quad \\
& \quad \quad 352414253 & \quad \quad 756343657 \quad \quad \quad \quad \\
& \quad \quad \quad \quad \quad \quad \quad 764535467 \quad \quad \quad \quad \quad \\
& \quad \quad \quad \quad \quad \quad \quad 765343567 \quad \quad \quad \quad \quad \\
\end{align*}
\]

The first prime in the fourth column, 345676543, is the only peak \[2\] nine-digit palindromic prime composed of consecutive digits \[3\].

The thirteen palindromic primes composed of five consecutive odd digits are:

\[
\begin{align*}
135979531 & \quad 319575913 & \quad 719535917 & \quad 913575319 \\
153979351 & \quad 371595173 & \quad 759313957 & \quad 971535179 \\
157393751 & \quad 395717593 & \quad 791535197 & \quad 973515379 \\
157939751 \\
\end{align*}
\]
As a central digit, 3 and 7 each appear four times, 1 appears three times, 9 appears twice, and 5 not at all.

There are forty-two nine-digit palindromic primes composed of digits that are powers of 2. The three composed of 1's and 2's, and the five composed of 1's and 8's, have been mentioned previously. The twenty-nine primes composed of three distinct powers, and the five containing the four digit-powers are:

<table>
<thead>
<tr>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
</tr>
</thead>
<tbody>
<tr>
<td>121414121</td>
<td>121282121</td>
<td>111484111</td>
<td>144818411e</td>
<td>122484221</td>
</tr>
<tr>
<td>122444221</td>
<td>121818121</td>
<td>111848111</td>
<td>144848411f</td>
<td>128444821</td>
</tr>
<tr>
<td>141242141</td>
<td>128121821</td>
<td>114484411a</td>
<td>148414841e</td>
<td>142888241</td>
</tr>
<tr>
<td>144212441</td>
<td>128181821</td>
<td>114818411b</td>
<td>148444841f</td>
<td>184212481</td>
</tr>
<tr>
<td></td>
<td>128282821</td>
<td>114848411c</td>
<td>148818841g</td>
<td>184222481</td>
</tr>
<tr>
<td></td>
<td></td>
<td>118414811b</td>
<td>148888841</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>118848811d</td>
<td>184414481e</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>141484141a</td>
<td>184818481g</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1418418141c</td>
<td>188141881d</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>144484441</td>
<td>188414881g</td>
<td></td>
</tr>
</tbody>
</table>

Primes followed by the same lower-case letter are permutations of the same digit set.

There are twenty-eight nine-digit palindromic primes in which every digit is a power of 3. The six primes composed of 1's and 3's, and the five primes composed of 1's and 9's have been previously mentioned. The other seventeen are:

<table>
<thead>
<tr>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
<th>Prime</th>
</tr>
</thead>
<tbody>
<tr>
<td>113939311a</td>
<td>191393191</td>
<td>319131913a</td>
<td>319999913b</td>
<td>391999193b</td>
<td></td>
</tr>
<tr>
<td>139131931a</td>
<td>193191391c</td>
<td>319191913c</td>
<td>331999133d</td>
<td>399191993b</td>
<td></td>
</tr>
<tr>
<td>139999931b</td>
<td>199393991b</td>
<td>319393913d</td>
<td>391333193</td>
<td>913939319</td>
<td></td>
</tr>
<tr>
<td>191313191</td>
<td>313999313d</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Primes followed by the same lower-case letter are permutations of the same digit set.

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3. Léo Sauvé, Editor's comment, this journal, 6 (November 1980) 289-290.

2404 Loring Street, San Diego, California 92109.

* * *

MAMA-THEMATICS

Frau Gauss, speaking of her son Carl Friedrich: "He likes to get down to the roots of things."

ALAN WAYNE, Holiday, Florida
The following problems, for which readers are invited to send me elegant solutions, are from Középiskolai Matematikai Lapok, 62 (May 1981) 208-209. They are labeled "Olympiad Preparatory Problems" and were collected by József Szikszai, Miskolc. I am grateful to Frank Papp for supplying the English versions of these problems.

1. Which of the following two numbers is larger:
   \[ \sqrt[7]{7+\sqrt{7}} - \sqrt[7]{7}, \quad \sqrt[7]{7} - \sqrt[7]{7-\sqrt{7}} \] ?

2. Justify the following assertion: If the positive numbers \( x_1, x_2, \ldots, x_n \) have product 1, then
   \[ \sum_{i=1}^{n} x_i^n \leq \prod_{i=1}^{n} x_i^{n+1} . \]

3. Determine the pairs \((m,n)\) of natural numbers for which the equation
   \[ \frac{1 - \sin^2 nx}{1 - \sin^2 \cos nx} = \sin nx \]
   has real solutions.

4. Show that
   \[ \sum_{k=1}^{n-1} \cot (kn/n) \cdot \cos^2 (kn/n) = 0 . \]

5. If \( n \) is a given natural number, solve the equation
   \[ (2x-1)^n + (1-x)^n = x^n . \]

6. If \( n \) is a given natural number, determine the largest and least values of the expression
   \[ \prod_{k=1}^{n} (2-\cos^2 \alpha_k) + \prod_{k=1}^{n} \cos^2 \alpha_k . \]

7. Show that, for nonnegative numbers \( a, b, c, d \),
   \[ (a+c)(b+d)(2a+c+d)(2b+c+d) \geq 4ac(2a+c)(2b+d) . \]

8. Let \( G \) denote the geometric mean of the \( n \) positive numbers \( a_i \) and, for natural numbers \( k \), let \( p_k \) denote the \( k \)th power mean, i.e.,
   \[ p_k = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^{k/n} \right)^{1/k} . \]
Show that
\[(n-1)a^n \leq nP_{n-1}^n - P_n^n.
\]

9. Show that, for an arbitrary pair \((n,k)\) of natural numbers, there is a unique natural number \(f(n,k)\) which satisfies the relation
\[(\sqrt{n+1} + \sqrt{n})^k = \sqrt{f(n,k)+1} + \sqrt{f(n,k)}.
\]

10. For which real numbers \(x, y\) is the following inequality satisfied?
\[\frac{3\sqrt{x^3+y^3}}{2} \geq \sqrt{\frac{x^2+y^2}{2}}.
\]

A large number of problems published earlier in this column are still awaiting a published solution. Space permitting, we would like to publish elegant solutions to as many of them as possible, and readers are invited to collaborate in this project by submitting their solutions to me. The following solution to one of the backlog problems is not particularly elegant. Readers are urged to find a better one.

J-33, [1981:144] A straight line CD and two points A and B not on the line are given. Locate the point M on this line such that \(\angle AMC = \frac{\angle BMD}{2}\).

Solution.

We begin by showing that there is on CD a unique point M such that \(\angle AMC = \frac{k}{BMD}\) for any \(k > 0\). We may assume that A and B are on the same side of CD, for otherwise we could replace one point by its mirror-image across CD.

When \(k = 1\), the construction of the point M is well known. It occupies the position \(M'\) shown in Figure 1. As M moves to the left on CD, the ratio \(\rho = \frac{\angle AMC}{\angle BMD}\) increases monotonically and becomes unbounded; and as M moves to the right on CD, \(\rho\) decreases monotonically to zero.

Hence, by continuity, for every \(k > 0\) there is a unique point M on CD for which \(\rho = k\). This point M is to the left or to the right of \(M'\) according as \(k > 1\) or \(k < 1\).

We show that when \(k = 2\), as in our problem, the point M can be constructed with straightedge and compass. Let \(A'\) and \(B'\) be the feet of the perpendiculars from A and B, respectively, upon CD, and set \(AA' = a\) and \(BB' = b\), as shown in Figure 2.
We orient line CD so that $A'B' = \sigma$ is positive or negative according as $B$ is to the right or to the left of $A$. In any case we have $a \cot 2\theta + b \cot \theta = \sigma$.

Using a familiar trigonometric identity for $\cot 2\theta$, this equation is easily shown to be equivalent to

$$(a + 2b) \cot^2 \theta - 2a \cot \theta - \sigma = 0,$$

from which we get

$$MB' = b \cot \theta = \frac{b(a + \sqrt{a^2 + a(a + 2b)})}{a + 2b},$$

which shows that the point $M$ is constructible with straightedge and compass. Note that we have used the positive root of (1) for $\cot \theta$ because $\angle AMC = 2\theta < 180^\circ$ implies that $\angle BMD = \theta < 90^\circ$. □

There may be an elegant construction that avoids the straightforward but tedious Euclidean construction of (2). Readers are invited to find one and send it to me.

---

**THE PUZZLE CORNER**

**Puzzle No. 6:** Rebus (*5 8)

EY, EY, EY, ...

The ALL we mathematicians may define
As certain fractions ordered in a line.

**Puzzle No. 7:** Rebus (*7)

H/O

Not everything, you see,
In ALL's philosophy.

ALAN WAYNE, Holiday, Florida

---

Readers are urged to verify on the front page of this issue that the addresses of the editor (Léo Sauvé) and managing editor (F.G.B. Maskell) are different, being on different campuses of Algonquin College. The appropriate address should be used in each case to ensure safe and prompt arrival of readers' communications.
PROBLEMS — PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

691. Proposed by J.A. McCallum, Medicine Hat, Alberta.

Here is an alphametic about the man who coined the word "alphametic", the well-known author of the syndicated column Fun With Figures, J.A.H. Hunter:

\[
\begin{align*}
H & E'S \\
T & H & A & T \\
F & U & N \\
M & A & T & H & S. \\
H & U & N & T & E & R
\end{align*}
\]

The apostrophe has no mathematical significance and the answer, like the man himself, is unique.

692. Proposed by Dan Sokolowsky, California State University at Los Angeles.

\( S_n \) is a set of \( n \) distinct objects. For a fixed \( k \geq 1 \), \( 2k \) subsets of \( S_n \) are denoted by \( A_i, B_i \), \( i = 1, \ldots, k \). Find the largest possible value of \( n \) for which the following conditions (a)-(d) can hold simultaneously for \( i = 1, \ldots, k \):

(a) \( A_i \cup B_i = S_n \).

(b) \( A_i \cap B_i = \emptyset \).

(c) For each pair of distinct elements of \( S_n \), there exists an \( i \) such that the two elements are either both in \( A_i \) or both in \( B_i \).

(d) For each pair of distinct elements of \( S_n \), there exists an \( i \) such that one of the two elements is in \( A_i \) and the other is in \( B_i \).

693*: Proposed by Ferrell Wheeler, student, Texas A & M University.

On a 4x4 tick-tack-toe board, a winning path consists of four squares in a row, column, or diagonal. In how many ways can three X's be placed on the board, not all on the same winning path, so that if a game is played on this partly-filled board, X going first, then X can absolutely force a win?
302

694: Proposed by Jack Garfunkel, Flushing, N.Y.

Three congruent circles with radical center R lie inside a given triangle with incenter I and circumcenter O. Each circle touches a pair of sides of the triangle. Prove that O, R, and I are collinear.

(This generalizes Problem 5 of the 1981 International Mathematical Olympiad [1981: 223], where it was specified that the three circles had a common point.)


For \( i = 1, 2, 3 \), \( A_i \) are the vertices of a triangle with sides \( a_i \) and excircles with centers \( I_i \) touching \( a_i \) in \( B_i \). For \( j, k \neq i \), \( M_i \) are the midpoints and \( m_i \) the right bisectors of \( B_i B_k \). Prove that the \( m_i \) are concurrent.

696: Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \( \triangle ABC \) be a triangle; \( a, b, c \) its sides; and \( s, r, R \) its semiperimeter, inradius and circumradius. Prove that, with sums cyclic over \( A, B, C \),

(a) \( \frac{a}{s} + \frac{1}{2} \cos \frac{1}{2}(B-C) \geq \cos A; \)
(b) \( \frac{\pi}{2} \cos \frac{1}{2}(B-C) \geq \frac{s}{1 + 2r/R}. \)

697: Proposed by G.C. Giri, Midnapore College, West Bengal, India.

Let

\[
\begin{align*}
 a &= \tan \theta + \tan \phi , \\
b &= \sec \theta + \sec \phi , \\
c &= \csc \theta + \csc \phi .
\end{align*}
\]

If the angles \( \theta \) and \( \phi \) are such that the requisite functions are defined and \( bc \neq 0 \), show that \( 2a/bc < 1 \).

698: Proposé par Hippolyte Charles, Waterloo, Québec.

Les sommes partielles de la série harmonique (laquelle, on le sait bien, est divergente) sont définies par

\[
\varepsilon_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.
\]

La série \( \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n} \) est-elle convergente ou divergente?

699: Proposed by Charles W. Trigg, San Diego, California.

A quadrilateral is inscribed in a circle. One side is a diameter of the circle and the other sides have lengths of 3, 4, and 5. What is the length of the diameter of the circle?

700: Proposed by Jordi Dou, Barcelona, Spain.

Construct the centre of the ellipse of minimum eccentricity circumscribed to a given convex quadrilateral.
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


Consider the following three inequalities for the angles A, B, C of a triangle:

\[ \cos \frac{B-C}{2} \cos \frac{C-A}{2} \cos \frac{A-B}{2} \geq 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, \]  
\[ \csc \frac{A}{2} \cos \frac{B-C}{2} + \csc \frac{B}{2} \cos \frac{C-A}{2} + \csc \frac{C}{2} \cos \frac{A-B}{2} \geq 6, \]  
\[ \csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2} \geq 6. \]

Inequality (3) is well-known (American Mathematical Monthly 66 (1959) 916) and it is trivially implied by (2). Prove (1) and show that (1) implies (2).

Solution by M.S. Klamkin, University of Alberta.

We will show that inequalities (1) and (2) are just disguised forms of the two well-known elementary inequalities

\[ (b+c)(c+a)(a+b) \geq 8abc, \]  
\[ bc(b+c) + ca(c+a) + ab(a+b) \geq 6abc, \]

which are valid for arbitrary nonnegative real values of a, b, c. These are easily established by the arithmetic-geometric mean inequality (or see Bottema et al. [1]). If a, b, c are the sides of a triangle, then (1') and (2') are equivalent to

\[ \frac{b+c}{a} \cdot \frac{a+c}{b} \cdot \frac{a+b}{c} \geq 8 \]  
\[ \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} \geq 6. \]

Now, by the law of sines,

\[ \frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2 \cos (A/2) \cos (B-C)/2}{2 \sin (A/2) \cos (A/2)} = \frac{\cos (B-C)/2}{\sin (A/2)}. \]

If we substitute this and two similar expressions in (1'') and (2''), we obtain (1) and (2).

To show that (1') implies (2'), and hence that (1) implies (2), we apply the arithmetic-geometric inequality: with sum and product cyclic over a, b, c, we have

\[ \Pi bc(b+c) \geq 3\Pi bc(b+c)^{1/3} \geq 6abc. \]

For the geometrical significance of (1), we refer to an exercise in Todhunter [2] and an inequality of Gridasov [3]. In the exercise, one is to show that if...
the bisectors of angles $A, B, C$ of a triangle meet the opposite sides in $D, E, F$, respectively, then
\[
\frac{[DEF]}{[ABC]} = 2 \pi \frac{\sin (A/2)}{\cos (B-C)/2}, \tag{5}
\]
where $[DEF]$ denotes the area of triangle $DEF$, etc. Gridasov's inequality is that $4[DEF] \leq [ABC]$, which follows from (1) and (5). This inequality has been extended by this author to arbitrary concurrent cevians for simplexes [4]. (See also this journal [1978: 255-256].)

To establish (5), we use an elementary result given in [4], viz.,
\[
\frac{[DEF]}{[ABC]} = \frac{2xyz}{(y+z)(z+x)(x+y)}, \tag{6}
\]
where $x, y, z$ are the barycentric coordinates of the point $P$ of concurrency of the three cevians. Here we have
\[
P = x\mathbf{A} + y\mathbf{B} + z\mathbf{C}, \quad x, y, z \geq 0, \quad x+y+z = 1,
\]
where the vectors to $P$ and the vertices are taken from an origin outside the plane of the triangle. If the cevians are angle bisectors, then $x = a, y = b, z = c$ (assuming that the side lengths have been normalized so that $a+b+c = 1$), and (5) follows from (6) and (4).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland (two solutions); J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus (two solutions); NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

The proposer gave without proof the following geometric equivalent of (1):

Let $I$ be the incenter of triangle $ABC$, and let the bisectors of angles $A, B, C$ meet the circumcircle again in $R, S, T$, respectively. Then
\[
AR \cdot BS \cdot CT \geq 8 \cdot IR \cdot IS \cdot IT.
\]

REFERENCES

(a) Given a natural number \( n \), show that the equation

\[
9n^3 = 6abn + ab(a+b)
\]

has no solution in natural numbers \( a \) and \( b \).

(b) Using (a), or otherwise, show that none of the following expressions is a perfect square for any natural number \( n \):

\[
\begin{align*}
36n^3 + 36n^2 + 12n + 1, \\
12n^3 + 36n^2 + 36n + 9, \\
4n^3 + 36n^2 + 108n + 81.
\end{align*}
\]

**Solution by the proposer.**

(a) Equation (1) is equivalent to

\[
(3n+a)^3 + (3n+b)^3 = (3n+a+b)^3
\]

which, by Fermat's Last Theorem, has no solution in natural numbers (or even in positive rationals). Hence, for any given \( n \), equation (1) has no solution in natural numbers \( a \) and \( b \).

(b) Solving (1) for \( a \), we get

\[
a = \frac{-b(b+6n) + \sqrt{\Delta}}{2b},
\]

where

\[
\Delta = b^2(b+6n)^2 + 36bn^3.
\]

A necessary condition for the natural number \( b \) to be part of a solution \((a,b)\) of (1) is that \( b|9n^3 \). For such a natural number \( b \), \( \Delta \) cannot be a perfect square; otherwise \((a,b)\), with \( a \) given by (2), would be a positive rational solution of (1), which is impossible. In particular, for \( b = 1, 3, 9 \), the values of \( \Delta \), \( \Delta/9 \), and \( \Delta/81 \), respectively, viz.,

\[
\begin{align*}
36n^3 + 36n^2 + 12n + 1, \\
12n^3 + 36n^2 + 36n + 9, \\
4n^3 + 36n^2 + 108n + 81,
\end{align*}
\]

are not perfect squares for any natural number \( n \).

Also solved by KENNETH M. WILKE, Topeka, Kansas.

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Let \( \sigma = A_0A_1\ldots A_n \) be an \( n \)-simplex in \( \mathbb{R}^n \). A straight line cuts the \((n-1)\)-dimensional faces

\[
\sigma_i = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n, \quad i = 0, 1, \ldots n
\]
in the points $B_i$. If $M_i$ is the midpoint of the straight line segment $A_iB_i$, show that all the points $M_i$ lie in the same $(n-1)$-dimensional plane.

I. Comment by M.S. Klamkin, University of Alberta.

Coincidentally, I had proposed the same problem in *Elemente der Mathematik*, and a simple solution by I. Paasche was published in that journal [31 (1976) 14-15]. This problem extends the known results for $n = 2, 3$ for which the midpoints are collinear and coplanar, respectively.

II. Comment by Hessel Pot, Woerden, The Netherlands.

The special case $n = 2$ brings a question to mind. Starting with a triangle and a line, the three midpoints are again on a line, so the process can be repeated with this new line. Is there any sort of convergence or regularity when the process is repeated indefinitely?

Solutions were received from M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; HESSEL POT, Woerden, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.


Given is a triangle $ABC$ with internal angle bisectors $t_a$, $t_b$, $t_c$ and medians $m_a$, $m_b$, $m_c$ to sides $a$, $b$, $c$, respectively. If

$$m_a \cap t_b = P, m_b \cap t_c = Q, m_c \cap t_a = R,$$

and $L, M, N$ are the midpoints of the sides $a, b, c$, respectively, prove that

$$\frac{AP \cdot BQ \cdot CR}{PL \cdot QM \cdot RN} = 8.$$

Solution by Poland H. Eddy, Memorial University of Newfoundland.

Since $BP$ bisects angle $B$ in triangle $ABL$, we have $AP/PL = c/(a/2) = 2c/a$. With this and two similar results, we have

$$\frac{AP \cdot BQ \cdot CR}{PL \cdot QM \cdot RN} = \frac{2c}{a} \cdot \frac{2a}{b} \cdot \frac{2b}{c} = 8. \quad \Box$$

We show that if we replace the medians by the altitudes (when the triangle is acute-angled), the Gergonne cevians, or the Nagel cevians, we obtain

$$\Pi \equiv \frac{AP \cdot BQ \cdot CR}{PL \cdot QM \cdot RN} \geq 8.$$

For the altitudes we have $AP/PL = c/c \cos B = \sec B$ and two similar results, from which

$$\Pi = \frac{\sec A \cdot \sec B \cdot \sec C}{a \cdot b \cdot c} \geq 8. \quad (1)$$

The Gergonne cevians join the vertices to the points of contact of the incircle with the opposite sides. They are concurrent in the Gergonne point of
the triangle. The Nagel cevians join the vertices to the points of contact with
the opposite sides of the excircles relative to those sides. They are concurrent
in the Nagel point of the triangle. In both the Gergonne and the Nagel cases, we
find \( AP/PL = \frac{c}{(s-c)} \) and two similar results, from which

\[
\Pi = \frac{abc}{(s-a)(s-b)(s-c)} \geq 8. \tag{2}
\]

Inequalities (1) and (2) can be found in O. Bottema et al., Geometric Inequalities,

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W.
DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; G.C. GIRI,
Midnapore College, West Bengal, India; J.T GROENMAN, Arnhem, The Netherlands;
FRED A. MILLER, Elkins, West Virginia; NGO TAN, student, J.F Kennedy H.S., Bronx,
N.Y.; HESSEL POT, Woerden, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal
Colony, Hyderabad, India; MALCOLM A. SMITH, Georgia Southern College, Statesboro,
Georgia; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A.
STUMP, Hopewell, Virginia; and the proposer.

Editor's comment.


\[ \ast \ast \ast \]


In a triangle ABC with semiperimeter \( s \), sides of lengths \( a, b, c \), and
medians of lengths \( m_a, m_b, m_c \), prove that:

(a) There exists a triangle with sides of lengths \( a(s-a), b(s-b), c(s-c) \).

(b) \( \left( \frac{m_a}{a} \right)^2 + \left( \frac{m_b}{b} \right)^2 + \left( \frac{m_c}{c} \right)^2 \geq \frac{9}{4} \), with equality if and only if the
triangle is equilateral.

Solution by M.S. Klamkin, University of Alberta.

(a) The desired result follows immediately if we set

\[ x = s-a > 0, \quad y = s-b > 0, \quad z = s-c > 0, \]

for then the triangle inequality

\[ b(s-b) + c(s-c) > a(s-a), \]

for example, becomes \( y(x+y) + z(x+z) > x(y+z) \), which is equivalent to \( 2yz > 0 \).

(b) With \( 4m_a^2 = 2b^2 + 2c^2 - a^2 \), etc., the required inequality is easily found to
be equivalent to

\[ \left( \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \right) + \left( \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \right) \geq 6, \]

which is true since each quantity in parentheses is at least 3 by the A.M.-G.M.
inequality, with equality if and only if \( a = b = c \). \( \square \)
A dual inequality to (b) is
\[(a/m_a)^2 + (b/m_b)^2 + (c/m_c)^2 \geq 4.\]

More generally, for any triangle inequality
\[I(a,b,c,m_a,m_b,m_c) \geq 0\]
we have the dual inequality
\[I(m_a,m_b,m_c,\frac{3a}{2},\frac{3b}{2},\frac{3c}{2}) \geq 0,\]
because the three medians of a triangle also form a triangle whose medians are the respective sides of the original triangle (see Nathan Altshiller Court, *College Geometry*, Barnes and Noble, New York, 1952, p.66).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; and the proposer.

Find all real solutions of the equation \([x^3] - 3[x^2] + 2[x] = 0,\) where the brackets denote the greatest integer function.

Solution by Hessel Pot, Woerden, The Netherlands.
The step function defined by \(f(x) = [x^3] - 3[x^2] + 2[x]\) for all real \(x\) obviously satisfies \(f(x) < 0\) for \(x < 0.\) If \(x \geq 3,\) then \([x^3] \geq [3x^2] \geq 3[x^2]\) and \(f(x) \geq 2[x] > 0.\)
So the solutions of the equation \(f(x) = 0\) all lie in the interval \([0,3),\) to which the subsequent discussion is restricted.
The step function is continuous (from the right) at \(x = 0,\) continuous at \(x = 1\) and \(x = 2\) since, for example,
\[f(1+\epsilon) = f(1) - f(1-\epsilon)+1-3+2 = f(1-\epsilon)\]
when \(\epsilon\) is a small positive number, and continuous from the right at all points of discontinuity, which are the square roots \(s\) and the cube roots \(c\) of integers in the intervals \((1,2)\) and \((2,3).\) As \(x\) increases from 1 to 3, \(f(x)\) decreases by 3 when \(x = s\) (i.e., when \(x^6\) is a cube not a square), and it increases by 1 when \(x = c\) (i.e., when \(x^6\) is a square not a cube). It is now easy to evaluate mentally \(f(x)\) when \(x^6\) ranges along the combined ascending sequence of squares and cubes less than \(3^6.\) Part of this sequence, with the corresponding values of \(f(x),\) is tabulated at the top of the next page.
\[ x^6 = 0 \quad 1^6 \quad 2^3 \quad 2^4 \quad 3^2 \quad 3^3 \quad 5^2 \quad 5^3 \quad 6^2 \quad 7^2 \]
\[ f(x) = 0 \quad 0 \quad 1 \quad -2 \quad -1 \quad 0 \quad 1 \quad -2 \quad -1 \quad 0 \]
\[ x^6 = 2^6 \quad 9^2 \quad 10^2 \quad 11^2 \quad 5^3 \quad 12^2 \quad 13^2 \quad 14^2 \quad 6^3 \quad 15^2 \]
\[ f(x) = 0 \quad 1 \quad 2 \quad 3 \quad 0 \quad 1 \quad 2 \quad 3 \quad 0 \quad 1 \]

The rest of the sequence, \(16^2, 17^2, \ldots, 26^2\), produces no more zeros for \(f\) because it contains at least three \(f\)-increasing squares between two succeeding \(f\)-decreasing cubes.

The above tabulation shows that the required solution set is

\[ \{0, \sqrt{2}\} \cup \{\sqrt{5}, \sqrt{5}\} \cup \{\sqrt{7}, \sqrt{9}\} \cup \{\sqrt{5}, \sqrt{12}\} \cup \{\sqrt{5}, \sqrt{15}\} \]

Also solved by JORDI DOU, Barcelona, Spain; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; HARRY L. NELSON, Livermore, California; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

editor's comment.

One solver wondered why December was a good month to publish this cryptarithm. The answer is, obviously, because the December 1980 issue of Crux was delivered to readers in early January 1981, just around the Feast of Epiphany.

(a) Given a segment AB of length \( l \), and a rusty compass of fixed opening \( r \), show how to find a point C such that the length of AC is the mean proportional between \( r \) and \( l \), by use of the rusty compass only, if \( \frac{1}{2}l \leq r \leq l \) but \( r \neq \frac{1}{2}l \).

(b) Show that the construction is impossible if \( r = \frac{1}{2}l \).

(c) Is the construction possible if \( r < \frac{1}{2}l \) or \( r > l \)?

(This problem was inspired by Dan Pedoe's Problem 492.)

Solution of parts (a) and (b) by the proposer.

Since only one radius is possible, it will be unambiguous and convenient to denote by (P) a circle with center P and radius \( r \).

(a) The circles (A) and (B) intersect the segment AB in unique points A' and B', respectively. Since \( \frac{1}{2}l \leq r \leq l \) and \( r \neq \frac{1}{2}l \), we have \( 0 < A'B' \leq 2r \), and so the circles (A') and (B') intersect. Let C be one of the points of intersection, and let D be the midpoint of AB. Then CD \( \perp \) AB and

\[
AC^2 = AD^2 + CD^2 = AD^2 + A'C^2 - A'D^2
\]

\[
= (\frac{1}{2}l)^2 + r^2 - (\frac{1}{2}l-r)^2 = lr.
\]

Hence AC is the mean proportional between \( l \) and \( r \). The rusty compass was used exactly four times. Note that if \( r = \frac{1}{4}l \), then the triangle ACD is degenerate, but the calculation goes through; if \( r = \frac{1}{2}l \), then A' = B and B' = A, so that C is a point of intersection of the first two circles drawn.

(b) If \( r = \frac{3}{4}l \), then the circles (A) and (B) are tangent at the midpoint D of AB. In the above notation, A' = B' = D, so that the circles (A') and (B') are not distinct, and the construction, if possible, must be continued in a different way. However, the only points which can be obtained successively as intersections of circles with centers already determined are those of the triangular lattice of side length \( r \). The three smallest distances between any two of these points are \( r, r\sqrt{3}, \) and \( 2r \), none of which is the mean proportional \( r\sqrt{2} \). Hence the mean proportional cannot be determined by rusty compass alone.

* * *


Grandpa is 100 years old and his memory is fading. He remembers that last year — or was it the year before that? — there was a big birthday party in his honor, each guest giving him a number of beads equal to his age. The total number of beads was a five-digit number, \( x67y2 \), but to his chagrin he cannot recall what \( x \) and \( y \) stand for. How many guests were at the party?

Suppose the big party occurred last year, when Grandpa was 99. Then, with $0 < x \leq 9$ and $0 \leq y \leq 9$, we have

$$x67y2 \equiv x + 10y + 69 \equiv x + 10y - 30 \equiv 0 \pmod{99},$$

which implies that $x+10y = 30$ and $10|x$. So there is no solution and the big party must have occurred the year before, when Grandpa was 98. Now we have

$$x67y2 \equiv 4x + 10y + 38 \equiv 4x + 10y - 60 \equiv 0 \pmod{98}.$$

The acceptable values of $x$ and $y$ must therefore satisfy $2x+5y = 30$, and the only solution is $x = 5$, $y = 4$.

The number of guests was thus $56742/98 = 579$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.C. GIRI, Midnapore College, West Bengal, India; HANS HAVERMANN, Weston, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec (deux solutions); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.


Let $N$ be a natural number which is not a perfect cube. Investigate the existence, nature, and number of solutions of either or both of the Diophantine equations

$$x^3 - Ny^3 = \pm 1$$

in positive integers $x$ and $y$.

Comment by Kenneth M. Wilke, Topeka, Kansas.

We will let $x$ and $y$ range over the set $\mathbb{Z}$ of all rational integers. The solutions of the two equations in the proposal, if any, are then included among those of the Diophantine equation

$$x^3 + dy^3 = 1, \quad (1)$$

which has been extensively studied, especially by B. Delaunay, whose methods were later refined and extended by T. Nagell.

We quote from Mordell [1]. "The integer solutions [of (1)] are trivial when $d$ is a perfect cube. Then if $|d| > 1$, the only solution is $x = 1, y = 0$, and when $|d| = 1$, there is another solution $x = 0, dy = 1$. We may suppose now that $d > 1$ and is free from cubed factors since these can be absorbed in $y^3$. We consider the
The cubic field $K = \mathbb{Q}(\sqrt[3]{d})$. The integers in $K$ of the form $x + y\sqrt[3]{d} + z\sqrt[3]{d^2}$, where $x, y, z$ are rational integers, form a ring $\mathbb{Z}[\sqrt[3]{d}]$, the units in which are those integers $\eta$ whose norms $N(\eta) = \pm 1$. Let $\varepsilon$ be the fundamental unit in the ring chosen so that $0 < \varepsilon < 1$. Then all the units in $\mathbb{Z}[\sqrt[3]{d}]$ are given by $\eta = \pm \varepsilon^n$, where $n$ takes all integer values.

Mordell then goes on to discuss and prove Delaunay's result:

The equation $x^3 + dy^3 = 1$ ($d > 1$) has at most one integer solution with $xy \neq 0$. This is given by the fundamental unit in the ring when it is a binomial unit, i.e., $\varepsilon$ takes the form $\varepsilon = x + y\sqrt[3]{d}$.

Mordell then gives Nagell's more comprehensive result as it applies to the more general equation $ax^3 + by^3 = c$. The Delaunay-Nagell Theorem as it applies to equation (1) is given by LeVeque [2] as follows (adjusted only for notation):

The equation $x^3 + dy^3 = 1$ has at most one solution in integers $x, y \neq 0$. If $(x_1, y_1)$ is a solution, the number $x_1 + y_1\sqrt[3]{d}$ is either the fundamental unit of $K = \mathbb{Q}(\sqrt[3]{d})$ or its square; the latter can happen for only finitely many values of $d$.

See Cohn [3] for a discussion of values of $d$ for which (1) has no nontrivial solution.

A comment was also received from HERMAN NYON, Paramaribo, Surinam.

REFERENCES


Let $f(x, y) = a^2 \cos x \cos y + a(\sin x + \sin y) + 1$. Prove that

$$f(\beta, y) = 0 \quad \text{and} \quad f(y, \alpha) = 0 \implies f(\alpha, \beta) = 0.$$ 

Solution by the proposer.

It follows from the hypothesis that $\theta = \alpha$ and $\theta = \beta$ are solutions of the equation

$$a^2 \cos \gamma \cos \theta + a(\sin \gamma + \sin \theta) + 1 = 0$$

and hence of

$$a^4 \cos^2 \gamma \cos^2 \theta = \{a(\sin \gamma + \sin \theta) + 1\}^2 \quad (1)$$

as well as of
\[ a^2 \sin^2 \theta = (a^2 \cos \gamma \cos \theta + a \sin \gamma + 1)^2. \quad (2) \]

With \( \cos^2 \theta = 1 - \sin^2 \theta \), (1) is equivalent to a quadratic in \( \sin \theta \), for which the sum of the roots is

\[ \sin \alpha + \sin \beta = -\frac{2(1 + a \sin \gamma)}{a(1 + a^2 \cos^2 \gamma)}. \quad (3) \]

With \( \sin^2 \theta = 1 - \cos^2 \theta \), (2) is equivalent to a quadratic in \( \cos \theta \) for which the product of the roots is

\[ \cos \alpha \cos \beta = \frac{(1 + a \sin \gamma)^2 - a^2}{a^2(1 + a^2 \cos^2 \gamma)}. \quad (4) \]

Now, from (3) and (4),

\[ a^2 \cos \alpha \cos \beta + a(\sin \alpha + \sin \beta) + 1 = 0, \]

that is, \( f(\alpha, \beta) = 0 \).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.


Automorphic numbers were discussed in my comment II to Crux 321 [1978: 2521]. An automorphic number (in base ten) is a positive integer \( k \) whose square ends in \( k \). (Initial zeros are permitted.) Not counting the trivial solutions 1, 01, 001, ..., there are exactly two \( n \)-digit automorphic numbers for each positive integer \( n \). Examples are

5, 6, 25, 76, 376, 625, 0625, 9376.

For an arbitrary positive integer \( n \), find explicit formulas for the two nontrivial \( n \)-digit automorphic numbers.

Solution by Robert A. Stump, Hopewell, Virginia (revised by the editor).

If \( k > 1 \) is an \( n \)-digit automorphic number, then, by definition,

\[ k^2 - k = k(k-1) \equiv 0 \pmod{10^n}; \quad (1) \]

so, since \((k, k-1) = 1\) and \(10 \nmid k\), either

\[ 5^n \mid k \quad \text{and} \quad 2^n \mid k-1 \quad (2) \]

or

\[ 2^n \mid k \quad \text{and} \quad 5^n \mid k-1. \quad (3) \]

There is at most one \( n \)-digit number \( k \) satisfying (2); for if \( k' \) is also such a number, then \( k-k' \equiv 0 \pmod{10^n} \), and \( k = k' \) since \(|k-k'| < 10^n\). This number, if it exists, will be denoted by \( a_n \). Similarly, there is at most one \( n \)-digit number \( b_n \) satisfying (3).

Having shown their uniqueness, we now show that \( a_n \) and \( b_n \) exist for every positive integer \( n \). In fact, we show that, explicitly,
\[ a_n = \text{the number formed by the last } n \text{ digits of } 5^{2^n - 1} \quad (4) \]

and

\[ b_n = \text{the number formed by the last } n \text{ digits of } 2^{4 \cdot 5^n - 1} \quad (5) \]

First, observe that if a positive integer \( k \) satisfies (2) or (3) for some positive integer \( n \), then \( k \) has at least \( n \) digits, it satisfies (1), and the number formed by its last \( n \) digits is automorphic. Now \( 2^n - 1 \geq n \) for every \( n \) and, from Euler's generalization of Fermat's Theorem,

\[ 5^{2^n - 1} = 5^{\phi(2^n)} \equiv 1 \pmod{2^n}; \]

so (2) holds for \( k = 5^{2^n - 1} \) and (4) is established. The proof of (5) is similar. It is based on

\[ 2^{4 \cdot 5^n - 1} = 2^{\phi(5^n)} \equiv 1 \pmod{5^n} \]

and on the fact that \( 4 \cdot 5^n - 1 \geq n \) for every \( n \). \( \square \)

It follows from (2) and (3) that

\[ 10^n | a_n b_n \quad \text{and} \quad 10^n | (a_n - 1)(b_n - 1); \]

hence

\[ a_n b_n - (a_n - 1)(b_n - 1) + 1 = a_n + b_n \equiv 1 \pmod{10^n}; \]

and since \( 1 < a_n < 10^n \) and \( 1 < b_n < 10^n \), so that \( 2 < a_n + b_n < 2 \cdot 10^n \), we conclude that

\[ a_n + b_n = 10^n + 1. \quad (6) \]

So when one of the two \( n \)-digit automorphic numbers has been calculated, the other can be found more easily from (6).

If the positive integer \( x \) has at least \( n \) digits, the number formed by its last \( n \) digits is

\[ x - 10^n \lfloor x/10^n \rfloor. \quad (7) \]

So if a more mathematically explicit formulation is required for (4) and (5), e.g., for a computer who does not "speak English", one can always substitute

\[ x = 5^{2^n - 1} \quad \text{or} \quad 2^{4 \cdot 5^n - 1} \]

in (7) to obtain \( a_n \) or \( b_n \).

Also solved by the proposer. Comments were received from HAYO AHLBURG, Benidorm, Alicante, Spain; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; HEPMAN NYON, Paramaribo, Surinam; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

The proposal was quite explicit in asking for explicit formulas for \( a_n \) and \( b_n \). Yet only our featured solver and the proposer addressed themselves specifically.
to that question. Of the other "comments" received, some gave references (the most important of which had already appeared in this journal [1978: 254]) where limited lists of automorphic numbers can be found, and others showed how to calculate a few automorphic numbers (or even infinitely many, by recurrence). Strictly speaking, all these discussions are beside the point here if they don't (and they didn't) lead to explicit formulas for $a_n$ and $b_n$.


Consider the equalities

$$\sqrt{\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \quad \text{and} \quad \sqrt{\frac{b}{c}} = a\sqrt{\frac{b}{a}}.$$ 

The first occurs in W. Knight's item "...But Don't Tell Your Students" [1980: 240], which inspired this problem. Find all positive integer triples $(a, b, c)$, with $b$ and $c$ square-free and $(b, c) = 1$, that satisfy the second.

Solution by Leroy F. Meyers, The Ohio State University.

Suppose that $a, b, c$ are positive integers such that $b$ and $c$ are square-free and relatively prime, and

$$\sqrt{a + \frac{b}{c}} = a\sqrt{\frac{b}{a}}. \quad (1)$$

Then $ac = b(a^2 - 1)$. Since $(a, a^2 - 1) = (b, c) = 1$, we must have $a | b$ and $b | a$, so $a = b$ and $c = a^2 - 1$. Hence $a$, like $b$, is square-free, and so are $a + 1$ and $a - 1$ since their product $c$ is square-free. Thus we have the necessary conditions:

$$a - 1, a, a + 1 \text{ are all square-free; } b = a; \; c = a^2 - 1. \quad (2)$$

These conditions are also sufficient. For suppose $a, b, c$ are positive integers satisfying (2). Then (1) holds, $b$ is square-free, $(b, c) = (a, a^2 - 1) = 1$, and we have only left to show that $c$ is square-free. Observe that the square-free numbers $a + 1$ and $a - 1$ must both be odd (otherwise one would be divisible by 4), so their gcd must be odd. Since this gcd divides their difference 2, it must be 1. The square-free numbers $a + 1$ and $a - 1$ are therefore relatively prime, and their product $c$ is square-free.

We conclude that the triple $(a, b, c)$ is a solution to our problem if and only if it satisfies (2). There are infinitely many solutions, for Sierpiński [1] affirms: "One can prove that there exist infinitely many triples of consecutive natural numbers such that each of the numbers is square-free." The first few values of $a$ leading to solutions are: 2, 6, 14, 22, 30, 34, 38, 42, 58, 66, 70, 78, 86, 94, 102, 106, 110, 114, 130, 138, 142, 158, 166, 178, 182, 186, 194.
Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; LEON BANKOFF, Los Angeles, California; JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; J.A. MCCALLUM, Medicine Hat, Alberta; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Editor's Comment,

The Sierpiński reference in the above solution was added for completeness by the editor, who took it from the solution of Bob Prielipp.

Most of the other solvers arrived at the necessary conditions (2). But also, the editor regrets to report, most did not seem to be aware that a proof of sufficiency was also required for completeness. Even a bit of feeble arm waving in that direction would have been welcome. They thus appeared to take an attitude towards mathematical proofs that they would not tolerate in their students.

Ahlburg and Nyon hinted at the following more general result:

Let \( n > 2 \). If \( a, b, c \) are positive integers such that \( b \) and \( c \) are \( n \)-th-power-free and relatively prime, and

\[
\sqrt[n]{a + \frac{b}{c}} = a \sqrt[n]{\frac{b}{c}},
\]

then

\( a-1, a, a^{n-1}+\ldots+a+1 \) are all \( n \)-th-power-free; \( b = a; c = a^{n-1}. \) \( (3) \)

The proof that conditions (3) are necessary follows the same pattern as in the case \( n = 2 \), but the proof of sufficiency breaks down when we try to show that \( c \) is \( n \)-th-power-free. For example, when \( n = 3 \) and \( a = 10 \), then 9, 10, 111 are all cube-free, but \( c = 999 = 3^3 \cdot 37 \) is not. Thus conditions (3) would have to be strengthened to make them sufficient. In any case, this emphasizes the fact that a proof of sufficiency was absolutely essential in the case \( n = 2 \).

REFERENCE


598. [2010: 18] Proposed by Jack Garfunkel, Flushing, N.Y.

Given a triangle ABC and a segment PQ on side BC, find, by Euclidean construction, segments RS on side CA and TU on side AB such that, if equilateral triangles PQJ, RSK, and TUL are drawn outside the given triangle, then JKL is an equilateral triangle.
Solutions or comments were received from JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

As several readers pointed out, this problem is ill-posed and additional conditions would have to be imposed to make it viable. As it stands now, with the point J known, if any equilateral triangle JKL is drawn with K on the opposite side of CA from B and L on the opposite side of AB from C, and then, by a trivial construction, equilateral triangles RSK and TUL are drawn with RS on CA and TU on AB, then the segments RS and TU constitute one of infinitely many solutions. Nothing to write home about. This problem, it is clear, should have been diverted to the circular file. But the proposer and the editor were both asleep at the switch.

A viable problem can, however, still be salvaged from the debacle. It is clear from the proposer's solution that the problem he should have proposed is the following:

Given a (not necessarily convex) hexagon PQRSTU in which two pairs of opposite sides (hence also the third pair) are equal and parallel, equilateral triangles PQJ, RSK, and TUL are drawn externally. Prove that triangle JKL is equilateral.

Solution adapted from the proposer's.

We represent vectors by complex numbers (denoted by Greek letters). Let \( \omega = e^{2\pi i/3} \) (so that \( \omega^3 = 1 \)) and

\[
PQ = \alpha, \quad OR = \beta, \quad RS = \gamma,
\]
as shown in the figure. Then

\[
JK = JQ + QR + RK
= -\alpha \omega^2 + \beta - \gamma \omega
\]
and

\[
KL = RS + ST + TL
= -\gamma \omega^2 - \alpha + \beta \omega.
\]

Thus \( \omega JK = KL \), which shows that triangle JKL is equilateral.
Propose that 36 divides the sum of the 36 integers composing a sixth-order magic square that is pandiagonal (magic also along the broken diagonals) or symmetrical (pairs symmetrical with respect to the center have a constant sum).

Solution by the proposer.

It suffices to show that 6 divides the magic sum $M$ of every sixth-order magic square that is pandiagonal or symmetrical.

**Proof for symmetrical squares.**

Let $T$ be the constant sum of pairs symmetrical with respect to the center. If the top row contains the numbers $A$, $B$, $C$, $D$, $E$, $F$, then the bottom row contains the numbers $T-F$, $T-E$, ..., $T-A$. These twelve numbers form two complete rows whose sum is $6T = 2M$. Hence $T = M/3$, which shows that $3|M$.

To show that also $2|M$, we partition the sixth-order square into four third-order squares, the nine numbers in each third-order square summing to $Q_1$, $Q_2$, $Q_3$, $Q_4$, respectively, as shown in Figure 1. Because the square is magic, we have

$$Q_1 + Q_2 = Q_2 + Q_4 = 3M;$$

and because it is symmetrical, we have

$$Q_1 + Q_4 = 9T = 3M.$$  

Hence $Q_1 = Q_4 = 3M/2$, and $2|M$.

**Proof for pandiagonal squares.**

To show that $2|M$, we consider the square in Figure 2, which is assumed to be magic and pandiagonal.

Let

$$A = \sum_{i=1}^{9} A_i, \quad B = \sum_{i=1}^{9} B_i, \quad D = \sum_{i=1}^{9} D_i.$$
Because the square is magic, if we add rows 1, 3, 5, and then separately add columns 2, 4, 6, we get

\[ A + B = B + D = 3M; \]

and because it is pandiagonal, adding the three northwest-southeast diagonals which begin at \( A_1, A_2, A_3 \) gives

\[ A + D = 3M. \]

Hence \( A = D = 3M/2 \), and \( 2|M \).

To show that \( 3|M \), we consider the square of Figure 3, which is assumed magic and pandiagonal.

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Here we define

\[ A = \sum_{i=1}^{4} A_i, \quad B = \sum_{i=1}^{4} B_i, \]

with similar definitions for \( C, D, E, F, G, H, I \). Adding separately rows 1, 4, then rows 2, 5, then rows 3, 6, we get

\[ A + B + C = 2M, \quad D + E + F = 2M, \quad G + H + I = 2M; \]

and adding separately columns 1, 4, then columns 2, 5, then columns 3, 6 gives

\[ A + D + G = 2M, \quad B + E + H = 2M, \quad C + F + I = 2M. \]

Finally, adding separately the northwest-southeast diagonals beginning at \( A_1, A_2 \), then the northeast-southwest diagonals beginning at \( C_1, C_2 \), we get

\[ A + E + I = 2M, \quad C + E + G = 2M. \]

If follows from these results that the square in Figure 4 is magic with magic sum \( 2M \). Since every third-order magic square has a magic sum equal to thrice the center number, we conclude that \( E = 2M/3 \), and so \( 3|M \). \( \square \)
By continuing to add broken diagonals in Figure 3, it is easy to show that the square of Figure 4 is also pandiagonal, which is possible only if all its entries are equal. This proves that the entries in a sixth-order pandiagonal magic square can be partitioned into nine disjoint quartets each of which sums to 2M/3.

The 36 consecutive integers \( m, m+1, \ldots, m+35 \) add up to \( 18(2m+35) \), so a magic square composed of these numbers must have magic sum \( M = 3(2m+35) \). Because this magic sum is odd, parity prevents this magic square from being pandiagonal or symmetrical. That 36 (more generally, \((4p+2)^2\)) consecutive integers cannot be arranged into a magic square that is pandiagonal or symmetrical was first proved over 60 years ago by Planck [1].

A nearly complete solution was submitted by KENNETH M. WILKE, Topeka, Kansas; and a somewhat inconclusive argument dealing with special cases was submitted by BIKASH K. GHOSH, Bombay, India.

REFERENCE


THE DOT POLKA

The editor has received, through the courtesy of Leon Bankoff, a generous extract from a book entitled *One Million*, by Hendrik Hertzberg, published in 1970 by Simon and Schuster. The book contains one million dots.

We quote from the introduction. "There are 5000 dots to a page—10000 on each double-page spread. ...Notes are scattered—like mileposts—here and there in the inside margins. Each note corresponds to a number, and the dot signifying that number is readily identifiable." The notes, of which there are several hundreds, range from 2 (population of the Garden of Eden), through 32500 (number of laps in Lapland) and 155024 (number of breasts in Brest), to 1000000 (number of dots in the book). One of the most stirring passages in the book is reproduced below.

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If the reader stares fixedly at the above for a few minutes, the dots will soon begin to dance before his eyes. They are dancing, of course, a polka.
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