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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name EUREKA.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name Crux Mathematicorum.
- Issues from Vol. 23, No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name Crux Mathematicorum with Mathematical Mayhem.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name Crux Mathematicorum.
CRUX MATHEMATICORUM

Vol. 6, No. 9
November 1980

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Mathematics Department, the Ottawa Valley Education Liaison Council, and the University of Ottawa Mathematics Department are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $12.00. Back issues: $1.20 each. Bound volumes with index: Vols. 1&2 (combined), $12.00; Vols. 3,4,5, $12.00 each. Cheques and money orders, payable to CRUX MATHEMATICORUM (in US funds from outside Canada), should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

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Problem 490, which is about palindromic primes, is discussed on pages 288-290 in this issue. As material supplementary to this discussion, I have, at the editor's request, prepared a list of the 93 five-digit and 668 seven-digit palindromic primes. The calculations were done on a PDP-11/45 at the University of Waterloo, and the computer time required was slightly more than one minute.

**FIVE-DIGIT PALINDROMIC PRIMES**

10301 10501 10601 11311 11411 12421 12721 12821 13331 13831
13931 14341 14741 15451 15551 16061 16361 16561 16661 17471
17971 18181 18481 19391 19991 30103 30203 30403 30703
30803 31013 31513 32323 33533 34543 34843 35053 35153
35353 35753 36263 36563 37273 37573 38083 38183 38783 39293
70207 70507 70607 71317 71917 72227 72727 73037 73237 73637
74047 74747 75557 76367 76667 77377 77477 77977 78487 78787
78887 79397 79697 90709 91019 93139 93239 93739 94049
94349 94649 94849 95959 96269 96469 96769 97379 97579
97879 98389 98689

**SEVEN-DIGIT PALINDROMIC PRIMES**

1003001 1008001 1022201 1028201 1035301 1043401 1055501 1062601
1065601 1074701 1082801 1085801 1092901 1093901 1114111 1117111
1120211 1123211 1126211 1129211 1134311 1145411 1150511 1153511
1160611 1163611 1175711 1177711 1178711 1180811 1183811 1186811
1190911 1193911 1196911 1201021 1208021 1212121 1215121 1218121
1221221 1235321 1242421 1243421 1245421 1250521 1253521 1257521
1262621 1268621 1273721 1276721 1278721 1280821 1281821 1286821
1287821 1300031 1303031 1311131 1317131 1327231 1328231 1333331
1335331 1338331 1343431 1360631 1362631 1363631 1371731 1374731
1390931 1407041 1409041 1411141 1412141 1422241 1437341 1444441
1447441 1452541 1456541 1461641 1463641 1464641 1469641 1486841
1489841 1490941 1496941 1508051 1513151 1520251 1532351 1535351
1542451 1548451 1550551 1551551 1556551 1557551 1565651 1572751
1579751 1580851 1583851 1589851 1594951 1597951 1598951 1600061
7314137 7324237 7327237 7347437 7352537 7354537 7362637 7365637
7381837 7388837 7392937 7401047 7403047 7409047 7415147 7434347
7436347 7439347 7452547 7461647 7466647 7472747 7475747 7485847
7486847 7493947 7507057 7508057 7518157 7519157 7521257
7527257 7540457 7562657 7564657 7576757 7586857 7592957 7594957
7600067 7611167 7619167 7622267 7630367 7632367 7644667 7654567
7662667 7665667 7666667 7669667 7674767 7681867 7690967
7693967 7696967 7715177 7718177 7722277 7729277 7733377 7742477
7747477 7750577 7758577 7764677 7772277 7774777 7778777 7782877
7783877 7791977 7794977 7807087 7819187 7820287 7821287 7831387
7832387 7838387 7843487 7850587 7856587 7865687 7867687 7868687
7873787 7884887 7891987 7897987 7913197 7916197 7930397 7933397
7935397 7938397 7941497 7943497 7949497 7957597 7958597 7960697
7977797 7984897 7985897 7987897 7996997 9002009 9015109 9024209
9037309 9042409 9043409 9045409 9046409 9049409 9067609 9073709
9076709 9078709 9091909 9095909 9103019 9109019 9110119 9127219
9128219 9136319 9149419 9169619 9173719 9174719 9179719 9185819
9196919 9199919 9200029 9209029 9212129 9217129 9222229 9223229
9230329 9231329 9255529 9269629 9271729 9277729 9280829 9286829
9289829 9318139 9320239 9324239 9329239 9332339 9338339 9351539
9357539 9375739 9384839 9397939 9400049 9414149 9419149 9433349
9439349 9440449 9446449 9451549 9470749 9477749 9492949 9493949
9495949 9504059 9514159 9526259 9529259 9547459 9556559 9558559
9561659 9577759 9583859 9585859 9586859 9601069 9602069 9604069
9610169 9620269 9624269 9626269 9632369 9634369 9645469 9650569
9657569 9670769 9686869 9700079 9709079 9711179 9714179 9724279
9727279 9732379 9733379 9743479 9749479 9752579 9754579 9758579
9762679 9770779 9776779 9779779 9781879 9782879 9787879 9788879
9795979 9801089 9807089 9809089 9817189 9818189 9820289 9822289
9836389 9837389 9845489 9852589 9871789 9888889 9889889 9896989
9902099 9907099 9908099 9916199 9918199 9919199 9921299 9923299
9926299 9927299 9931399 9932399 9935399 9938399 9957599 9965699
9978799 9980899 9981899 9989899

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* * *

* * *
TWO SLICING PROBLEMS

Two planar areas that can be placed so that they cut off equal chords on each member of a family of parallel lines, or two solids that can be placed so that they intercept equiareal sections on each member of a family of parallel planes, are said to be \textit{Cavalieri congruent}. Two figures that are Cavalieri congruent have, of course, equal areas (in the one case) or equal volumes (in the other case). The following two problems point out a curiosity related to Cavalieri congruence.

\textbf{Problem 1.}
Show that there cannot exist a polygon to which a given circle is Cavalieri congruent.

\textbf{Problem 2.}
On the other hand, show that there exists a polyhedron (actually a tetrahedron) to which a given sphere is Cavalieri congruent.

\textit{Howard Eves, University of Maine.}

\textit{Solutions to these problems appear on page 270 in this issue.}  

\*

ON MATHEMATICAL BEHAVIOUR

The logician Raymond Smullyan gives the following two examples of mathematical behaviour ([2], see also [1]):

\textbf{Example 1.}
1. To get hot water given an unlighted stove, matches, cold water, and an empty pot: fill pot, light stove, put pot on stove, and wait.
2. To get hot water given an unlighted stove, matches, and a pot filled with cold water: pour out water, thereby reducing the problem to the already solved Case 1.
3. To get hot water given a lighted stove and a pot filled with cold water: turn off stove and pour out water, thereby reducing to Case 1; alternatively, turn off stove, thereby reducing to Case 2.

\textbf{Example 2.}
1. To put out the fire given a hydrant, a disconnected hose, and a house on fire: connect hose and squirt house.
2. To put out the fire given a hydrant, a disconnected hose, and a house not on fire: set fire to house, thereby reducing to Case 1.

The behaviour of the mathematician in Example 1 is perfectly logical if he has a robot programmed to solve Case 1 from start to finish (with no variations, robots being what they are). Hence, in Case 2, it makes sense for him to perform the simple task of emptying the pot and then to turn the job over to the robot, rather than to light the stove, put the pot on the stove, and wait. In Case 3, emptying the pot and turning off the stove still beat putting the pot on the stove and waiting for the water to get hot. Alternatively, if the robot is programmed to solve Case 2, it makes sense to just turn off the stove and leave the rest to the robot.

It is in Example 2 that a lesson may be learned. Many mathematicians have the habit of making things out to be more difficult than they really are, creating difficulties where there are none. While the effect may be blazing, it is hardly worth doing.

REFERENCES


A. LIU, University of Alberta.

*SOLUTIONS TO "TWO SLICING PROBLEMS" (see page 269)*

Solution to Problem 1. Parallel chords cut off by a pair of coplanar lines vary linearly in length, whereas parallel chords cut off by a circle do not.

Solution to Problem 2. Let AB and CD be two line segments in space such that:
(1) $AB = CD = 2r\sqrt{\pi}$, where $r$ is the radius of the sphere; (2) $AB$ and $CD$ are each perpendicular to the line joining their midpoints, this join having length $2r$; (3) $AB$ is perpendicular to $CD$. It can easily be shown that the tetrahedron $ABCD$ may serve as the comparison solid. □

The interested reader may care to try to establish the following more difficult curiosity: Though there exist pairs of tetrahedra of the same volume that are not Cavalieri congruent, any pair of triangles of the same area are Cavalieri congruent.

HOWARD EVE
On Mathematical Olympiads.

The International Mathematical Olympiad (IMO) was initiated by Rumania in 1959 [1]. It is for students who have not yet started at university and have not reached their 20th year. So far it has run for 21 consecutive years. The last one was held in England in 1979. Unfortunately, no country was willing to be the host country for 1980 (this involves a lot of work, organization, and expense). It is still uncertain whether or not the IMO will continue to be a yearly event. There are pressures to change it to a biennial event. The 1981 IMO is set to be held in the U.S.A. in July (in Washington, D.C.). More information concerning this will be given in a subsequent issue.

The U.S.S.R. teams have had outstanding results in the IMO's. A measure of their success is the number of individual prizes won as well as the unofficial team placement. A good part of the reason for this success is the support given in the U.S.S.R. to mathematics (as well as to other intellectual endeavours and sports) for a long time. This should be contrasted with the relatively meager support the U.S.A. and Canada have provided for their gifted children. By and large, programs for gifted children are still considered here to be elitist, notwithstanding the fact that many of our future intellectual leaders and statesmen should be coming from this pool of gifted children. Unfortunately, many of them either "drop out" or else fail to live up to their potential because they are bored with many of their programs which present little or no challenge. Considering the present competition between the "Communist System" and the "Free Enterprise System", we can ill afford this neglect. That the U.S.S.R. is in dead earnest about education is indicated by recent initiatives taken by them for all secondary school students [2].

H. Freudenthal [3] has given a highly recommended report on mathematical olympiads in many countries. The report also contains many problems from various countries as well as a list of 105 references to papers mostly in English. There is in [3] a particularly interesting section on local and national olympiads in the Soviet Union, and I regret that there is no room here to quote it in extenso. It shows to what lengths Soviet authorities are prepared to go to discover and foster the development of mathematical talent.¹ I give below a few excerpts from this section, but

¹Unfortunately, there is also discrimination in mathematics for certain minority groups. See [1980: 145].
readers are urged to study the full report.

*Local olympiads.*

The name 'olympiads' for mathematical contests among high school students seems to have been first used in Russia...

The first was held in Leningrad in the spring of 1934. Moscow followed in 1935... The first Moscow Olympiad had 314 participants, in 1964 the number was about 4000.

The Moscow olympiads (and most of the other local olympiads) are run in two rounds, in spring, with a fortnight interval... the 1953 Moscow olympiad was entered by 1350 students, 517 of which were admitted to the second round, in which 262 succeeded, 3 got a first prize, 15 a second prize, 24 a third prize, and 69 got certificates of merit. The prizes consist of small mathematical libraries. Certificates of merit are also awarded to the teachers of winning students.

... The time allowed for solving the problems is 4-5 hours in both the first and second round. The degree of difficulty of the problems has much increased in the course of the years. Their character is much like that of the Eötvös contest problems though in general the appeal to a creative mind is perhaps less strong and they often require more familiarity with intentionally cultivated techniques. They are traditional mathematics, except for the use of some artifices like Dirichlet's drawer principle.

... The Committee read the participant's answers and grade them according to a not too formal system. Such criteria as elegance and originality of the solution play an important role.

... Another new feature are Moscow olympiads for the 4th-6th grades (10 to 13-year-olds), separated according to these grades.

... In the course of the years, in particular after 1945, the olympiads and related activities have spread to other major cities of the Soviet Union, where universities or pedagogical institutes existed...

*National olympiads.*

In 1960 the Moscow Olympiads Committee took an initiative to organize a geographically broader olympiad, the first all-Russian (actually all-Union) olympiad, in which teams of 13 oblast* (provinces) of the R.S.F.S.R. and 9 Union-states participated... This enterprise has annually been repeated, in different places...

These olympiads are played on four geographic levels: 1st round, school olympiads; 2nd, city and 'rayon' olympiads; 3rd, oblast*, krai, republic (province or state) olympiads; 4th, final round. In this pattern the Moscow olympiad is a third round competition. There are also 'correspondence instruction olympiads' and television olympiads... The level of the television olympiads is subjected to great variations. People who succeeded in the correspondence instruction olympiad are admitted to the 3rd round of the national olympiads. The problems of the final round are differentiated according to the four highest school classes...

The final round of the 1967 U.S.S.R. mathematical olympiad took place in Tbilisi... It seems that the first round is entered by hundreds of thousands of competitors.

Thus in the Soviet Union. The following "Notice to Canadian Students" reflects the Canadian reality. It would be appreciated if teachers would post a copy of the Notice on their school bulletin boards.
NOTICE TO CANADIAN STUDENTS

So far the Canadian Mathematical Olympiad Committee has not succeeded in obtaining financial support for travel to send an 8-student Canadian team to participate in the July 1981 IMO to be held in the U.S.A. One possible alternative is to participate with that subset of students who would be selected to be team members on the basis of their performance in the 1981 Canadian Mathematical Olympiad and who can finance their own travel to New York City and return from Washington, D.C. Once in the U.S.A., all expenses (room, board, travel) will be provided for by the host country.

It may also be possible for the selected team members to participate in a pre-training session for approximately one month starting around 8 June 1981 at the U.S.A. Military Academy at West Point, N.Y. together with the U.S.A. team. This will probably entail a cost of approximately $5.00/day for room and board.

REFERENCES


2. I. Wirszup, Preliminary report on the present status of Soviet mathematics and science training at the pre-university level, Mathematics Department, University of Chicago, Chicago, Illinois 60637.


*Recently, through the courtesy of Willie Yong, I received a number of U.S.S.R. National Olympiad and Moscow Olympiad preparation problem sets. These were translated by a teacher in New York whom I will acknowledge as soon as I learn his name. Since I believe that these problems are quite good and challenging, I now include the Ninth U.S.S.R. National Olympiad set of 1974 and expect to include other sets in subsequent issues. Since there are quite a few problems in each set, I will only publish selected elegant solutions which are submitted to me. Solvers are encouraged to reveal the approximate time they spent on each problem.*
1. Triangle $ABC$ is rotated about the centre of its circumscribed circle by an angle less than $180^\circ$ to form triangle $A_1B_1C_1$. If $BC \cap B_1C_1 = A_2$, $CA \cap C_1A_1 = B_2$, and $AB \cap A_1B_1 = C_2$, prove that triangles $ABC$ and $A_2B_2C_2$ are similar.

2. Two players play the following game on a triangle $ABC$ of unit area. The first player picks a point $X$ on side $BC$, then the second player picks a point $Y$ on $CA$, and finally the first player picks a point $Z$ on $AB$. The first player wants triangle $XYZ$ to have the largest possible area, while the second player wants it to have the smallest possible area. What is the largest area that the first player can be sure of getting?

3. The vertices of a convex 32-gon lie on the points of a square lattice whose squares have sides of unit length. Find the smallest perimeter such a figure can have.

4. On a $13 \times 13$ square piece of graph paper the centres of 53 of the 169 squares are chosen. Show that there will always be 4 of these 53 points which are the vertices of a rectangle whose sides are parallel to those of the paper.

5. Three ants crawl along the sides of a triangle $ABC$ in such a way that the centroid of the triangle they form at any given moment remains fixed. Show that this centroid coincides with the centroid of triangle $ABC$ if one of the ants travels along the entire perimeter of triangle $ABC$.

6. A certain number of 0's, 1's, and 2's are written on a blackboard. Two unequal digits are erased and the third digit is written in their place (e.g., 2 is written if 0 and 1 are erased). This operation is repeated until no two distinct digits remain on the blackboard. Show that if only one digit remains at the end of the game, then this digit is independent of the order in which the digits were erased.

7. In a convex hexagon $A_1A_2A_3A_4A_5A_6$, let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of diagonals $A_6A_2, A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1$, respectively. Show that if hexagon $B_1B_2B_3B_4B_5B_6$ is convex, then its area is $\frac{1}{4}$ the area of $A_1A_2A_3A_4A_5A_6$.

8. Show that with the digits 1 and 2 one can form $2^{n+1}$ numbers, each having $2^n$ digits, and every two of which differ in at least $2^{n-1}$ places.

9. On a $7 \times 7$ square piece of graph paper, the centres of $k$ of the 49 squares
are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest $k$ for which this is possible?

10. A large cube measuring $k$ units on each edge is to be formed of smaller unit cubes, each coloured either black or white. Can this be done so that for any unit cube exactly two of its neighbours have the same colour as the unit cube itself? (Two cubes are called *neighbours* if they share a common face.)

11. A horizontal strip is given in the plane, bounded by straight lines, and $n$ lines are drawn intersecting this strip. Every two of these lines intersect inside the strip and no three of them are concurrent. Consider all paths starting on the lower edge of the strip, passing along segments of the given lines, and ending on the upper edge of the strip, which have the following property: travelling along such a path, we are always going upward, and when we come to the point of intersection of two of the lines we must change over to the other line to continue following the path. Show that, among these paths,

(a) at least $\frac{1}{2}n$ of them have no point in common;
(b) there is some path consisting of at least $n$ segments;
(c) there is some path passing along at most $\frac{1}{2}n + 1$ of the lines;
(d) there is some path which passes along each of the $n$ lines.

12. Given is a polynomial $P(x)$ whose coefficients are (i) natural numbers, (ii) integers. Denote by $a_n$ the sum of the digits in the decimal representation of $P(n)$. Show that there is some number which occurs infinitely often in the sequence $a_1, a_2, a_3, \ldots$.

13. In a plane is given a finite set of polygons, every two of which have a common point. Show that there exists a line which intersects all the polygons.

14. Prove that, for positive $a, b, c$, we have

$$a^3 + b^3 + c^3 + 3abc \geq bc(b+c) + ca(c+a) + ab(a+b).$$

15. Quadrilateral $ABCD$ is inscribed in a circle. It is rotated about the centre of the circle through an angle less than $180^\circ$ to form quadrilateral $A_1B_1C_1D_1$. Show that the points $AB \cap A_1B_1, BC \cap B_1C_1, CD \cap C_1D_1, DA \cap D_1A_1$ are the vertices of a parallelogram.
16. Twenty teams are participating in the competition for the championships both of Europe and the world in a certain sport. Among them, there are \( k \) European teams (the results of their competitions for world champion count also towards the European championship). The tournament is conducted in round robin fashion. What is the largest value of \( k \) for which it is possible that the team getting the (strictly) largest number of points towards the European championship also gets the (strictly) smallest number of points towards the world championship, if the sport involved is

(a) hockey (0 for a loss, 1 for a tie, 2 for a win);
(b) volleyball (0 for a loss, 1 for a win, no ties).

17. Given real numbers
\[ a_1, a_2, \ldots, a_m \quad \text{and} \quad b_1, b_2, \ldots, b_n, \]
and positive numbers
\[ p_1, p_2, \ldots, p_m \quad \text{and} \quad q_1, q_2, \ldots, q_n, \]
we form an \( m \times n \) array in which the entry in the \( i \)th row \((i=1,2,\ldots,m)\) and \( j \)th column \((j=1,2,\ldots,n)\) is
\[ \frac{a_i + b_j}{p_i + q_j}. \]
Show that in such an array there is some entry which is no less than any other in the same row and no greater than an other in the same column

(a) when \( m=2 \) and \( n=2 \),
(b) for arbitrary \( m \) and \( n \).


\[ \ast \]

**SOLUTIONS TO PRACTICE SET 15**

15-1. Determine an \( n \)-digit number (in base 10) such that the number formed by reversing the digits is nine times the original number.

What other multiples besides nine are possible?
Solution.

We use the notation \( \overline{ab} = 10a + b, \overline{abc} = 100a + 10b + c, \) etc., and we exclude (for now) leading and final zeros. We first show that if

\[
N \equiv \overline{ab...xy} \cdot k = \overline{yx...ba} \equiv N', \quad k \neq 1,
\]

then \( k = 4 \) or \( 9. \)

If \( k = 5, \) then \( a = 1 \) for otherwise \( N \) would have more digits than \( N'; \) and \( N' \) is not divisible by 5 when \( a = 1. \) Similarly, \( k \neq 6 \) or \( 8. \) If \( k = 7, \) then again \( a = 1. \) But then we must have \( y = 3 \) for \( N' \) to end in 1, with the resulting contradiction

\[
N = \overline{1b...x3} \cdot 7 > \overline{3x...b1} = N'.
\]

If \( k = 2, \) then \( \alpha \leq 4, \) and \( a = 2 \) or \( 4 \) since \( N' \) is even. If \( \alpha = 4, \) then \( y, \) the initial digit of \( N', \) must equal 8 or 9, but neither \( \overline{4b...x8} \cdot 2 \) nor \( \overline{4b...x9} \cdot 2 \) ends in 4.

If \( a = 2, \) then \( \alpha = 4 \) or 5, but neither \( \overline{2b...x4} \cdot 2 \) nor \( \overline{2b...x5} \cdot 2 \) ends in 2. Finally, if \( k = 3, \) then \( \alpha \leq 3. \) If \( \alpha = 1, \) then \( \alpha = 7 \) and \( N < N'. \) If \( \alpha = 2 \) then \( \alpha = 4, \) and if \( \alpha = 3 \) then \( \alpha = 1; \) and in each case \( N > N'. \) So \( k = 4 \) or \( 9. \)

We now assume \( k = 4 \) and find all numbers \( \overline{ab...xy} \) such that

\[
N \equiv \overline{ab...xy} \cdot 4 = \overline{yx...ba} \equiv N'.
\]

For \( N \) and \( N' \) to have the same number of digits, we must have \( \alpha = 1 \) or 2; hence \( \alpha = 2 \) since \( N' \) is even. Now \( y, \) the initial digit of \( N', \) must equal 8 or 9. In fact, since \( 4y \) ends in 2, we must have \( \alpha = 8 \) and

\[
\overline{2b...x8} \cdot 4 = \overline{6x...b2}.
\]

Since \( 23 \cdot 4 > 90, \) we now have \( b = 0, 1, \) or 2. At the same time, the digit in the tens' place in the product \( \overline{x8} \cdot 4 \) is odd for any \( x, \) so \( b = 1. \) Knowing the last two digits 12 of the product \( \overline{21...x8} \cdot 4, \) we conclude that \( x = 2 \) or 7. Since \( 21 \cdot 4 > 82, \) it follows that \( x = 7 \) and the required numbers are of the form \( 21...78. \) The smallest satisfactory answer is \( 2178. \)

Answers with more than four digits must satisfy

\[
21uv...rs78 \cdot 4 = 87sr...vu12. \tag{1}
\]

If there are \( k \) unassigned digits on each side, we have

\[
84 \cdot 10^{k+2} + 312 + \overline{uv...rs00} \cdot 4 = 87 \cdot 10^{k+2} + 12 + \overline{sr...vu00},
\]

from which

\[
M \equiv \overline{uw...rs} \cdot 4 + 3 = \overline{3sr...vu} \equiv M'. \tag{2}
\]
We conclude from (2) that $M - 3$ starts with 29 or 3 and that $u = 7, 8, \text{ or } 9$. Indeed, since $M'$ is odd, we must have $u = 7$ or 9. We consider these two cases separately.

If $u = 9$, we have

$$M \equiv 9\ldots rs \cdot 4 + 3 = 3sr\ldots v9 \equiv M'.$$

Since $4s + 3$ ends in 9, we have $s = 9$ for the only alternative $s = 4$ implies $M > M'$.

As in going from (1) to (2), with $s = 9$ we find that (3) implies

$$v\ldots r \cdot 4 + 3 = 3r\ldots v.$$

It follows from (2) and (4) that if $uv\ldots rs$ satisfies (2) with $u = s = 9$, then dropping the initial and final 9's yields a number $v\ldots r$ with the same property as $uv\ldots rs$. In particular, $uv\ldots rs$ can be any one of the numbers 9, 99, 999, ..., from which we get the numbers

$$21978, \ 219978, \ 2199978, \ ..., $$

all of which satisfy (1).

If $u = 7$, then we get from (2)

$$7v\ldots rs \cdot 4 + 3 = 3sr\ldots v7.$$

An analysis similar to that for the case $u = 9$ shows that we must have $s = 1, v = 8,$ and $r = 2$, so that $uv\ldots rs$ is of the form 78...21. This analysis also shows that if the initial 78 and final 21 are dropped, then the remaining integer is one of the answers to our problem, that is, multiplying it by 4 reverses the digits.

To recapitulate, we have shown that, for the case $k = 4$, any answer to the problem which differs from all the numbers in the sequence

$$0, \ 2178, \ 21978, \ 219978, \ 2199978, \ ..., $$

has the same combination of digits at the beginning and at the end, that this combination is one of the numbers (5), and that if this combination is dropped from the beginning and the end the resulting number (here we allow leading and final zeros) is also an answer to the problem. Thus all answers to the problem consist of concatenations of one of the types

$$P_1 P_2 \cdots P_{n-1} P_n P_{n-1} \cdots P_2 P_1$$

or

$$P_1 P_2 \cdots P_{n-1} P_n P_{n-1} \cdots P_2 P_1,$$

where each $P_i$ is one of the numbers (5). Here are some examples:
A similar analysis for the case \( k = 9 \) (which we leave to the reader) shows that all answers are concatenations of one of the types (6) or (7), where each \( P_i \) is a number in the sequence

\[
0, \ 1089, \ 10989, \ 109989, \ 1099989, \ldots
\]

This problem appears in [1], and the above solution was edited from the one given in that excellent reference, which consists of 350 problems (with solutions) from Russian Olympiads and mathematics hobby groups in Moscow.

**REFERENCE**


15-2. Solve the following system of equations:

\[
\begin{align*}
ax_1 + bx_2 + bx_3 + \ldots + bx_n &= c_1, \\
bx_1 + ax_2 + bx_3 + \ldots + bx_n &= c_2, \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \\
bx_1 + bx_2 + bx_3 + \ldots + ax_n &= c_n.
\end{align*}
\]

(In the left member of each equation, all the coefficients except one are \( b \)'s and the remaining one is \( a \).)

**Solution.**

The system is trivial if \( a = b \), so we assume \( a \neq b \). Subtracting the \( r \)th equation from the first, we get

\[
(a - b)(x_1 - x_r) = c_1 - c_r, \quad r = 2, 3, \ldots, n,
\]

from which

\[
x_r = x_1 - \frac{c_1 - c_r}{a - b}, \quad r = 2, 3, \ldots, n.
\]
Adding all the equations gives
\[ \{a + (n-1)b\} \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} c_k \]
which, by using (1), becomes
\[ \{a + (n-1)b\} \left\{ nx_1 - \sum_{k=1}^{n} \frac{c_1 - c_k}{a - b} \right\} = \sum_{k=1}^{n} c_k. \]  \( \text{(2)} \)

If \( a + (n-1)b \neq 0 \), then \( x_1 \) is uniquely determined by (2) and the remaining \( x_n \) by (1). Alternatively, any \( x_n \) can be obtained from (2) by replacing \( x_1 \) and \( c_1 \) by \( x_n \) and \( c_n \), respectively. Finally, if \( a + (n-1)b = 0 \), there is no solution unless \( \sum c_k = 0 \), in which case \( x_1 \) is arbitrary and the remaining \( x_n \) are then uniquely determined by (1).

In this problem, the row vectors of the coefficient matrix are cyclic permutations of
\[ (a,b,b,\ldots,b) \quad (n-1 \text{ b's}). \]
As a rider, solve the more general system where the row vectors of the coefficient matrix are cyclic permutations of
\[ (a,a,\ldots,a,b,b,\ldots,b) \quad (m \text{ a's and } n-m \text{ b's}). \]

15-3. Three circular arcs BC, CA, AB, of fixed total length \( l \), are constructed outwardly on the sides of a given triangle ABC, each passing through two vertices, so that the area they enclose is a maximum (for the given \( l \)). Show that the radii of the three arcs are equal. (The problem has been restated for greater clarity.)

Solution.

It is assumed that the total length \( l \) is not less than the perimeter of the triangle but not too large (we shall consider this point subsequently). Although it is intuitively clear that a maximum area does exist for a given \( l \), nevertheless we establish this fact by a continuity argument. Let the lengths of the arcs AB, BC be \( l_1, l_2 \), respectively; then the length of arc CA is \( l - l_1 - l_2 \). Since the area of a segment of a circle is a continuous function of the lengths of its bounding arc and bounding chord, the area bounded by the three arcs is a continuous function of \( l_1 \) and \( l_2 \) over the closed domain
\[ l_1 \geq AB, \quad l_2 \geq BC, \quad l_1 + l_2 \leq l - CA. \]  \( \text{(1)} \)
Consequently this area takes on its maximum value (also its minimum value) for some values of \( l_1, l_2 \) in the domain (1).
Assume that arc CA has length \( z_3 \) when the area bounded by the three arcs is a maximum. We will show that the arcs AB and BC, of fixed total length \( z - z_3 \), have equal radii if the sum of the areas of the circular segments AB and BC is a maximum. Increase or decrease angle B, keeping AB and BC fixed in length, until the length of the portion of the circumcircle through A, B, C (above AC) is also \( z - z_3 \), as shown in the figure. This can be done if \( z \) is not too large (see discussion of this point at the end).

It now follows by the Isoperimetric Theorem for the circle (see [1]) that the sum of the areas of the segments AB and BC, of fixed total arc length, is a maximum when they are segments of the same circle. Similarly, the arc CA (in the original configuration) must have the same radius. There also result equal radii if we consider the corresponding problem for an arbitrary convex polygon instead of a triangle.

We now show that the above proof imposes an upper bound on \( z \). For simplicity, we consider the case when \( AB = BC \). Letting angle B vary while keeping AB and BC fixed in length, it is not difficult to show that the maximum length of the part of the circumcircle of ABC above AC occurs when angle B = 0. Thus our proof requires that

\[
\pi AB \geq \text{arc AB} + \text{arc BC} \geq 2AB.
\]

Presumably, one could by calculus establish the desired result for larger \( z \). However, if \( z \) gets too large we run into a complication. For sufficiently large \( z \), two of the circular arcs become tangent at a vertex. Then, for still larger \( z \), the two corresponding segments have a nonempty intersection whose area is counted twice. This can be corrected for, but it makes finding the solution much more difficult.

This problem appeared in the Spring 1960 issue of Pi Mu Epsilon Journal, where it was proposed by M.S. Klamkin and D.J. Newman, and the proposers' solution, equivalent to the above but more succinct, appeared in the Spring 1963 issue of the same journal. The more general problem of determining a closed curve of given length which passes through a set of given points and encloses a maximum area was treated by Steiner [21]. For the Steiner problem in the case of an equilateral triangle ABC with large enough \( z \) it is to be expected that the maximizing configuration will con-
sist of three congruent segments AA', BB', CC' along extensions of the three medians and three congruent circular arcs B'C', C'A', A'B'. If this be true, there still remains the problem of determining the lengths of the arcs and the segments. This is a standard calculus problem but most likely one without an explicit solution. It would be interesting, in the corresponding Steiner problem for an arbitrary triangle, if the three circular arcs of the maximizing configuration still turn out to have the equal radii property.

As a rider, consider the original problem in which "maximum" is replaced by "minimum".

REFERENCES


Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

LETTER TO THE EDITOR

Dear Editor:

I was interested to see the discussion by Kenneth S. Williams [1980: 204] of the identity

\[ \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2n-1}. \]

It is essentially the same as

\[ \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n} \]

which I asked my analysis class to prove last term. Of course induction was the most popular method, but Miss Judith Lum Wan gave a different solution which seemed to me to be rather elegant. Consider the difference between each side of the equation and the harmonic sum \( 1 + 1/2 + 1/3 + \ldots + 1/2n \). For the left-hand side it is the harmonic sum \( 1 + 1/2 + \ldots + 1/n \), and for the right-hand side it is \( 2/2 + 2/4 + \ldots + 2/2n \) which is the same.

B.C. RENNIE, James Cook
University of North Queensland.
Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.


The Soldier's Farewell

My love, since we must PART
There's ALOE in my soul;
Oh, hear the drumbeats ROLL
That TELL how throbs my heart!

In the "word square" above, each word represents a four-digit decimal integer which is a perfect square. The letters are one-to-one images of the digits. Restore the digits.

582, Proposed by Allan Wm. Johnson Jr., Washington, D.C.

In how many ways can five distinct digits A, B, C, D, E be formed into four decimal integers AB, CDE, EDC, BA for which the mirror-image multiplication

\[ AB \cdot CDE = EDC \cdot BA \]

is true? (For example, the mirror-image multiplication \( AB \cdot CD = DC \cdot BA \) is true for \( 13 \cdot 62 = 26 \cdot 31 \).)

583, Proposed by Charles W. Trigg, San Diego, California.

A man, being asked the ages of his two sons, replied: "Each of their ages is one more than three times the sum of its digits." How old is each son?

584, Proposed by F.G.B. Maskell, Algonquin College, Ottawa.

If a triangle is isosceles, then its centroid, circumcentre, and the centre of an escribed circle are collinear. Prove the converse.

Consider the following three inequalities for the angles A, B, C of a triangle:

\[
\begin{align*}
\cos \frac{B-C}{2} \cos \frac{C-A}{2} \cos \frac{A-B}{2} & \geq 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, \\
csc \frac{A}{2} \cos \frac{B-C}{2} + \csc \frac{B}{2} \cos \frac{C-A}{2} + \csc \frac{C}{2} \cos \frac{A-B}{2} & \geq 6, \\
csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2} & \geq 6. 
\end{align*}
\]  

Inequality (3) is well-known (American Mathematical Monthly 66 (1959) 916) and it is trivially implied by (2). Prove (1) and show that (1) implies (2).

586. Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.

(a) Given a natural number \( n \), show that the equation

\[9n^3 = 6abn + ab(a+b)\]

has no solution in natural numbers \( a \) and \( b \).

(b) Using (a), or otherwise, show that none of the following expressions is a perfect square for any natural number \( n \):

\[
\begin{align*}
36n^3 + 36n^2 + 12n + 1, \\
12n^3 + 36n^2 + 36n + 9, \\
4n^3 + 36n^2 + 108n + 81.
\end{align*}
\]

\*

VERNER E. HOGGATT, Jr.

(In Memoriam)

There was a game one played a few years ago in which one attempted to represent randomly chosen positive integers by arithmetic expressions that involved each of the ten digits 0, 1, ..., 9 once and only once. The game was completely solved when Verner E. Hoggatt, Jr.

discovered that, for any nonnegative integer \( n \),

\[
\log_{10} \sqrt[6]{\sqrt[5]{\sqrt[4]{\sqrt[3]{\sqrt[2]{\sqrt[1]{0+1+2+3+4}}}}}} = n,
\]

where there are \( n \) square roots in the second logarithmic base. (Notice that the ten digits appear in their natural order.)

HOWARD EVEES

Verner E. Hoggatt, Jr., a former student of Howard Eves and the founder of The Fibonacci Quarterly, died on 12 August 1980. (Editor)
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


483. [1979: 265; 1980: 223] Late comment from SAHIB RAM MANDAN, Bombay, India.


Let \( A \) and \( B \) be two independent events in a sample space, and let \( \chi_A \) and \( \chi_B \) be their characteristic functions (so that, for example, \( \chi_A(x) = 1 \) or \( 0 \) according as \( x \in A \) or \( x \notin A \)). If \( F = \chi_A + \chi_B \), show that at least one of the three numbers

\[
a = P(F=2), \quad b = P(F=1), \quad c = P(F=0)
\]

is not less than \( \frac{4}{9} \).

II. Further comments by M.S. Klamkin, University of Alberta.

As extensions of this problem, we had in solution I considered the determination of

\[
A(m,n) = \min_{0 \leq \sum_{i,j} P_{ij} \leq 1} \max_k P_k
\]

and

\[
B(m,n) = \max_{0 \leq \sum_{i,j} P_{ij} \leq 1} \min_k P_k,
\]

where

\[
P_0 + P_1 t + \ldots + P_m t^m = \prod_{i=1}^m (P_{i0} + P_{i1} t + \ldots + P_{in} t^n)
\]

and

\[
0 \leq P_{i,j} \leq 1, \quad \sum_{j=0}^n P_{i,j} = 1, \quad i = 1, 2, \ldots, m.
\]
We had conjectured that
\[ A(m,1) = \binom{m}{\lfloor m/2 \rfloor}, \quad m \text{ odd} \]  
(2)

and
\[ B(m,1) = \frac{1}{2^m}, \quad \text{all } m. \]  
(3)

First we give a simple proof of (3). Here we have
\[ P_0 + P_1 t + \ldots + P_m t^m = (P_{10} + P_{11} t)(P_{20} + P_{21} t)\ldots(P_{m0} + P_{m1} t), \]
from which
\[ P_0 = P_{10}P_{20}\ldots P_{m0} \quad \text{and} \quad P_m = (1-P_{10})(1-P_{20})\ldots(1-P_{m0}). \]

Now \( x(1-x) \leq \frac{1}{4} \) in \([0,1]\), so \( P_0P_m \leq \frac{1}{4^m} \); hence
\[ \min_k P_k \leq \min\{P_0, P_m\} \leq \max_{0 \leq i,j \leq 1} \min\{P_0, P_m\} = \sqrt{\frac{1}{4^m}} = \frac{1}{2^m}, \]
and equality holds throughout when \( P_{ij} = \frac{1}{4} \) for all \( i,j \), thus establishing (3).

The conjecture for \( A(m,1) \) in (2) is valid and its value for all \( m \) was established by J.D. Dixon [1] in a somewhat different context when he generalized the following problem of L. Moser and J.R. Pounder [2]:

If \( ax^2 + bx + c \) is a polynomial with real coefficients and real roots, then
\[ \max\{a, b, c\} \geq \frac{4(a+b+c)}{9}. \]

Dixon showed more generally that if \( a_0 + a_1 x + \ldots + a_m x^m \) is a polynomial of degree \( m \) with real coefficients and only real roots, then
\[ \max_k a_k \geq \binom{m}{s} \frac{(m-s)^{m-s}(s+1)^s}{(m+1)^m} (a_0 + a_1 + \ldots + a_m), \]  
(4)

where \( s = \lfloor m/2 \rfloor \), and equality is actually attained when all the roots are equal.

When \( m \) is odd and the sum of the \( a_k \) is 1 (as in our problem), (4) reduces to (2). Thus we have, for all \( m \),
\[ A(m,1) = \binom{m}{s} \frac{(m-s)^{m-s}(s+1)^s}{(m+1)^m}, \quad s = \lfloor m/2 \rfloor. \]  
(5)

Seven years later and unaware of the Dixon result, W.O.J. Moser [3] proposed a problem which is a special case of (5):

Let \( m \) identical weighted coins, each falling heads with probability \( x \), be tossed, and let \( P_k(x) \) be the probability that exactly \( k \) of them fall heads. Evaluate
\[ F_m = \min_{0 \leq x \leq 1} \max_{k=0,1,\ldots,m} P_k(x). \]
The published solution by D.Ž. Djoković gives the result

$$f_m = \binom{m}{s} \frac{8^s (m+1-s)^{m-s}}{(m+1)^{m}}, \quad s = \lfloor m/2 \rfloor. \quad (6)$$

Although (6) agrees with (5) for even $m$, it is incorrect for odd $m$. The error comes from failing to consider one of the possible cases.

We now give conjectured values for $A(2,n)$ and $B(2,n)$ for $n > 1$. Even if it turns out that our values are incorrect, they will at least provide good bounds.

For $A(2,n)$, we choose the $P_{i,j}$'s as in solution I [1980: 255]. This gives $P_0 = 0$ and $P_k = 1/2n$ for $k > 0$. Thus our conjecture is

$$A(2,n) = \frac{1}{2n}. \quad (7)$$

We also obtain (7) with the choice

$$P_{1,j} = \begin{cases} \frac{1}{2n}, & \text{for } j = 0, n, \\ \frac{1}{n}, & \text{for } 0 < j < n \end{cases} \quad \text{and} \quad P_{2,j} = \begin{cases} \frac{1}{2}, & \text{for } j = 0, n, \\ 0, & \text{for } 0 < j < n. \end{cases}$$

For $B(2,n)$, we set $P_{1,j} = 1/(n+1)$ for all $j$ and $P_{2,j} = \frac{1}{2}$ or $0$ as above. This gives $P_0 = 1/(n+1)$ and all the remaining $P_k = 1/(2n+2)$. Thus our conjecture is

$$B(2,n) = \frac{1}{2n+2}. \quad (8)$$

Since we must have $P_0 + P_1 + \ldots + P_{2n} = 1$ for any choice of the $P_{i,j}$'s (just set $t = 1$ in (1)), the average value of the $P_k$'s is always $1/(2n+1)$. Hence the exact values of $A(2,n)$ and $B(2,n)$ must satisfy

$$A(2,n) \geq \frac{1}{2n+1} \geq B(2,n).$$

Consequently, even if (7) and (8) are incorrect, they do provide good bounds.

Finally, we bring attention to a paper of MacLeod and Roberts [4] dealing with similar problems but with different norms.

REFERENCES

Inadvertently omitted from the list of solvers: VIKTORS LINIS, University of Ottawa.

Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.

Are there infinitely many palindromic primes (e.g., 131, 70207)?

I. Comment by Friend H. Kierstead, Cuyahoga Falls, Ohio.

For even \( n \), every \( n \)-digit palindrome is divisible by 11 since the alternating sum of its digits, which is 0, is divisible by 11. So 11 is the only palindromic prime with an even number of digits.

We now consider odd \( n \). As in Gabai and Coogan [1], we define

- \( \alpha_n \) = number of \( n \)-digit palindromic primes,
- \( A_n \) = number of (positive) \( n \)-digit palindromes (= \( 9 \cdot 10^{(n-1)/2} \)),
- \( \beta_n \) = number of \( n \)-digit primes,
- \( B_n \) = number of (positive) \( n \)-digit integers (= \( 9 \cdot 10^{n-1} \)),

and

\[ P(n) = \frac{\alpha_n}{A_n} \cdot \frac{\beta_n}{B_n} = 10^{(n-1)/2} \left( \frac{\alpha_n}{\beta_n} \right). \]

From the exact data given in [1] and updated by Nelson in [2], we have

\[ P(1) = \frac{n}{4} = 1, \]
\[ P(3) = 10^2 \left( \frac{15}{143} \right) \approx 1.04895, \]
\[ P(5) = 10^3 \left( \frac{93}{8363} \right) \approx 1.11204, \]
\[ P(7) = 10^3 \left( \frac{668}{586081} \right) \approx 1.13977, \]
\[ P(9) = 10^4 \left( \frac{5172}{45086079} \right) \approx 1.14714. \]

Several people (e.g., Beiler [3] and Card [4]) have stated flatly that there are infinitely many palindromic primes but, to the best of our knowledge, no proof of this statement has ever been adduced. The above evidence, however, shows that this statement is probably true. Indeed it suggests that, at least for small \( n \), there are proportionally more primes in the set of \( n \)-digit palindromes than there are primes in the set of \( n \)-digit integers, and that the proportion increases with \( n \).

II. Unedited extract from a comment submitted by Professor X at University Y.

In 1950, Moser [5] remarked that primes (>5) end in 1, 3, 7, or 9, and palindromic primes (other than 11) must have an odd number of digits. He listed 14 three-digit and 88 five-digit palindromic primes, along with the larger
1818181, 7878787, 3535353, and 7272727.
(The last one, as printed, had an additional 27 tacked on, a proofing error.)

Comments were also received from HIPPOLYTE CHARLES, Waterloo, Québec; and CHARLES W. TRIGG, San Diego, California.

Editor's comment.

The least that can be said about Professor X is that he has exhibited stupefying naïveté in parroting Moser's statements without checking at least some of them. Why should Moser list only 14 three-digit palindromic primes when there are 15, and only 88 five-digit ones when there are 93? These facts can easily be checked in even a small table of primes. As to Moser's sample of exactly 4 seven-digit palindromic primes, behold:

\begin{align*}
1818181 &= 31 \cdot 89 \cdot 659, \\
7878787 &= 7 \cdot 19 \cdot 59239, \\
3535353 &= 3^3 \cdot 23 \cdot 5693, \\
7272727 &= 7^2 \cdot 11 \cdot 103 \cdot 131.
\end{align*}

Misery loves company. So Professor X will be glad to learn that, fourteen years after Moser, Beiler [3] also gives a sample of exactly 4 seven-digit palindromic primes. They are (you guessed it) Moser's "primes". What is more, Beiler gives them without reference, thereby announcing, at least by implication, that they are his own "discoveries", since he can hardly claim that they are "well-known" primes. It is only fitting that he should now be hoist by his own petard.

One can only conjecture what Moser and the editor of Scripta Mathematica were thinking of when they published such baloney. The late Leo Moser was a mathematician and problemist of the first rank. He also had, as those who knew him can testify (this editor was privileged to meet him once), a highly-developed sense of humor and he loved a jape surpassing well. So perhaps ... (we leave the thought unfinished). Wherever he is right now, Leo Moser is probably having a good laugh at the gullibility of mortals.

In his famous Rede Lecture in 1959, the late C.P. Snow delineated and brought the world's consciousness to bear upon the gap between the "two cultures" (science and the humanities). As a result of efforts made since then to bridge the gap (not least by Snow himself), one would expect that by now every cultured person should know enough mathematics to recognize a small multiple of 3 (one of Leo Moser's "primes" was a multiple of 3). Yet in the April 1980 issue of the august Atlantic, a magazine that is just dripping with culture and history, Horace Judson (in "The
Rage To Know") claims that 1023 is a prime! The gap is widening, not being bridged.

As far as we know, the 93 five-digit palindromic primes and the 668 seven-digit ones have never been actually listed in the literature. Because of the misinformation floating around as a result of Leo Moser's "jape", and because lists of palindromes are so convenient to read by people who, like Professor X and Beiler, never know whether they are coming or going, it will be useful to record them in this issue. Rather than burying them in the Problem Section, we give them in a separate article (pages 266-268) so that they can later be more easily located in the index to this volume. We leave it to a more voluminous publication to publish the list of the 5172 nine-digit palindromic primes, one of which, a lovely example of the up-hill-and-down-dale type, is

345676543.

(Readers, remembering that the editor is another Leo, will probably want to check this.)

REFERENCES


(Dédie au souvenir de Victor Thébault, jadis inspecteur d'assurances à Le Mans, France.)

Résoudre la cryptarithmie décimale suivante:

UN + DEUX + DEUX + DEUX + DEUX + DEUX = ONZE.

I. Solution by Kenneth M. Wilke, Topeka, Kansas.
Clearly, D = 1 and 0 ≥ 5. Since X even implies N = E, it follows that X is odd
and $|E-N| = 5$. The problem is equivalent to
\[ \frac{10 \cdot \text{DEUX}}{2} = \text{ONZE} - \text{UN}, \]
from which $[\text{DE}/2] = 0$, and we have only the possibilities

$$(N,E,0) = (9,4,7), (8,3,6), (7,2,6), (3,8,9), (2,7,8), \text{ or } (0,5,7).$$

If $E$ is even then $N < 5$, and so $(N,E,0) = (3,8,9)$. But then $N = 3$ requires $U = 5$, which leaves $X = 7$ and produces no solution. If $E$ is odd then $N > 5$, and so $(N,E,0) = (8,3,6)$. Now $N = 8$ requires $U = 5$ and $X = 7$, from which $\text{DEUX} = 1357$, $\text{UN} = 58$, and $\text{ONZE} = 6843$. The unique solution is

$$58 + 1357 + 1357 + 1357 + 1357 + 1357 = 6843.$$ 

II. Comment by Donval R. Simpson, Fairbanks, Alaska.

Variants of this problem can be obtained by changing base 10 to some other base $b \geq 7$. An even more fundamental change is to change the language. In Spanish, for example, we have in base 10:

$$\text{UNO} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} = \text{ONCE}$$

with (at least) the solution

$$891 + 213 + 213 + 213 + 213 + 213 = 1956.$$ 

Also solved by J.A.H. Hunter, Toronto, Ontario; Allan Wm. Johnson Jr., Washington, D.C.; Edgar Lachance, Ottawa, Ontario; J.A. McCallum, Medicine Hat, Alberta; Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.; Herman Nyon, Paramaribo, Surinam; Donval R. Simpson, Fairbanks, Alaska; Charles W. Trigg, San Diego, California; and the proposer.

* * *

492. Proposed by Dan Pedoe, University of Minnesota.

(a) A segment $AB$ and a rusty compass of span $r \geq \frac{1}{3} AB$ are given. Show how to find the vertex $C$ of an equilateral triangle $ABC$ using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{3} AB$?

Solution by William A. McWorter, Jr., and Leroy F. Meyers, both from The Ohio State University (jointly).

We can draw only circles with fixed radius $r$, so the notation $(P)$ for a circle with center $P$ and radius $r$ will be unambiguous and convenient.

(a) The only centers for drawing circles which are available at the beginning of the construction are $A$ and $B$, and so we draw $(A)$ and $(B)$, which must intersect
since $r \geq \frac{1}{2}AB$. Let the points of intersection be $Z$ and $Z'$ (where $Z' = Z$ if $r = \frac{1}{2}AB$). If $B \in (A)$, then $r = AB$, and so $Z$ (or $Z'$) is the third vertex of the equilateral triangle $ABZ$ (or $ABZ'$), obtained by exactly two uses of the rusty compass. (This is Book I, Proposition 1 of Euclid.)

Otherwise, the required third vertex is not on any of the circles already drawn. The only new points which can be used as centers are $Z$ and $Z'$. (If $\frac{1}{2}AB < r < AB$, then $(A)$ and $(B)$ intersect $AB$ in two additional points $A'$ and $B'$. The circles $(A')$ and $(B')$ will then intersect in two points $K$ and $K'$ whose distance from $A$ and $B$ is less than $AB$, as is easy to check. Hence these four uses of the rusty compass will not suffice.) We now draw $(Z)$, and let $(A) \cap (Z) = \{X, X'\}$ and $(B) \cap (Z) = \{Y, Y'\}$, with the notation chosen so that $XY$ and $X'Y'$ are parallel to $AB$. (The intersections consist of two points, since $(A,B) \subset (Z)$.) None of the points on any of the three circles drawn so far is the required third vertex, and so we need to draw at least two more circles to determine it. Hence at least five circles are needed. In fact, if $(X) \cap (Y) = \{Z, C\}$, then $C$ is the required third vertex, as we will show. (Note that these five circles are just the ones anyone would try first.)

Now $YZXC$ and $YZB'$ are rhombi, since all sides have length $r$. Hence $CX \parallel YZ \parallel BY'$, and so $CXY'B$ is a parallelogram and $CB = XY'$. Using directed arcs on $(Z)$, we have

$$\text{arc } XY' = \text{arc } XA + \text{arc } AY' = \text{arc } Y'B + \text{arc } AY' = \text{arc } AB,$$

since triangles $AZX$ and $BZY$ are equilateral of side $r$. Hence $XY' = AB$, and so $CB = AB$. By the symmetry of the construction, also $CA = AB$, since $C$ lies on the perpendicular bisector of $AB$. Hence $ABC$ is equilateral. It may be noted that if $(X') \cap (Y') = \{Z, C'\}$, then $ABC'$ is also equilateral.

(b) We lay off points $A_1, A_2, \ldots, A_k$, and $B_1, B_2, \ldots, B_k$ on straight lines at the angle of 60° to $AB$ on the same side of it, beginning at $A$ and $B$, respectively, so that

$$AA_1 = A_1A_2 = \ldots = A_{k-1}A_k = BB_1 = B_1B_2 = \ldots = B_{k-1}B_k = r.$$

This is done by constructing, in order,

$$X_1 = AB \cap (A), \ A_1 \in (A) \cap (X_1), \ X_2 \in (X_1) \cap (A_1), \ A_2 \in (A_1) \cap (X_2), \ etc.$$

and similarly

$$Y_1 = AB \cap (B), \ B_1 \in (B) \cap (Y_1), \ Y_2 \in (Y_1) \cap (B_1), \ B_2 \in (B_1) \cap (Y_2), \ etc.$$

We choose $k$ so that $A_kB_k \leq 2r$, as must happen for some $k$. We use part (a) to construct an equilateral triangle $A_kB_kC$ on the appropriate side of $A_kB_k$. Then $ABC$ is equilateral. □
We have been unable to construct the third vertex C of an equilateral triangle ABC given only the points A and B (not the entire segment AB) if \( r < \frac{1}{2}AB \).

Also solved by CLAYTON W. DODGE, University of Maine at Orono; NGO TAN, student, J.F.Kennedy H.S., Bronx, N.Y.; and HERMAN NYON, Paramaribo, Surinam.

Solutions to part (a) only were submitted by JACK GARFUNKEL, Flushing, N.Y.; G.C. Giri, Midnapore College, West Bengal, India; FRIEND H. KIERSTEAD, Jr.,
Editor's comment.

All solutions to part (a) were essentially the same. The proposer noted that this part of the problem and its solution were suggested to him by Kevin Panzer, a student at the University of Minnesota. The problem of solving part (b) when only the points A and B, not the entire segment AB, are given (which may have been what the proposer intended since only the points A and B are needed for part (a)) remains open. For more information about geometry with a rusty compass, see [1] and [2].

REFERENCES


(a) A, B, C are the angles of a triangle. Prove that there are positive \( y, z, x \), each less than \( \frac{1}{2} \), simultaneously satisfying

\[
y^2 \cot \frac{B}{2} + 2yz + z^2 \cot \frac{C}{2} = \sin A,
\]

\[
z^2 \cot \frac{C}{2} + 2zx + x^2 \cot \frac{A}{2} = \sin B,
\]

\[
x^2 \cot \frac{A}{2} + 2xy + y^2 \cot \frac{B}{2} = \sin C.
\]

(b) In fact, \( \frac{1}{2} \) may be replaced by a smaller \( k > 0.4 \). What is the least value of \( k \)?

Partial solution by the proposer.

(a) Let \( S_1, S_2, S_3 \) be three circles each touching the other two externally and each touching a different pair of (unextended) sides of a triangle ABC, as shown in the figure. If \( r_1, r_2, r_3 \) are the radii of these circles, then we have

\[
r_2 \cot \frac{B}{2} + 2\sqrt{r_2 r_3} + r_3 \cot \frac{C}{2} = a = 2R \sin A,
\]

where \( R \) is the circumradius of the triangle, and two similar equations. For a given triangle, there is only one ratio \( r_1 : r_2 : r_3 \). The circles are called Malfatti circles and their unique existence is easily proved by continuity, as we now show. Remove the restriction that \( S_2 \) and \( S_3 \) touch each other and start with \( r_1 = r \), the inradius of the triangle. Then \( S_2 \) and \( S_3 \) do not meet. As \( r_1 \) is decreased, \( S_2 \) and \( S_3 \) approach each other and when \( r_1 \) is sufficiently small they intersect in
two points. Thus there is one and only one value of \( r_1 \) between 0 and \( r \) for which all three circles touch one another.

If we set

\[
\begin{align*}
x^2 &= r_1/2R, \\
y^2 &= r_2/2R, \\
z^2 &= r_3/2R,
\end{align*}
\]

we obtain the system of equations in the proposal. This system therefore has a unique solution in positive reals \( x, y, z \). Since

\[
r_i/2R < r/2R < \frac{1}{4}, \quad i = 1, 2, 3,
\]

we have \( 0 < x, y, z < \frac{1}{2} \).

(b) For given angle \( A \) and Malfatti radius \( r_1 \), we will assume that the length of \( BC = a \) is least when \( r_2 = r_3 \), that is, when \( B = C \). This seems obvious but I cannot find a simple proof. Since \( 2R \sin A \) is least when \( r_2 = r_3 \), it follows that \( x = \sqrt{r_1/2} \) is greatest when \( y = z \). Now the first equation in the proposal becomes

\[
2y^2(1 + \cot \frac{B}{2}) = \sin A
\]

or, since \( A = \pi - 2B \),

\[
y^2 = \frac{1}{2} \sin 2B/(1 + \cot \frac{B}{2}), \quad (1)
\]

and the third becomes

\[
x^2 \tan B + 2xy + y^2 \cot \frac{B}{2} = \sin B. \quad (2)
\]

If we eliminate \( y \) from (1) and (2), e.g., by expressing the trigonometric ratios in terms of \( t = \tan (B/2) \), we obtain
\[ x = \sqrt{\frac{1}{2}(1 - \tan \frac{B}{2})} \cdot (-\cos B + \cos \frac{B}{2} + \sin \frac{B}{2}). \]

Using numerical techniques, we find the approximation
\[ x_{\text{max}} \approx 0.409148 \text{ when } B \approx 67^\circ 03'. \]

Editor's comment.

Interested readers are invited to try to tie up the following loose ends in the solution to this problem:

i) Prove that, when A and r are fixed, BC is least when B = C.

ii) If \( M = x_{\text{max}} \approx 0.409148 \) then, for all triangles, \( 0 < x, y, z < M. \)

iii) It is probably too much to expect that the exact value of \( M \) can be found, but it may be possible to characterize geometrically the triangle in which \( x = M. \)

The following information may be useful:

It was recently proved in this journal [1980: 242] that, when A and the inradius \( r \) are fixed, BC is least when B = C.

Lob and Richmond (quoted by Goldberg in [1]) have shown that, for any triangle with inradius \( r = 1 \), the Malfatti radii are

\[ r_1 = \frac{(1+u)(1+v)}{2(1+u)}, \quad r_2 = \frac{(1+w)(1+u)}{2(1+v)}, \quad r_3 = \frac{(1+u)(1+v)}{2(1+w)}, \]

where \( u = \tan \left( \frac{A}{4} \right), \quad v = \tan \left( \frac{B}{4} \right), \quad w = \tan \left( \frac{C}{4} \right). \)

The Malfatti Problem dates from 1803. Solutions and historical information about the problem can be found, e.g., in Coolidge [2], Dörrie [3], and F. G.-M. [4].

REFERENCES


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Let \( r_j, \ j = 1, \ldots, k \), be the roots of a polynomial with integral coefficients and leading coefficient 1.

(a) For \( p \) a prime, show that
(Note that the sum is an integer, since it is a symmetric polynomial of the roots, and hence a polynomial of the coefficients.)

This generalizes Fermat's Little Theorem.

(b) Prove or disprove the corresponding extension of Gauss's generalization of Fermat's Theorem: for any positive integer $n$,

$$n|\sum_{j} \left( \sum_{d|n} r_{i}^{d} \mu(n/d) \right),$$

where $\mu$ is the Möbius function.

Solution of part (a) by the proposer.

In the expansion of $(\sum r_{i})^{p}$, each multinomial coefficient is of the form

$$\frac{p!}{i_{1}! \cdots i_{k}!},$$

with $\sum_{j=1}^{k} i_{j} = p$.

If some $i_{j} = p$, then the remaining $i_{j}$'s are all zero and the corresponding coefficient is 1; in all other cases the coefficient is a multiple of $p$. Thus

$$\left( \sum_{j=1}^{k} r_{i}^{j} \right)^{p} = \sum_{j=1}^{k} r_{i}^{p} + pQ, \quad (1)$$

where $Q$ is a polynomial in the $r_{i}^{j}$. Since the left member and the sum on the right in (1) are polynomials symmetric in the $r_{i}^{j}$, so is $Q$, and thus $Q$ is an integer. Accordingly,

$$\left( \sum_{j=1}^{k} r_{i}^{j} \right)^{p} \equiv \sum_{j=1}^{k} r_{i}^{p} \pmod{p}. \quad (2)$$

By Fermat's Theorem, the left side of (2) is congruent to $\sum r_{i}^{j}$ modulo $p$, and our conclusion follows. □

When the polynomial of the proposal is of degree 1, say $x - a$, our conclusion is the well-known $p|a^{p} - a$.

Editor's comment.

The proposer wrote that he discovered the generalization in part (a) many years ago and took some pride in it until he showed it to the late A.A. Albert, who stared at it intently for ten seconds and then uttered one word: "Trivial!" Our other readers, none of whom submitted a solution, apparently thought otherwise.

Part (b) remains open. The proposer wrote that he verified it for a number of numerical cases.

*   *   *
REVIEW


Subscribers and nonsubscribers to the Journal of Recreational Mathematics alike should welcome this first volume in the Excursions in Recreational Mathematics series. Nonsubscribers have a golden opportunity to sample a choice selection of articles from the Journal. These should whet their appetite for Recreational Mathematics. Each section is preceded by a lucid introduction to the subject matter. Subscribers will value the grouping together of articles on the same topic or similar topics for easy reference and comparison. The chronology of a problem and its solution (as far as that may go) may be traced. Moreover, some of the articles are from the earlier volumes of the Journal as well as from its predecessor, the Recreational Mathematics Magazine, and these would be difficult to find elsewhere.

Mathematical Solitaires and Games consists of 27 articles grouped under the following headings:
Section 1: Solitaire Games with Toys.
Section 2: Competitive Games.
Section 3: Solitaire Games.
Bonus Section: The Four-Color Problem.

The articles are of rather uneven quality, inevitable in a publication of this nature, but collectively the effect is quite inspiring. The publisher, the editor, and the Advisory Committee do maintain very stringent criteria in article selection. The line of division between Sections 1 and 3 may appear a little thin. The inclusion of the Bonus Section is an excellent concept.

A.C.-F. LIU,
University of Alberta.

MAMA-THEMATICS

Mrs. Hamilton to son William: "And I thought you learned your times table correctly years ago."

Mrs. Occam to son William: "And I thought you were going to grow a beard and had thrown away your shaving equipment."

Mrs. Boole to son George: "I only hope that some day you will learn something about the simple process of thought."

HOWARD EVES