# Crux Mathematicorum

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# E U R E K A

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## RAPPORT DE FIN D'ANNÉE

LÉO SAUVÉ, Collège Algonquin

Avec la présente livraison de décembre 1975, EUREKA touche à la fin de sa première année. A cette occasion, au lieu de me retrancher derrière l'anonymat de l'éditeur, j'ai décidé de signer cet article afin de pouvoir remercier, face à face pour ainsi dire, tous ceux qui m'ont aidé et encouragé depuis la publication d'EUREKA No. 1 en mars 1975, soit en m'envoyant des articles (trop peu, en vérité), soit en proposant des problèmes ou en envoyant des solutions.

Je dois d'abord des remerciements tout particuliers à M. Jose R. Holmes, qui était directeur du département de mathématiques du Collège Algonquin à l'époque des débuts d'EUREKA, dont l'encouragement ne s'est jamais démenti. Je remercie aussi par anticipation son successeur actuel, M. Ken H. Williams, qui m'a assuré de son entière collaboration pour faciliter la publication continue et la diffusion d'EUREKA.

Tous les lecteurs ont, sans le savoir, une lourde dette de reconnaissance envers mes deux collègues MM. H.G. Dworschak et F.G.B. Maskell, qui m'ont puissamment aidé dans la préparation et la distribution de la revue.

Le nombre des abonnés à la revue est maintenant tout près de trois cent vingt-cinq. (Il est permis de supposer que le nombre des *lecteurs* est beaucoup plus grand.) La plupart des abonnés sont évidemment dans la région d'Ottawa, mais il y en a aussi un bon nombre ailleurs en Ontario et au Québec, et quelques copies de la revue sont envoyées en Colombie Britannique, au Saskatchewan, au Nouveau Brunswick, aux Etats-Unis, et en Angleterre.

J'aimerais recevoir en abondance des lecteurs (en particulier, des universitaires, dont l'apport jusqu'à présent s'est très peu fait sentir) des articles, des problèmes, des solutions aux problèmes proposés, et des citations, poèmes, ou bons mots se rattachant de près ou de loin aux mathématiques.

Les langues officielles de la revue sont l'anglais et le français. Rien n'y est traduit: toutes les communications sont publiées telles que reçues, dans leur langue originale. L'éditeur, qui lui-même écrit en anglais ou en français, plus ou moins au gré de sa fantaisie, se réserve cependant le droit traditionnel de changer quelques virgules de temps en temps.

Je vous souhaite à tous une bonne et heureuse année. Que vos cogitations mathématiques soient des plus fructueuses en 1976.

* * *

## IN REVERSE

*I used to notch with him life's stick*
*And understand arithmetic;*
*But now it seems I've lost the chap*
*In the generation gap.*

*For every year I say plus one,*
*And that leaves me with quite a sum;*
*But he, I find, subtracts instead ...*
*I am thus some years ahead.*

*'Tis this new math. without a doubt,*
*By that I'm always put to rout:*
*I understand the simple rules,*
*But for the rest — I'm with the fools!*

*Now while it seems to me quite wise*
*To take my years in graceful stride,*
*I must admit the mathematician*
*Is in fact a neat magician.*

*But fig'ring may I fear just take*
*Our clever prof. back to the place*
*Where adolescence he will meet*
*Above his academic feet!*

*So come old man, just grow some more,*
*Ere babyhood is at your door:*
*Claim bravely now your total score ...*
*Second childhood is a bore!*

MONICA MASKELL (1970)

* * *

*There once was a brainy baboon*
*Who always breathed down a bassoon,*
*For he said, "It appears*
*That in billions of years*
*I shall certainly hit on a tune."*

SIR ARTHUR EDDINGTON

# PROBLEMS--PROBLÈMES

*Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 95.*

*For the problems given below, solutions, if available, will appear in EUREKA Vol.2, No.3, to be published around April 15, 1976. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than April 5, 1976.*

**91.** *Proposé par Léo Sauvé, Collège Algonquin.*

Si $a$, $a'$, $b$, et $b'$ sont des entiers positifs, montrer qu'une condition suffisante pour que la fraction $\frac{a+a'}{b+b'}$ soit irréductible est que

$$|ab' - ba'| = 1.$$

Cette condition est-elle aussi nécessaire?

**92.** *Proposé par Léo Sauvé, Collège Algonquin.*

Si $a$ est un entier positif, montrer que la fraction

$$\frac{a^3 + 2a}{a^4 + 3a^2 + 1}$$

est irréductible.

**93.** *Proposed by H.G. Dworschak, Algonquin College.*

Is there a convex polyhedron having exactly seven edges?

**94.** *Proposed by H.G. Dworschak, Algonquin College.*

If, in a tetrahedron, two pairs of opposite sides are orthogonal, is the third pair of opposite sides necessarily orthogonal?

**95.** *Proposed by Walter Bluger, Department of National Health and Welfare.*

Said a math teacher, full of sweet wine:
"Your house number's the exact square of mine."
—"You are tight and see double
Each digit. That's your trouble!"
These two-digit numbers you must divine.

**96.** *Proposed by Viktors Linis, University of Ottawa.*

By Euclidean methods divide a 13° angle into 13 equal parts.

**97.** *Proposed by Viktors Linis, University of Ottawa.*

Find all primes $p$ such that $p^3 + p^2 + 11p + 2$ is a prime.

**98.** *Proposed by Viktors Linis, University of Ottawa.*

Prove that, if $0 < a < b$, then $\ln\frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}$.

99. *Proposed by H.G. Dworschak, Algonquin College.*

If $a$, $b$, and $n$ are positive integers, prove that there exist positive integers $x$ and $y$ such that

$$(a^2+b^2)^n = x^2+y^2. \qquad (1)$$

*Application:* If $a=3$, $b=4$, and $n=7$, find at least one pair $\{x,y\}$ of positive integers which verifies (1).

100. *Proposé par Léo Sauvé, Collège Algonquin.*

Soit $f$ une fonction numérique continue et non négative pour tout $x \geq 0$. On suppose qu'il existe un nombre réel $a > 0$ tel que, pour tout $x > 0$,

$$f(x) \leq a\int_0^x f(t)dt.$$

Montrer que la fonction $f$ est nulle.

* * *

# SOLUTIONS

61. *Proposed by Léo Sauvé, Algonquin College.*

Find autological adjectives other than those given in the article on page 55 of EUREKA Vol. 1, No. 7.

*Editor's composite list selected from the contributions of H.G. Dworschak, Algonquin College; F.G.B. Maskell, Algonquin College; George W. Maskell, Huddersfield, England; and the proposer.*

The first group of adjectives, given below in no particular order, consists of those I consider to be essentially autological, although readers may perhaps quarrel with some of the borderline ones I have included.

Readable, translatable, typographical, finite,
adjectival, polysyllabic, understandable, pro-
nounceable, spellable, intelligible, visible,
sesquipedalian, inflected, lettered, *recherché*,
erudite, legible, consonantal, pidgin, abstract.

The adjectives in the next group are not essentially autological but can be rendered temporarily so by some device. They are, so to speak, transients or tourists in the land of Autology, not full citizens like those in the first group.

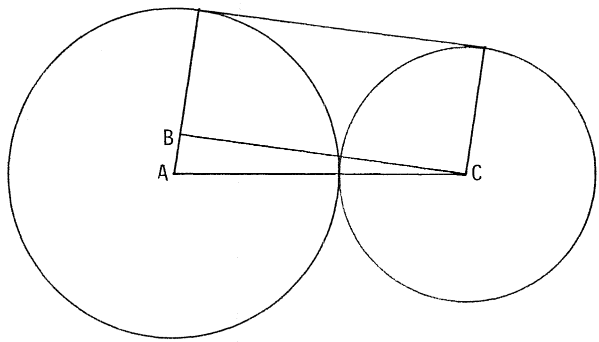
First, second, third, ..., black, printed, hyphen-
ated, *italicized*, CAPITALIZED, fō nĕt′ĭk, abbrev.,
central,
syl - lab - i - fied, mispelled, $\boxed{\text{boxed}}$, penultimate,
last.

62. *Proposed by F.G.B. Maskell, Algonquin College.*

Prove that if two circles touch externally, their common tangent is a mean proportional between their diameters.

*Solution de Richard Atlani, Collège Algonquin.*

Soit $t$ la longueur de la tangente commune des deux cercles et $d_1$, $d_2$ leurs diamètres. Dans le triangle rectangle ABC de la figure, il est clair que l'on a




$$|BC| = t, \quad |AB| = \frac{|d_1 - d_2|}{2}, \quad |AC| = \frac{d_1 + d_2}{2}.$$

Donc

$$t^2 = \left(\frac{d_1 + d_2}{2}\right)^2 - \left(\frac{d_1 - d_2}{2}\right)^2 = d_1 d_2,$$

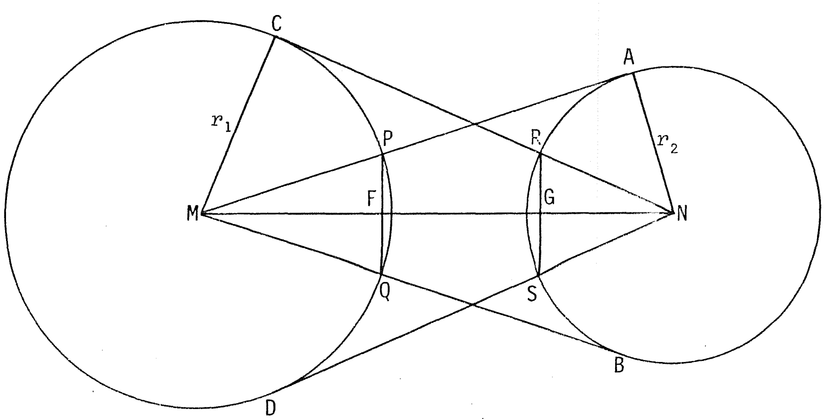
et $t$ est la moyenne proportionnelle entre $d_1$ et $d_2$.

*Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; Léo Sauvé, Collège Algonquin; and the proposer.*

63. *Proposed by H.G. Dworschak, Algonquin College.*

From the centres of each of two nonintersecting circles tangents are drawn to the other circle, as shown in the diagram below. Prove that the chords PQ and RS are equal in length. (I have been told that this problem originated with Newton, but have not been able to find the exact reference.)

*Solution by F.G.B. Maskell, Algonquin College.*




The line of centres MN intersects the chords PQ and RS at their midpoints

F and G. It is clear that

$$\Delta MFP \sim \Delta MAN \quad \text{and} \quad \Delta NGR \sim \Delta NCM.$$

Hence, if $r_1$ and $r_2$ denote the radii, as shown in the figure, we have

$$\tfrac{1}{2}\,PQ = FP = \frac{r_1 r_2}{MN} \qquad \text{and} \qquad \tfrac{1}{2}\,RS = GR = \frac{r_1 r_2}{MN},$$

and so $PQ = RS$.

*Also solved by Richard Atlani, Collège Algonquin; G.D. Kaye, Department of National Defence; B. Vanbrugghe, Université de Moncton; and the proposer.*

64. *Proposé par Léo Sauvé, Collège Algonquin.*

Décomposer 10,000,000,099 en un produit d'au moins deux facteurs.

*Solution by H.G. Dworschak, Algonquin College.*

If we set $x = 100$, then

$$\begin{aligned} 10{,}000{,}000{,}099 &= x^5 + x - 1 \\ &= (x^3 + x^2 - 1)(x^2 - x + 1) \\ &= 1{,}009{,}999 \times 9{,}901. \end{aligned}$$

A bit of trial and error division by a few low primes soon yields $1{,}009{,}999 = 23 \times 43913$ so that

$$10{,}000{,}000{,}099 = 23 \times 9901 \times 43913.$$

This is the complete factorization into primes, as can be verified from any table of primes. (For example, Chemical Rubber's *Standard Mathematical Tables*, 22nd edition, contains a table giving all primes to 100,000.)

*Also solved by F.G.B. Maskell, Algonquin College; and the proposer. Each of them found only two factors.*

65. *Proposed by Viktors Linis, University of Ottawa.*

Find all natural numbers whose square (in base 10) is represented by odd digits only.

*Solution by G.D. Kaye, Department of National Defence.*

We need consider only odd natural numbers. 1 and 3 are the only natural numbers having the desired property. For if $n = 10x + y$, where $y = 1,3,5,7$, or 9, then

$$n^2 = (10x + y)^2 = 100x^2 + 20xy + y^2,$$

and it is clear that the tens digit of $n^2$ is the same as the tens digit of $y^2$, which is even in every case.

*Also solved by F.G.B. Maskell, Algonquin College; and Léo Sauvé, Collège Algonquin.*

66. *Proposed by John Thomas, University of Ottawa.*

What is the largest non-trivial subgroup of the group of permutations on $n$ elements?

*Solution by Léo Sauvé, Algonquin College.*

We may without loss of generality assume that the group in question is the symmetric group $S_n$ of all permutations of the set $E = \{1,2,\ldots,n\}$.

The order of $S_n$ is $n!$ and, by Lagrange's Theorem, the order of any subgroup of $S_n$ must be a divisor of $n!$. Hence no non-trivial subgroup of $S_n$ can have order greater than $\frac{n!}{2}$. Now the order of the alternating group $A_n$, consisting of the even permutations of E, is precisely $\frac{n!}{2}$; and so $A_n$ furnishes an answer to the problem.

A discussion of this question can be found in most good algebra books. A particularly clear presentation of it can be found on pp. 145 - 146 of *A Survey of Modern Algebra*, by Birkhoff and MacLane (Macmillan, 1953).

67. *Proposed by Viktors Linis, University of Ottawa.*

Show that in any convex $2n$-gon there is a diagonal which is not parallel to any of its sides.

*Solution by the proposer.*

The total number of diagonals in a convex $2n$-gon is $\frac{1}{2}(2n)(2n-3) = n(2n-3)$. The diagonals which can be parallel to a fixed side must connect two vertices. The $2n-2$ vertices not on the fixed side must include at least one vertex not on a diagonal parallel to the fixed side; hence there can be at most $\frac{1}{2}(2n-4)$ diagonals parallel to the fixed side, or altogether at most $2n\cdot\frac{1}{2}(2n-4) = n(2n-4)$ diagonals which are parallel to some side. Of the remaining diagonals, at least

$$n(2n-3) - n(2n-4) = n$$

are not parallel to any side.

68. *Proposed by H.G. Dworschak, Algonquin College.*

It takes 5 minutes to cross a certain bridge and 1000 people cross it in a day of 12 hours, all times of day being equally likely. Find the probability that there will be nobody on the bridge at noon.

*Solution by G.D. Kaye, Department of National Defence.*

If there is nobody on the bridge at noon, no one has entered it in the 5-minute interval before noon. Since there are 144 intervals of 5 minutes in 12 hours, the probability that an individual enters the bridge in a specific one is $\frac{1}{144}$. The probability that none of the 1000 individuals enters in that interval is

$$\left(1 - \frac{1}{144}\right)^{1000} = \left[\left(1 - \frac{1}{144}\right)^{144}\right]^{\frac{125}{18}} \approx e^{-\frac{125}{18}} \approx 0.00096.$$

*Also solved by Léo Sauvé, Algonquin College; and the proposer.*

69. *Proposé par Léo Sauvé, Collège Algonquin.*

Existe-t-il une permutation $n \longmapsto a_n$ de l'ensemble N des entiers naturels telle que la série $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converge?

*Solution du proposeur.*

Il n'existe pas de telle permutation, car une condition nécessaire pour la convergence de la série est

$$\lim_{k\to\infty} \sum_{n=k+1}^{2k} \frac{a_n}{n^2} = 0,$$

tandis que, pour chaque permutation $n \longmapsto a_n$ et pour chaque $k \geq 1$ on a

$$\sum_{n=k+1}^{2k} \frac{a_n}{n^2} > \sum_{n=k+1}^{2k} \frac{a_n}{4k^2} = \frac{1}{4k^2} \sum_{n=k+1}^{2k} a_n \geq \frac{1}{4k^2} \sum_{n=1}^{k} n = \frac{1}{4k^2} \cdot \frac{k(k+1)}{2} = \frac{1}{8}(1+\frac{1}{k}) > \frac{1}{8}.$$

70. *Proposed by Viktors Linis, University of Ottawa.*

Show that for any 13-gon there exists a straight line containing only one of its sides. Show also that for every $n > 13$ there exists an $n$-gon for which the above statement does not hold.

*Solution by the proposer.*

Assume that each line determined by the sides of a 13-gon contains 2 or more sides. The number of such lines is at most six. Each of these six lines intersects the other lines in at most five points and therefore the number of sides on each line cannot be more than 2 (each side determines an *even* number of points) or $2 \times 6 = 12$ at most which is a contradiction to the 13 sides of a 13-gon. Consequently there must be a line having only one side on it.

For any even-numbered $n$-gon, construct a convex $\frac{n}{2}$-gon, and "bend" each side on a diagonal as indicated in Figure 1. Each line contains 2 sides on it.

For any odd-numbered $n$-gon start construction from the 15-gon as given by Figure 2. Then "cut off" by a line two corners, thus increasing the number of sides by two at each stage.

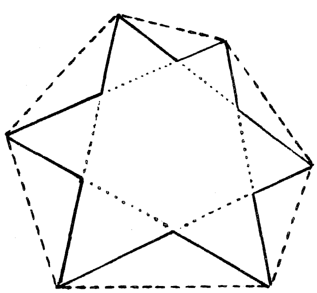


Figure 1

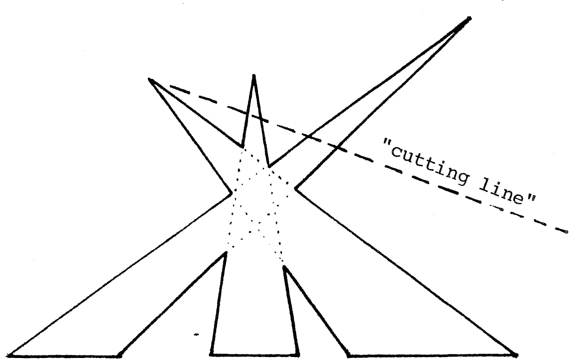



Figure 2

*Also solved by G.D. Kaye, Department of National Defence.*