Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

E U R E K A

Vol. 1, No. 9

November 1975

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education

Published by Algonquin College

Send all communications to the editor: Léo Sauvé, Math—Architecture, Algonquin College, Col. By Campus, 281 Echo Drive, Ottawa, Ont., K1S 5G2.

* * *

NOTE FROM THE EDITOR

The postal strike hit the country just as our issue No. 8 (October 1975) was being printed. We managed nevertheless to have many copies delivered to subscribers by using the internal mailing systems of the various school boards and universities in the Ottawa region. The remaining copies were mailed shortly after postal service resumed. It is probable that in spite of our efforts some copies did not reach their destinations. If you have not yet received EUREKA No. 8, please drop a note to the Secretary-Treasurer of COMA

Mr. F.G.B. Maskell,
Algonquin College,
Rideau Campus,
200 Lees Ave.,
Ottawa,
K1S 0C5.

He will send you a copy forthwith.

Regular publication resumes with the present November issue, which should reach readers around January 15, 1976. The next four or five issues will be published at a slightly accelerated pace to make up for the time lost because of the postal strike.

* * *

> "Let us sit on this log at the roadside," says I, "and forget the inhumanity and ribaldry of the poets. It is in the glorious columns of ascertained facts and legalised measures that beauty is to be found. In this very log we sit upon, Mrs. Sampson," says I, "is statistics more wonderful than any poem. The rings show that it was sixty years old. At the depth of two thousand feet it would become coal in three thousand years. The deepest coal mine in the world is at Killingworth, near Newcastle. A box four feet long, three feet wide, and two feet eight inches deep will hold one ton of coal. If an artery is cut, compress it above the wound. A man's leg contains thirty bones. The Tower of London was burned in 1841."
>
> "Go on, Mr. Pratt," says Mrs. Sampson. "Them ideas is so original and soothing. I think statistics are just as lovely as they can be."
>
> O. HENRY, *The Handbook of Hymen.*[1]

---

[1]Submitted by F.G.B. Maskell, Algonquin College.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 83.

*For the problems given below, solutions, if available, will appear in EUREKA Vol.*2, *No.*2, *to be published around March* 25, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than March* 15, 1976.

81. *Proposed by H.G. Dworschak, Algonquin College.*

Which of the following are divisible by 6 for all positive integers n?

(i) $n(n+1)(n+2)$

(ii) $n(n+1)(2n+1)$

(iii) $n(n^2+5)$

(iv) $(n+1)^{2k}-(n^{2k}+2n+1)$, k a positive integer.

82. *Proposé par Léo Sauvé, Collège Algonquin.*

Soit E un ensemble fini qui contient n éléments. Les faits suivants sont bien connus ou faciles à démontrer:

(a) Le nombre de sous-ensembles de E est 2^n.

(b) Le nombre de relations de la forme $A \subseteq B$, où $A \subseteq E$ et $B \subseteq E$, est $(2^n)^2 = 4^n$.

Combien des relations de (b) sont vraies?

83. *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que le produit de deux, trois ou quatre entiers positifs consécutifs n'est jamais un carré parfait?

84. *Proposed by Viktors Linis, University of Ottawa.*

Prove that for any positive integer n

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}\ .$$

85. *Proposed by Viktors Linis, University of Ottawa.*

Find n natural numbers such that the sum of any number of them is never a square.

86. *Proposed by Viktors Linis, University of Ottawa.*

Find all rational Pythagorean triples (a,b,c) such that

$$a^2+b^2=c^2 \quad \text{and} \quad a+b=c^2.$$

87. *Proposed by H.G. Dworschak, Algonquin College.*

(a) If $u_n = x^{2n}+x^n+1$, for which positive integer n is u_n divisible by u_1?

(b) For which positive integer n does $x+\frac{1}{x}=1$ imply $x^n+\frac{1}{x^n}=1$?

88. *Proposé par F.G.B. Maskell, Collège Algonquin.*

Evaluer l'intégrale indéfinie

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}}$$

89. *Proposed by Vince Bradley, Algonquin College and, independently, by Christine Robertson, Canterbury High School.*

A goat is tethered to a point on the circumference of a circular field of radius r by a rope of length ℓ. For what value of ℓ will it be able to graze over exactly half of the field?

90. *Proposed by Léo Sauvé, Algonquin College.*

(a) Determine, as a function of the positive integer n, the number of odd binomial coefficients in the expansion of $(a+b)^n$.

(b) Do the same for the number of odd multinomial coefficients in the expansion of $(a_1+a_2+\cdots+a_r)^n$.

* * *

SOLUTIONS

43. [1975, *p.* 73] *Proposed by André Bourbeau, École Secondaire Garneau.*

In a 3×3 matrix, the entries a_{ij} are randomly selected integers such that $0 \le a_{ij} \le 9$. Find the probability that

(a) the three-digit numbers formed by each row will be divisible by 11;

(b) the three-digit numbers formed by each row and each column will be divisible by 11.

Solution of (b) by Viktors Linis, University of Ottawa.

A three-digit number $100a+10b+c$ (allowing also zeros for a, b, c) is divisible by 11 if and only if $a-b+c \equiv 0 \pmod{11}$. The required matrix can be expressed in the form

$$\begin{bmatrix} s & s+t & t \\ s+u & s+t+u+v & t+v \\ u & u+v & v \end{bmatrix}$$

with four independent parameters s, t, u, v taking the values 0, 1, ..., 9 and all other entries $\not\equiv 10 \pmod{11}$. The problem is to find the number N_0 of quadruples (s,t,u,v) satisfying these conditions.

The total number N of quadruples is 10^4. Let S_i be the set of quadruples which satisfies the condition (i), $i = 1, 2, \ldots, 7$ where

(1) $s + t = 10$

(2) $s + u = 10$

(3) $t + v = 10$

(4) $u + v = 10$

(5) $s + t + u + v = 10$

(6) $s + t + u + v = 21$

(7) $s + t + u + v = 32$

If we set $S_{ij\ldots m} = S_i \cap S_j \cap \ldots \cap S_m$, then by the inclusion-exclusion principle of combinatorics the number of quadruples *not* satisfying conditions (1) - (7) is

$$N_0 = N - \Sigma \pm N_{ij\ldots m} \qquad (*)$$

where $N = 10^4$, $N_{ij\ldots m} = |S_{ij\ldots m}|$, and the summation is over all combinations of set-intersections with $i < j < \ldots < m$, + sign for odd number of sets, - sign for even. (To obtain the number of solutions of linear equations (1) - (7), see Problem 6, EUREKA 1975, p. 16; however, note the restriction $0 \le s,t,u,v \le 9$.)

We find

$N_i = 900$, $i = 1,2,3,4$; $N_5 = 282$, $N_6 = 592$, $N_7 = 35$;

$N_{12} = N_{13} = N_{24} = N_{34} = 90$; $N_{14} = N_{23} = 81$;

$N_{i5} = 9$, $N_{i6} = 72$, $N_{i7} = 0$, $i = 1,2,3,4$; $N_{56} = N_{57} = N_{67} = 0$;

$N_{123} = N_{124} = N_{134} = N_{234} = 9$; $N_{ij5} = 0$;

$N_{126} = N_{136} = N_{246} = N_{346} = 8$; all other $N_{ijk} = 0$;

$N_{1234} = 9$; all other $N_{ij\ldots m} = 0$.

Substituting all these values in (*) gives $N_0 = 6278$.

The probability that every three-digit number formed by all rows and columns in a 3×3 matrix is divisible by 11 is therefore 6.278×10^{-6}.

Part (b) was also solved by H.G. Dworschak, Algonquin College, who also found $N_0 = 6278$; *by another reader who found* $N_0 = 6254$; *and by a student who set up a computer program to find* N_0 *and obtained* $N_0 = 6537$.[1]

51. *Proposed by H.G. Dworschak, Algonquin College.*

Solve the following equation for the positive integers x and y:

$$(360 + 3x)^2 = 492{,}y04.$$

Solution d'André Ladouceur, École Secondaire De La Salle.

Le membre gauche de l'équation donnée étant divisible par 9, il en est de même de la somme des chiffres du membre droit, d'après le critère bien connu de divisibilité par 9. On doit donc avoir $y = 8$, et un calcul simple donne ensuite $x = 114$.

Also solved by G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; Léo Sauvé, Algonquin College; and the proposer.

---

[1] *LAST-MINUTE ADDITION: John Thomas, University of Ottawa, also set up a computer program and found* $N_0 = 6278$.

52. *Proposed by Viktors Linis, University of Ottawa.*

The sum of one hundred positive integers, each less than 100, is 200. Show that one can select a partial sum equal to 100.

Solutions were submitted by G.D. Kaye, Department of National Defence; and F.G.B. Maskell, Algonquin College.

Editor's comment.

This problem is a special case of the following, proposed by Judith Q. Longyear, Pennsylvania State University, and published as Problem E 2374 in the *American Mathematical Monthly* [1972, p. 905]:

Suppose that $a_1 \le a_2 \le \ldots \le a_n$ *are natural numbers such that* $a_1 + \ldots + a_n = 2n$ *and such that* $a_n \ne n+1$. *Show that if* n *is even, then for some subset* K *of* $\{1,2,\ldots,n\}$ *it is true that* $\Sigma_{i\in K} a_i = n$. *Show that this is true also if* n *is odd when we make the additional assumption that* $a_n \ne 2$.

The following solution by J.G. Mauldon, Amherst College, was later published in the *Monthly* [1973, p. 946]:

Let $s_k = a_1 + \ldots + a_k$ for $k = 1,2,\ldots,n-1$ and consider the set of $n+1$ (distinct) numbers

$$\{0,\ a_1 - a_n,\ s_1,\ s_2,\ \ldots,\ s_{n-1}\}.$$

By the Pigeonhole Principle, at least one pair of elements are congruent to each other modulo n. We distinguish four cases.

(1) If $a_1 - a_n \equiv 0 \pmod{n}$, then $a_1 = a_n$ since $0 \ge a_1 - a_n \ge -n+1$. Thus in this case it follows that $a_1 = \ldots = a_n = 2$. If $n = 2m$ is even, then obviously any m-element subset K will do, whereas if n is odd, this circumstance is prohibited by assumption.

(2) If $s_i \equiv s_k \pmod{n}$ for some $1 \le i < k \le n-1$, then since $2n-2 \ge s_k - s_i \ge 1$ it follows that $s_k - s_i = n$, i.e., $a_{i+1} + \ldots + a_k = n$, and we can take

$$K = \{i+1,\ i+2,\ \ldots,\ k\}.$$

(3) If $s_k \equiv 0 \pmod{n}$ for some $1 \le k \le n-1$, then $1 \le s_k \le 2n-1$, so that necessarily $s_k = n$, and we can take $K = \{1,\ 2,\ \ldots,\ k\}$.

(4) If $s_k \equiv a_1 - a_n \pmod{n}$ for some $1 \le k \le n-1$, then either $k = 1$, implying $a_n \equiv 0 \pmod{n}$ and thus $a_n = n$ (since $a_n \le n$) so that $K = \{n\}$ will do, or else $k \ge 2$ in which case $a_2 + \ldots + a_k + a_n \equiv 0 \pmod{n}$. In this latter case,

$$1 \le a_2 + \ldots + a_k + a_n < a_1 + a_2 + \ldots + a_k + \ldots + a_n = 2n$$

so that $a_2 + \ldots + a_k + a_n = n$, and we can take $K = \{2,\ 3,\ \ldots,\ k,\ n\}$.

The editor of the *Monthly* adds the following interesting comment:

> This problem has the following interesting interpretation. Suppose that one has a number of blocks of integral weight, which average 2 weight units apiece, and which have the property that the heaviest block is not heavier than all the rest of the blocks

put together. Then, with the exception of the case where one has an odd number of blocks all of weight 2, it is always possible to separate the blocks into two groups of equal weight. Moreover, it can be shown that the following algorithm will always lead to a solution of the problem: Take a two-pan balance, and starting with the heaviest weight and proceeding step-by-step with the next heaviest, etc., place the weights one at a time on that pan of the balance which is the lighter (if the pans are balanced, choose either). Then the placing of the last (lighest) block must necessarily make the two pans balance exactly.

53. *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que la somme de tous les entiers positifs inférieurs à $10n$ qui ne sont pas des multiples de 2 ou 5 est $20n^2$.

I. *Solution by H.G. Dworschak, Algonquin College.*

The required sum is that of an arithmetic progression:

$$S = (1+3+7+9) + (11+13+17+19) + \dots \text{ to } n \text{ terms}$$
$$= 20 + 60 + \dots \text{ to } n \text{ terms}$$
$$= \frac{n}{2}[40 + (n-1)40]$$
$$= 20n^2.$$

II. *Solution d'André Ladouceur, École Secondaire De La Salle.*

La somme proposée S_n est la différence de deux progressions arithmétiques dont la première contient $5n$ termes et la seconde n termes:

$$S_n = (1+3+5+\dots+10n-1) - (5+15+\dots+10n-5)$$
$$= \frac{5n}{2}[1+(10n-1)] - \frac{n}{2}[5+(10n-5)]$$
$$= 20n^2.$$

III. *Solution du proposeur.*

La démonstration est par induction. La propriété est certainement vraie pour $n=1$. Les entiers permis entre $10n$ et $10(n+1)$ sont

$$10n+1, \quad 10n+3, \quad 10n+7, \quad 10n+9,$$

dont la somme $40n+20$ ajoutée à $20n^2$ donne $20(n+1)^2$.

Also solved by G.D. Kaye, Department of National Defence. One incorrect solution was received.

54. *Proposé par Léo Sauvé, Collège Algonquin.*

Si $a,b,c>0$ et $a<b+c$, montrer que

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

I. *Solution by F.G.B. Maskell, Algonquin College.*

We have

$$\frac{b}{1+b} + \frac{c}{1+c} - \frac{a}{1+a} = \frac{abc + 2bc + (b+c-a)}{(1+a)(1+b)(1+c)} > 0,$$

since every term in the numerator is positive.

II. *Solution du proposeur.*

L'hypothèse donne $\frac{1}{a} > \frac{1}{b+c}$, d'où $1 + \frac{1}{a} > 1 + \frac{1}{b+c}$, c'est-à-dire

$$\frac{1+a}{a} > \frac{1+b+c}{b+c}.$$

Donc

$$\frac{a}{1+a} < \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} < \frac{b}{1+b} + \frac{c}{1+c}.$$

Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; and André Ladouceur, École Secondaire De La Salle.

Editor's comment.

The proposition in this problem can be used to prove that if E is a space with metric d, then the function $\delta: E \times E \to R$ defined by

$$\delta(A,B) = \frac{d(A,B)}{1 + d(A,B)}$$

is also a metric for E. It is, in fact, a *bounded* metric, since $0 \le \delta(A,B) < 1$.

55. *Proposed by Viktors Linis, University of Ottawa.*

What is the last digit of $1 + 2 + \ldots + n$ if the last digit of $1^3 + 2^3 + \ldots + n^3$ is 1?

Solution by G.D. Kaye, Department of National Defence.

Since the number

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2 = [\Sigma n]^2$$

ends in 1, the number

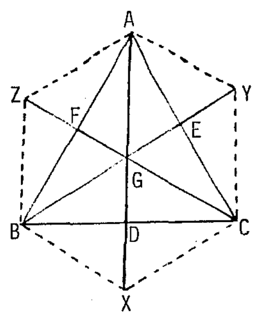
$$\Sigma n = \frac{n(n+1)}{2} \qquad (*)$$

must end in 1 or 9. Since the product of two consecutive integers can only end in 0, 2, or 6, the number (*) cannot possibly end in 9; it must therefore end in 1.

Also solved by H.G. Dworschak, Algonquin College; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and Léo Sauvé, Collège Algonquin.

56. *Proposed by F.G.B. Maskell, Algonquin College.*

The area of a triangle in terms of its sides is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. What is the area in terms of its medians m_1, m_2, m_3?



Solution by the proposer.

In ΔABC extend the medians AD, BE, CF, which meet in G, to X, Y, Z respectively, so that

$$DX = GD, \quad EY = GE, \quad FZ = GF.$$

Then BGCX, CGAY, and AGBZ are parallelograms, with

$$GX = AG = \frac{2}{3}m_1, \qquad GY = BG = \frac{2}{3}m_2, \qquad GZ = CG = \frac{2}{3}m_3,$$

so that

$$\Delta BGX = \Delta CGY = \Delta AGZ.$$

Now if $m = \frac{1}{2}(m_1 + m_2 + m_3)$, we have

$$\begin{aligned}\Delta ABC &= \Delta BGC + \Delta CGA + \Delta AGB\\ &= \Delta BGX + \Delta CGY + \Delta AGZ\\ &= 3\Delta BGX\\ &= 3\sqrt{\frac{2}{3}m\cdot\frac{2}{3}(m-m_1)\cdot\frac{2}{3}(m-m_2)\cdot\frac{2}{3}(m-m_3)}\\ &= \frac{4}{3}\sqrt{m(m-m_1)(m-m_2)(m-m_3)}.\end{aligned}$$

Also solved by H.G. Dworschak, Algonquin College; and G.D. Kaye, Department of National Defence.

Editor's comment.

The above Heron-like formula for the area K of a triangle in terms of its medians can be found, for example, on p. 204 in *College Geometry*, by David C. Kay (Holt, Rinehart and Winston, 1969).

It is natural to wonder whether there is also a Heron-like formula for the area K in terms of the altitudes h_1, h_2, h_3. There is. The same reference gives (on the same p. 204) the following:

$$\frac{1}{K} = 4\sqrt{\frac{1}{h}\left(\frac{1}{h}-\frac{1}{h_1}\right)\left(\frac{1}{h}-\frac{1}{h_2}\right)\left(\frac{1}{h}-\frac{1}{h_3}\right)} \qquad (1)$$

where $\frac{1}{h} = \frac{1}{2}\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)$.

Dworschak also mentioned formula (1) and submitted the following proof:

If r is the inradius and r_1, r_2, r_3 the exradii of the triangle, we have the well-known formulas:

$$K^2 = r r_1 r_2 r_3, \qquad \frac{1}{r} = \frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}, \text{ [cf. EUREKA 1975, p.75]}$$

$$\frac{1}{r_1} = \frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}, \qquad \frac{1}{r_2} = \frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}, \qquad \frac{1}{r_3} = \frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}.$$

Thus if we set $\frac{1}{h} = \frac{1}{2}\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)$, we have

$$\begin{aligned}\frac{1}{K^2} &= \frac{1}{r}\cdot\frac{1}{r_1}\cdot\frac{1}{r_2}\cdot\frac{1}{r_3}\\ &= \left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}\right)\left(\frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}\right)\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}\right)\\ &= 16\cdot\frac{1}{h}\left(\frac{1}{h}-\frac{1}{h_1}\right)\left(\frac{1}{h}-\frac{1}{h_2}\right)\left(\frac{1}{h}-\frac{1}{h_3}\right),\end{aligned}$$

and (1) follows.

57. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit G un groupe d'ordre pn ou p est premier et $p \geq n$. Si H est un sous-groupe d'ordre p alors H est normal dans G.

Solution du proposeur.

Soit $E = \{xH : x \in G\}$ l'ensemble des classes à gauche de H. Soit $P = \{\Gamma_g : g \in G\}$ le groupe de permutations sur E définies par $\Gamma_g(xH) = gxH$ et $\phi : G \longrightarrow P$ l'homomorphisme défini par $\phi(g) = \Gamma_g$.

Si $g \in \text{Ker}\,\phi$, Γ_g est la permutation identité sur E, c'est-à-dire $\forall x \in G$, $gxH = xH$. En particulier, quand $x = e$, $gH = H$, d'où $g \in H$. Donc $\text{Ker}\,\phi \subset H$. Mais d'autre part $\text{Ker}\,\phi$ est un sous-groupe normal de G, donc de H. Vu que $|H| = p$ (premier), ou bien $\text{Ker}\,\phi = \{e\}$, ou bien $\text{Ker}\,\phi = H$. Si $\text{Ker}\,\phi = \{e\}$, alors

$$G / \text{Ker}\,\phi \approx G \approx \text{Im}\,\phi = P < S_n$$

(la dernière relation se lit: P est un sous-groupe du groupe symétrique S_n), car $|E| = [G : H] = n$; on aurait donc que $|G| = |P|$ divise $n!$, c'est-à-dire $pn \mid n!$ ou $p \mid (n-1)!$ Ceci est impossible car p est premier et $p \geq n$. Par conséquent $\text{Ker}\,\phi = H$ et H est normal.

58. [1975, p.49]

Editor's comment.

No solution has been received for this problem, which is therefore still open. John Thomas, University of Ottawa, pointed out that the problem seems to be a reformulation of Problem 7 on p.49 of *Fonctions Analytiques*, by S. Saks and A. Zygmund (Masson, Paris, 1970).

59. *Proposed by John Thomas, University of Ottawa.*

Find the shortest proof to the following proposition: every open subset of R is a countable disjoint union of open intervals.

I. *Solution by Léo Sauvé, Algonquin College.*

My candidate for the shortest proof is taken from p.231 in Klambauer [2]:

> If A is an open set of real numbers, we say that the real numbers x and y are "equivalent" and write $x \sim y$ if and only if there is an open interval (a,b) with the set $\{x,y\} \subset (a,b) \subset A$. The relation $\sim$ is clearly an equivalence relation on A and the resulting equivalence classes are disjoint open intervals whose union is A. The fact that there can be only countably many follows since each must contain a distinct rational number.

This proof may or may not be the shortest, but it is at least shorter than some, since the same reference (Klambauer [2]) contains on p.148 another proof of the same result which takes up 24 lines. The problem occurs also on p.25 of Klambauer [3], where the *hints* for the solution take up 22 lines.

II. *Solution by the proposer (oral communication to the editor).*

I propose the following theorem and proof from p.17 in Conway [1], which can be easily modified for *R* instead of C:

THEOREM. *Let* G *be open in* C; *then the components of* G *are open and there are only a countable number of them.*

PROOF. Let A be a component of G and let $x_0 \in A$. Since G is open there is an $\varepsilon > 0$ with $B(x_0;\varepsilon) \subset G$. By Lemma 2.6 [whose proof takes up 7 lines], $B(x_0;\varepsilon) \cup A$ is connected and so must be A. That is $B(x_0;\varepsilon) \subset A$ and A is, therefore, open.

To see that the number of components is countable let $S = \{a + ib$: a and b are rational and $a + bi \in G\}$. Then S is countable and each component of G contains a point of S, so that the number of components is countable.

Editor's comment.

To find the shortest proof, one would have to examine and *measure* all possible proofs. Now the set of known proofs of this theorem is certainly finite, but it is not known if the set of all possible proofs is even countable, let alone measurable.

The problem can be considered as closed, unless some reader comes up with a proof shorter than the two given above.

This seems to be a good place to point out that reference [2], by University of Ottawa professor Gabriel Klambauer, is a remarkable book which is full of interesting and unusual problems. It is an ideal bedside book for a budding analyst. Reference [3], which is equally remarkable, is more advanced.

REFERENCES

1. John B. Conway, *Functions of One Complex Variable*, Springer-Verlag, 1973.
2. Gabriel Klambauer, *Mathematical Analysis*, Marcel Dekker Inc., 1975.
3. Gabriel Klambauer, *Real Analysis*, American Elsevier, 1973.

60. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit f une fonction analytique sur le disque fermé $\bar{B}(0,R) = \{z : |z| \le R\}$ telle que $|f(z)| < M$, et $|f(0)| = a > 0$. Montrer que le nombre de zéros de f dans $B(0,\frac{R}{3})$ est inférieur ou égal à $\frac{1}{\log 2}\log\frac{M}{a}$.

Solution du proposeur.

Si z_k, $k = 1,2,\ldots,n$ sont les zéros de f dans $B(0,\frac{R}{3})$, la fonction

$$g(z) = \frac{f(z)}{\prod_{k=1}^{n}\left(1 - \frac{z}{z_k}\right)}$$

est analytique sur $\bar{B}(0,R)$ et $|g(0)| = |f(0)| = a$. Or, pour tout $z \in \bar{B}(0,R)$ tel que $g(z) \neq 0$, on a

$$\prod_{k=1}^{n}\left|1 - \frac{|z|}{|z_k|}\right| \leq \prod_{k=1}^{n}\left|1 - \frac{z}{z_k}\right| = \frac{|f(z)|}{|g(z)|} \leq \frac{M}{|g(z)|}.$$

Vu que g est analytique sur $\bar{B}(0,R)$, $|g|$ atteint son maximum sur $\partial B(0,R)$, c'est-à-dire qu'il existe un $z_0 \in \partial B(0,R)$ tel que $|g(z_0)| \geq |g(0)| = a$. De plus, $|z_0| = R$ et, pour $k = 1,\ldots,n$, $|z_k| < \frac{R}{3}$, de sorte que

$$|z_0| > 3|z_k| \qquad \text{et} \qquad \left|1 - \frac{|z_0|}{|z_k|}\right| = \frac{|z_0|}{|z_k|} - 1 > 2.$$

Donc

$$2^n \leq \prod_{k=1}^{n}\left|1 - \frac{|z_0|}{|z_k|}\right| \leq \frac{M}{|g(z_0)|} \leq \frac{M}{a},$$

et

$$n \leq \frac{1}{\log 2}\log\left(\frac{M}{a}\right).$$

Editor's comment.

The proposer, in an attached note, pointed out that this problem, with a hint for its solution, can be found on p.126 in *Functions of One Complex Variable*, by John B. Conway (Springer-Verlag, 1973).

* * *

Mathematics is a madness
That gets into the bones.
It captivates the scientists —
And desolates the homes!

MONICA MASKELL

* * *

From *The Globe and Mail*, January 12, 1976.

PROFESSORS CONCERNED OVER LACK OF MATHEMATICAL, LANGUAGE SKILLS

OTTAWA (CP) — More emphasis should be placed on the basics in high school to combat growing illiteracy and lack of mathematical skills among university students, two Carleton University professors say.

Keith North, chairman of Carleton's science faculty committee on admissions and studies, says as many as 40 per cent of the students taking preliminary mathematics courses do not have a chance of passing.

> The problem is serious, he says, because first-year courses in mathematics are a prerequisite for all science majors...
>
> The mathematics department at Carleton has restructured first-year courses and is offering half courses so students can repeat their fall work in the spring if they fail the Christmas examination...

Editor's comment.

This is not the first time nor, we suspect, the last, that such complaints have been voiced. But what is being done about it, besides a periodic wringing of hands?

Undoubtedly many people and many groups are working in the background to find a solution to the vexing problem of mathematical illiteracy, and all of them deserve our unstinted support.

The key to the problem, as we see it, is to increase the mathematical competence of high school teachers and university undergraduates (the high school teachers of tomorrow). This is, of course, a truism. What is also a truism is that, to increase one's mathematical competence, it is not sufficient to read about mathematics, to talk about mathematics, or to yearn for the mathematical climate of yesteryear; one must *do* mathematics.

There is one local initiative, familiar to all the readers of EUREKA, which tries to do just that: get its readers to *do* mathematics. It is EUREKA itself, published by Algonquin College and sponsored by the Carleton-Ottawa Mathematics Association, a chapter of the Ontario Association for Mathematics Education. Algonquin College is ideally situated to accomplish this task, being intermediate in level between the high schools on the one hand, and the universities on the other, able to help the one by drawing sustenance from the other, and helping itself in the process.

EUREKA is an attempt by the high school and community college teachers to regenerate their own mathematical skills. But there is little likelihood that they can accomplish this regeneration of and by themselves. A continuing infusion of appropriately pitched articles, interesting problems, and model solutions is needed from the university contingent to ensure eventual success.

We hope that the universities will not let this appeal go unanswered.

* * *

I had a feeling once about Mathematics — that I saw it all. Depth beyond Depth was revealed to me — the Byss and the Abyss. I saw — as one might see the transit of Venus or even the Lord Mayor's Show — a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why the tergiversation was inevitable — but it was after dinner and I let it go.

— WINSTON CHURCHILL[1]

[1] *The Mathematical Magpie*, by Clifton Fadiman, Simon and Schuster, 1962.