Crux

Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

E U R E K A

Vol. 1, No. 8

October 1975

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education

Published by Algonquin College

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POPULAR MISCONCEPTIONS

LÉO SAUVÉ, Algonquin College

Debunking is always delightful. It is a pleasure to watch the bubbles burst when a trained mathematician applies his steel-trap mind and directs the cold light of his intellect upon the conventional wisdom of his age. Some of the most undisputed aphorisms of the populace then stand revealed as arrant misconceptions. For example:

1. *Any stick will do to beat a dog.*

Wrong. A cane (from *canis*, the Latin word for dog) is the only appropriate weapon to use. However, it *is* true that *any stigma will do to beat a dogma*.

2. *One man's meat is another man's poison.*

Wrong. Clearly what is meat for one man is meat for another man, just as what is sauce for the goose is sauce for the gander. Some have stretched plausibility to the limit by claiming that the thinly disguised meaning is: *the butcher's meat is the customer's poison*; but since the saying greatly antedates the recent Quebec rotten meat scandals, some other explanation must be found. The only rational explanation, and good philologists will hasten to agree, is that it is merely a phonetic corruption of the historical axiom: *one man's Mede is another man's Persian*.

3. *SUGAR and SUMAC are the only two words in the English language which begin with SU and are pronounced* shu'-.

Sure, everybody knows that.

4. *Some natural numbers are not interesting.*

Wrong. Every natural number is interesting, for otherwise the set of uninteresting natural numbers would be nonempty and, by the well-ordering principle, would contain a least uninteresting natural number, which is interesting.

5. *That is a horse of a different colour.*

Impossible. All horses are of the same colour, since the statement: *in every collection of n horses all horses are of the same colour*, is true for every natural number n. The statement certainly holds if $n = 1$. Suppose it holds for some n, and consider a collection of $n+1$ horses. If we remove the first, there is left a collection of n horses which are all of the same colour by the induction assumption; if we put the first back in and remove the last, the remaining horses are all of the same colour for the same reason. Hence the first, last, and all the other horses, that is, all $n+1$ horses, are of the same colour, and the induction is complete.

6. *The smallest positive integer is* 1.

Wrong. 1 is the *largest* positive integer. For if $N \neq 1$ is the largest positive integer, then $N+1 > N$, which contradicts the hypothesis.

7. Euclid said: *Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

Wrong. What Euclid said was: Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

* * *

NOTE FROM THE EDITOR

The editor would be happy to receive your reactions and comments about EUREKA, as well as your suggestions for the improvement of its contents and presentation. Unless you request otherwise, your letters may be published in a *Letters to the Editor* column. In this way, contacts may be established between readers through the pages of this magazine.

There is a desperate need for interesting short articles (two to four pages) pitched at an appropriate level accessible to high school teachers and university undergraduate students.

The few readers who have so far submitted problem proposals could probably continue to do so indefinitely, but greater participation by the other readers would be desirable. Send in any interesting problem you come across, giving the source if known, even if, or perhaps especially if, you don't know how to solve it.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 69.

For the problems given below, solutions, if available, will appear in EUREKA Vol. 2, *No.* 1, *to be published around January* 15, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than January* 1, 1976.

71. *Proposed by Léo Sauvé, Algonquin College.*

If ten sheep jump over a fence in ten seconds, how many would jump over the fence in ten minutes?

72. *Proposé par Léo Sauvé, Collège Algonquin.*

Déterminer le couple (p,q) sachant que p et q

(a) sont *les* racines de l'équation $x^2+px+q=0$;

(b) sont chacun racine de l'équation $x^2+px+q=0$.

73. *Proposed by Viktors Linis, University of Ottawa.*

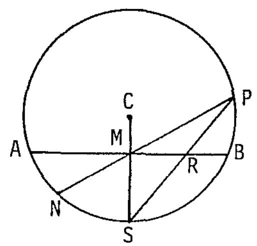
Is there a polyhedron with exactly ten pentagons as faces?

74. *Proposed by Viktors Linis, University of Ottawa.*

Prove that if the sides a, b, c of a triangle satisfy $a^2+b^2=kc^2$, then $k>\frac{1}{2}$.

75. *Proposed by R. Duff Butterill, Ottawa Board of Education.*

M is the midpoint of chord AB of the circle with centre C shown in the figure below. Prove that RS > MN.



76. *Proposed by H.G. Dworschak, Algonquin College.*

What is the remainder when 23^{23} is divided by 53?

77. *Proposed by H.G. Dworschak, Algonquin College.*

Let A_n, G_n, and H_n denote the arithmetic, geometric, and harmonic means of the n positive integers $n+1$, $n+2$, ..., $n+n$. Evaluate

$$\lim_{n\to\infty}\frac{A_n}{n},\ \lim_{n\to\infty}\frac{G_n}{n},\ \lim_{n\to\infty}\frac{H_n}{n}.$$

78. *Proposed by Jacques Sauvé, student, University of Ottawa.*

There is a well-known formula for the sum of all the combinations of n objects: $\sum_{r=0}^{n} C(n,r) = 2^n$. But is there a simple formula for the sum of all the *permutations* $\sum_{r=0}^{n} P(n,r)$?

The need for such a formula arose in a study of reliability in systems engineering.

79. *Proposed by John Thomas, University of Ottawa.*

Show that, for $x > 0$,

$$\left| \int_x^{x+1} \sin (t^2)\, dt \right| < \frac{2}{x^2} .$$

80. *Proposé par Jacques Marion, Université d'Ottawa.*

Existe-t-il une suite d'entiers $\{a_n\}$ telle que $\lim_{n\to\infty} a_n = \infty$ et que la suite $\{\sin a_n x\}$ converge pour tout $x \varepsilon [0,2\pi]$?

* * *

SOLUTIONS

41. *Proposé par Léo Sauvé, Collège Algonquin.*

Ayant donné $\log_8 3 = p$ et $\log_3 5 = q$, exprimer $\log_{10} 5$ et $\log_{10} 6$ en fonction de p et q.

Solution du proposeur.

On a $2^{3p} = 3$ et $3^q = 5$. Si $\log_{10} 5 = x$, alors $10^x = 5$ donne $2^x \cdot 2^{3pqx} = 2^{3pq}$, d'où $(3pq + 1)\,x = 3pq$ et

$$x = \frac{3pq}{3pq + 1} .$$

De même, si $\log_{10} 6 = y$, alors $10^y = 6$ donne $2^y \cdot 2^{3pqy} = 2^{3p+1}$, d'où $(3pq + 1)y = 3p + 1$ et

$$y = \frac{3p + 1}{3pq + 1} .$$

Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College; and Jim Mattice, Ancaster, Ont.

Editor's comment.

Maskell, with characteristic thoroughness, gave the following table, in which the entry in row a and column b is $\log_a b$. The inverse symmetry about the principal diagonal illustrates the well-known property $\log_a b \cdot \log_b a = 1$.

$a \backslash b$	2	3	5	6	10
2	1	$3p$	$3pq$	$3p+1$	$3pq+1$
3	$\frac{1}{3p}$	1	q	$\frac{3p+1}{3p}$	$\frac{3pq+1}{3p}$
5	$\frac{1}{3pq}$	$\frac{1}{q}$	1	$\frac{3p+1}{3pq}$	$\frac{3pq+1}{3pq}$
6	$\frac{1}{3p+1}$	$\frac{3p}{3p+1}$	$\frac{3pq}{3p+1}$	1	$\frac{3pq+1}{3p+1}$
10	$\frac{1}{3pq+1}$	$\frac{3p}{3pq+1}$	$\frac{3pq}{3pq+1}$	$\frac{3p+1}{3pq+1}$	1

42. *Proposed by Viktors Linis, University of Ottawa.*

Find the area of a quadrilateral as a function of its four sides, given that the sums of opposite angles are equal.

Solution de Jacques Sauvé, étudiant, Université d'Ottawa.

Puisque les sommes des angles opposés sont égales, les angles opposés sont supplémentaires et le quadrilatère est cyclique. D'après la formule classique de Brahmagupta, l'aire du quadrilatère est donc

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}\ ,$$

où a, b, c, d sont les longueurs des côtés et s est le demi-périmètre.

Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; and the proposer.

43. *Proposed by André Bourbeau, École Secondaire Garneau.*

In a 3×3 matrix, the entries a_{ij} are randomly selected integers such that $0 \le a_{ij} \le 9$. Find the probability that

(a) the three-digit numbers formed by each row will be divisible by 11;

(b) the three-digit numbers formed by each row and each column will be divisible by 11.

Solution of (a) by H.G. Dworschak, Algonquin College.

The number of multiples of 11 from 000 to 999 is $1 + [999/11] = 91$; hence the probability that a row is divisible by 11 is $p = 91/1000$, and the probability that all three rows are divisible by 11 is $p^3 = 0.000753571$.

Part (a) was also solved correctly by F.G.B. Maskell, Algonquin College, and incorrectly by one additional reader.

Editor's comment.

Only one solution to part (b) was received. It did not appear convincing to the editor, who could be wrong, and was returned to the sender for additional clarification. Part (b) must therefore still be considered as open.

44. *Proposed by Viktors Linis, University of Ottawa.*

Construct a square ABCD given its centre and any two points M and N on its two sides BC and CD respectively.

Solution by the proposer.

The construction uses symmetry properties of the square. Rotating the triangle OMN by 90° clockwise and then counterclockwise we obtain additional points M', N', M'', N'' which all lie on the sides of the square. Also by central symmetry with respect to O (or by 180° rotation) two more points M''', N''' can be generated, as shown in Figure 1. The rest is then obvious and the solution is unique. If after 90° rotation N and M' coincide, this means that the angle MON is 90° and OM = ON. In this case, there are infinitely many solutions: the vertex C is any point on the semicircle with MN as a diameter. This is illustrated in Figure 2.

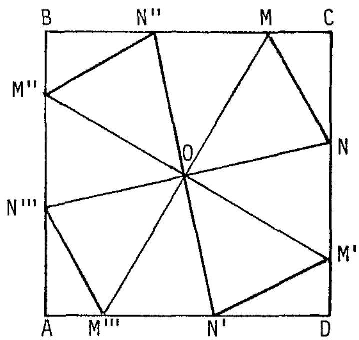


Figure 1.

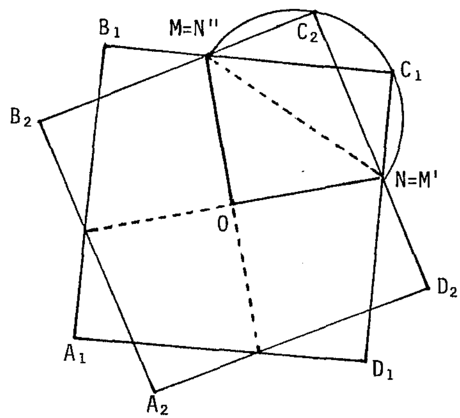


Figure 2.

Also solved by G.D. Kaye, Department of National Defence; and Léo Sauvé, Algonquin College. One additional incorrect solution was received.

45. *Proposed by H.G. Dworschak, Algonquin College.*

Find constants A, B, C, D, p, q such that

$$A(x-p)^2 + B(x-q)^2 = 5x^2 + 8x + 14,$$

$$C(x-p)^2 + D(x-q)^2 = x^2 + 10x + 7.$$

Solution by the proposer.

Substituting successively $x = q$ and $x = p$ in the first given equation, we get

$$A(q-p)^2 = 5q^2 + 8q + 14, \qquad (1)$$

$$B(p-q)^2 = 5p^2 + 8p + 14; \qquad (2)$$

and doing the same with the second given equation yields

$$C(q-p)^2 = q^2 + 10q + 7, \qquad (3)$$

$$D(p-q)^2 = p^2 + 10p + 7. \qquad (4)$$

Adding (1) and (2), and noting that $A+B=5$, gives

$$4(p+q)+5pq=-14; \qquad (5)$$

and similarly the sum of (3) and (4), with $C+D=1$, reduces to

$$5(p+q)+pq=-7. \qquad (6)$$

Solving (5) and (6) yields $(p+q,pq)=(-1,-2)$, from which $(p,q)=(1,-2)$ or $(-2,1)$. If we now substitute $(p,q)=(1,-2)$ into (1), (2), (3), and (4), we obtain $(A,B,C,D)=(2,3,-1,2)$, so that

$$2(x-1)^2+3(x+2)^2=5x^2+8x+14,$$
$$-(x-1)^2+2(x+2)^2=x^2+10x+7.$$

Substituting $(p,q)=(-2,1)$ gives the same solution, which is therefore unique.

Also solved by G.D. Kaye, Department of National Defence; and Léo Sauvé, Collège Algonquin.

Editor's comment.

The problem as originally proposed had 17 in the second equation instead of the 7 given here. 7 was the intended number, and the 17 was a misprint for which the editor apologizes. The method of solution is not affected, but with 17 the results are considerably messier.

46. *Proposed by F.G.B. Maskell, Algonquin College.*

If p_1, p_2, p_3 are the altitudes of a triangle and r is the radius of its inscribed circle, show that

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}.$$

Solution by Viktors Linis, University of Ottawa.

For the area A of the triangle with sides a, b, c, we have the identities

$$2A=ap_1=bp_2=cp_3=(a+b+c)r.$$

Dividing the first and the last by $2Ar$, we obtain

$$\frac{1}{r}=\frac{a+b+c}{2A}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}.$$

Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; Jim Mattice, Ancaster, Ont.; Léo Sauvé, Collège Algonquin; and the proposer.

Editor's comment.

Many geometry and trigonometry textbooks contain dozens of formulas of a similar character involving the inradius r, the exradii r_1, r_2, r_3, and the altitudes p_1, p_2, p_3. Representative samples were given by Mattice and Maskell.

47. *Proposé par Jacques Sauvé, étudiant, Université d'Ottawa.*

Si $a > 1$, calculer la somme de la série $\sum_{k=1}^{\infty} \frac{k^2}{a^k}$.

Solution de Jacques Marion, Université d'Ottawa.

Sur l'identité

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \qquad |z| < 1$$

on effectue l'opération qui consiste à calculer la dérivée de chaque membre puis multiplier chaque membre de l'équation qui résulte par z. Cette opération est permise et peut être répétée à volonté à l'intérieur de $|z| < 1$. Une première application de l'opération donne

$$\frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k\, z^k ,$$

et une deuxième application amène

$$\frac{z(z+1)}{(1-z)^3} = \sum_{k=1}^{\infty} k^2 z^k .$$

Si l'on pose maintenant $z = \frac{1}{a} < 1$, on obtient

$$\sum_{k=1}^{\infty} \frac{k^2}{a^k} = \frac{a(a+1)}{(a-1)^3} .$$

On pourrait de même calculer $\sum_{k=1}^{\infty} \frac{k^n}{a^k}$ en répétant l'opération n fois.

Also solved by G.D. Kaye, Department of National Defence; Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College; Léo Sauvé, Collège Algonquin; and the proposer.

48. *Proposé par Léo Sauvé, Collège Algonquin.*

La fonction $f : R \to R$ est définie par les relations

$$f(x) = 2 + \sin x \cos \frac{1}{x}, \qquad \text{si } x \neq 0,$$

$$f(0) = 2.$$

Pour tout entier $n \geq 1$, on considère l'intégrale

$$I_n = \int_{-\frac{2}{n}}^{\frac{2}{n}} \{n + (\tfrac{1}{n} - n)\, \chi_n(x)\}\, f(x)\, dx,$$

où χ_n désigne la fonction caractéristique de l'intervalle $[-\frac{1}{n}, \frac{1}{n}]$. Calculer I_n en fonction de n et en déduire la valeur de $\lim_{n\to\infty} I_n$.

Solution by Viktors Linis, University of Ottawa.

Let $\phi(x) = \sin x \cos\frac{1}{x}$ if $x \neq 0$ and $\phi(0) = 0$; then $\phi(-x) = -\phi(x)$. Therefore

$$J_n = \int_{-\frac{2}{n}}^{\frac{2}{n}} \{n + (\frac{1}{n} - n)\,\chi(x)\}\,\phi(x)\,dx = 0$$

and

$$I_n = \int_{-\frac{2}{n}}^{\frac{2}{n}} \{n + (\frac{1}{n} - n)\,\chi(x)\}\,2\,dx = 4n\int_{\frac{1}{n}}^{\frac{2}{n}} dx + \frac{2}{n}\int_{-\frac{1}{n}}^{\frac{1}{n}} dx$$

$$= 4 + \frac{4}{n^2},$$

so that $\lim_{n\to\infty} I_n = 4$.

Also solved by Jacques Marion, Université d'Ottawa; and the proposer.

49. *Proposed by H.G. Dworschak, Algonquin College.*

The series

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \ldots + \frac{1}{6n-5} - \frac{1}{6n-1} + \ldots$$

clearly converges. Find its sum.

I. *Solution by Léo Sauvé, Algonquin College.*

A routine expansion of the constant function

$$f(x) = \frac{\pi}{4}, \qquad 0 < x < \pi$$

into a Fourier sine series gives

$$\frac{\pi}{4} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots,$$

and setting $x = \frac{\pi}{3}$ gives

$$\frac{\pi}{4} = \frac{\sqrt{3}}{2}(1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \ldots + \frac{1}{6n-5} - \frac{1}{6n-1} + \ldots),$$

from which the required sum is seen to be $\frac{\pi}{2\sqrt{3}}$.

II. *Solution by Viktors Linis, University of Ottawa.*

Using the standard technique define a power series

$$p(x) = \sum_{n=1}^{\infty} \left(\frac{x^{6n-5}}{6n-5} - \frac{x^{6n-1}}{6n-1}\right)$$

which converges for $|x| \leq 1$. Verify that

$$p'(x) = \frac{1 + x^2}{1 + x^2 + x^4}$$

and $p(0) = 0$. Integrating we find

$$p(x) = \frac{1}{\sqrt{3}}(\arctan\frac{2x-1}{\sqrt{3}} + \arctan\frac{2x+1}{\sqrt{3}})$$

and

$$p(1) = \frac{1}{\sqrt{3}}(\arctan\frac{1}{\sqrt{3}} + \arctan\sqrt{3}) = \frac{\pi}{2\sqrt{3}}.$$

The passage $p(1) = \lim_{x\to 1} p(x)$ is justified by Abel's theorem.

Also solved by Jacques Marion, Université d'Ottawa; and the proposer.

50. *Proposed by John Thomas, Digital Methods Ltd.*

I found the following fascinating two-part problem in Martin Eisen's *Introduction to Mathematical Probability Theory* (Prentice-Hall 1969). Information as to its origin and history would be appreciated.

(a) Show that 2^n can begin with any sequence of digits.

(b) Let N be an r-digit number. What is the probability that the first r digits of 2^n represent N?

Solution by Léo Sauvé, Algonquin College.

In solving Eisen's problem, I soon became aware that the base 2 had no essential role to play, and that it could be replaced by other positive numbers. The only positive real numbers which could be safely excluded were the rational powers of 10, for their positive integral powers only generate a finite number of distinct sequences of significant digits. I will show that all other positive numbers can be used in place of 2 without affecting the conclusion.

Let N be a positive integer. If x is a positive real number, I shall for convenience use the expression "x begins with N" to mean that the sequence of significant digits in the decimal representation of x begins with the sequence of digits in the decimal representation of N. Logarithms to base 10 are used throughout.

(a) The following theorem generalizes the first part of Eisen's problem:

THEOREM 1. If N is a positive integer and α is a positive real number that is not a rational power of 10, then infinitely many terms of the sequence $S = \{\alpha^i\}_{i=1}^{\infty}$ begin with N.

The proof of Theorem 1 will hinge on the following lemma, which expresses a property of real numbers that is interesting in its own right.

LEMMA. Let α and S be as in Theorem 1. If, for $x > 0$, $\mu(x)$ represents the mantissa of $\log x$, that is,

$$\mu(x) = \log x - [\log x],$$

then $\mu(S)$ is dense in the open interval $(0,1)$.

The lemma is actually a special case of Kronecker's Theorem whose proof, in full generality, is quite difficult; but as stated here, a relatively easy proof is possible.

Proof of the Lemma. It will be sufficient to show that, for arbitrary a and b such that $0 < a < b < 1$, there exists at least one positive integer n such that

$$a < \mu(\alpha^n) < b.$$

We first observe that the restriction of μ to S is injective, and hence that $\mu(S)$ forms an infinite set, since

$$\mu(\alpha^m) = \mu(\alpha^n) \quad \Rightarrow \quad \frac{\alpha^m}{\alpha^n} = 10^k \quad \Rightarrow \quad m = n,$$

for $m \neq n$ implies $\alpha = 10^{k/(m-n)}$, contrary to the hypothesis. Since the interval $[0,1]$ is of finite length, it contains at least one point of accumulation of $\mu(S)$. In a neighbourhood of such a point, there are at least two elements of $\mu(S)$, say $\mu(\alpha^r)$ and $\mu(\alpha^{r+s})$, whose distance apart is $|d| < b - a$, where $d = \mu(\alpha^{r+s}) - \mu(\alpha^r)$.

Now the values under μ of the geometric progression

$$\alpha^r,\ \alpha^{r+s},\ \alpha^{r+2s},\ \alpha^{r+3s},\ \ldots$$

form an arithmetic progression modulo 1 of common difference d. It follows that at least one of these values, say $\mu(\alpha^{r+ks})$, must fall inside the interval (a,b), and we have

$$a < \mu(\alpha^{r+ks}) < b,$$

which completes the proof.

Proof of Theorem 1. We assume for the time being that the digits of N are not all 9's. It is clear that any $x > 0$ begins with N if and only if

$$\mu(N) \leq \mu(x) < \mu(N+1). \tag{1}$$

Since, from the lemma, $\mu(S)$ is dense in the interval $(0,1)$, infinitely many elements of $\mu(S)$ fall in the interval $[\mu(N), \mu(N+1))$; hence, by (1), infinitely many elements of S begin with N.

If the digits of N are all 9's, the proof is still valid, provided we replace $\mu(N+1)$ by 1 wherever it occurs.

(b) The following theorem generalizes the second part of Eisen's problem:

THEOREM 2. *If n is a positive integer, the probability that α^n begins with N is independent of α. It is, in fact,* $\log(1 + \frac{1}{N})$.

Proof. The probability that α^n begins with N is the probability that $\mu(\alpha^n)$ lies in the interval $I = [\mu(N), \mu(N+1))$, whose length is

$$|I| = \mu(N+1) - \mu(N) = \log(N+1) - \log N = \log(1 + \tfrac{1}{N}).$$

If $f(n)$ represents the number of points of the sequence $\mu(\alpha), \mu(\alpha^2), \ldots, \mu(\alpha^n)$ which lie in I, the required probability is clearly $\lim_{n\to\infty} \frac{f(n)}{n}$. By Theorem 445 on page 390 of Hardy and Wright's *Introduction to the Theory of Numbers* (Oxford 1960), the points of $\mu(S)$ are uniformly distributed in the interval $(0,1)$, which means precisely that

$$\lim_{n\to\infty}\frac{f(n)}{n} = |I| = \log\left(1+\frac{1}{N}\right).$$

Here again, we must replace $\mu(N+1)$ by 1 if the digits of N are all 9's.

Part (a) was also solved by Viktors Linis, University of Ottawa.

Editor's comments.

1. Linis gave the following examples for the case $\alpha = 2$, using the notation (x) for $x - [x]$, the fractional part of x.

(i) If $N = 25$, then (1) becomes (using 3 decimals):

$$0.398 \le (0.301n) < 0.415.$$

By straightforward calculations (checking $n = 1, 2, \ldots$), one finds $n = 8$ the smallest positive integer satisfying the inequality, and we have $2^8 = \underline{25}6$.

(ii) If $N = 16$, then $n = 4$ is an obvious solution. However, there are others: the inequality (1) becomes

$$0.204 \le (0.301n) < 0.230$$

(the equality corresponds to $n = 4$). Direct calculations give $n = 14$ and $n = 24$ as solutions, and we have $2^{14} = \underline{16}384$, $2^{24} = \underline{16}777216$.

2. As a further example, if $\alpha = \pi$ and $N = 217$, then (1) becomes

$$0.336459734 \le (0.497149873n) < 0.338456494.$$

We find $n = 57$ as the smallest positive integer satisfying this inequality, and indeed we have

$$\pi^{57} = \underline{2.17}5418\ldots \times 10^{28}.$$

The next solution is $n = 759$, and it turns out, wonder of wonders, that

$$\pi^{759} = \underline{2.17}1467\ldots \times 10^{377}.$$

Is it not a reassuring thought, in these uncertain times, to *know* that there are infinitely many powers of π that begin with, say, a million 1's?

3. *References to this problem in the literature.*

(i) Leo Moser and Nathaniel Macon have shown that every finite sequence of digits appears as the first digits of some power of 2, in their paper, *On the distribution of first digits of powers*, published in *Scripta Mathematica*, Vol. XVI (1950), pp. 290 - 2. This is the earliest reference I have been able to uncover.

(ii) Those famous Russian mathematicians and expositors, the twin brothers Akiva and Isaak Yaglom, proved both parts of the problem for the case $\alpha = 2$ in their book, *Neelementarnye Zadachi v Elementarnom Izlozhenii*, originally published in 1954. Their proof can also be found on pp. 199 - 202 in Volume 1 of an English translation of their book entitled *Challenging Mathematical Problems with Elementary Solutions*, published in 1964 by Holden-Day. It is characteristic

of the expository work of the Yagloms that they prove both parts of the problem *ab ovo* in a painstaking, immensely detailed fashion which should surely be within the reach of the average high school genius.

(iii) On pp. 38 - 45 of his fascinating book, *Ingenuity in Mathematics*, in the New Mathematical Library series (Random House, 1970), University of Waterloo professor Ross Honsberger gives a very clear paraphrase of the Yagloms' proof of part (a). He makes the interesting remark that, although fairly low powers of 2 begin with 1, 2, 3, 4, 5, 6, or 8, the smallest power of 2 beginning with 7 is 2^{56} and the smallest beginning with 9 is 2^{53}.

(iv) This last reference was pointed out to me by Jacques Marion and John Thomas. It is the article, *Integers with given initial digits*, by R.S. Bird, published in the *American Mathematical Monthly*, Vol. 79 (1972), pp. 367 - 70. Here Bird restricts himself to sequences of positive integers. He proves that a wide category of sequences of positive integers have the property that at least one member of the sequence begins with a given N, giving as examples the sequences of primes, squares, powers of 2, and factorials.

* * *

A PIECE OF PI

LÉO SAUVÉ, Algonquin College

In 1889 a Hamburg mathematics professor named Hermann Schubert wrote:[1]

> Conceive a sphere constructed with the earth as its center, and imagine its surface to pass through Sirius, which is 8.8 light years distant from the earth. Then imagine this enormous sphere to be so packed with microbes that in every cubic millimeter millions of millions of these diminutive animalcula are present. Now conceive these microbes to be all unpacked and so distributed singly along a straight line that every two microbes are as far distant from each other as Sirius from us, 8.8 light years. Conceive the long line thus fixed by all the microbes as the diameter of a circle, and imagine its circumference to be calculated by multiplying its diameter by π to 100 decimal places. Then, in the case of a circle of this enormous magnitude even, the circumference so calculated would not vary from the real circumference by a millionth part of a millimeter.
>
> This example will suffice to show that the calculation of π to 100 or 500 decimal places is wholly useless.

The uselessness of the exercise, which would have delighted G.H. Hardy, has never prevented mathematicians over the centuries from trying to find more and more digits in the decimal expansion of π, which is now known to at least one million decimal places.[2]

As a service to the readers of EUREKA, I will give below a small sample of this expansion, a mere 300 decimal places. But first, in order to make the task of remembering a piece of π a piece of cake, I give a few mnemonic devices in which the number of letters in a word represents the corresponding digit in the expansion.

[1]Quoted in *A History of* π, by Petr Beckmann, The Golem Press 1971, p. 98.
[2]See EUREKA, Vol.1 (1975), p. 5.

1. Fifteen significant digits.[3]

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics!

2. Thirty-one significant digits.[4]

Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, artiste ingénieur,
Qui de ton jugement peut priser la valeur?
Pour moi, ton problème eut de pareils avantages.

3. Thirty-one significant digits.

Dir, o Held, o alter Philosoph, du riesen Genie!
Wie viele Tausende bewundern Geister
Himmlisch wie du und Göttlich!
Noch reiner in Aeonen
Wird das uns strahlen
Wie im lichten Morgenrot!

4. Thirty decimal places.[5]

To Mr. H. Bailey
(who has to have recourse to Greek Iambics when he wants to remember the value of π)

I nunc, O Baili, Parnassum et desere rupem;
Dic sacra Pieridum deteriora quadris!
Subsidium hoc ad vos, quamquam leve, fertur ab hymnis
Quos dat vox Sophocli (non in utroque probrumst?)[6]

5. Thirty decimal places.

ὁ παῖς ὁ κύκλῳ περιφορὰν πρ γράφων[8]
οὐκ εὐθὺς ηὐπόρησε διάμετρον μετροῦν.
ἀναλογίας γὰρ ἦν μὲν εὔφορτον κάρα,
ἀχηνία δὲ μνῆσις· ἀλλὰ νῦν ἔφη,
‘σκέψασθε πῶς με *Κνόξιος* παρηγορεῖ.’

Budding poets will please note that, since the 32nd decimal place of π is a zero, there is mercifully a limit to the possible lengths of such mnemonic verses.

And now here is the promised expansion:

π = 3.1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679
8214808651 3282306647 0938446095 5058223172 5359408128
4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091
4564856692 3460348610 4543266482 1339360726 0249141273 ...

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[3] Attributed by Petr Beckmann (op. cit.) to C.B. Boyer (1968).

[4] This poem and the next are due to the pen of that prolific polyglot author *Anonymous*. They are both quoted by Petr Beckmann (op. cit.).

[5] This poem and the next are by Monsignor Ronald A. Knox (1917). They are quoted by Clifton Fadiman in *The Mathematical Magpie*, Simon and Schuster 1962, pp. 287 - 8.

[6] "Will you dare any longer, Bailey, to turn your back on Parnassus hill, telling us that the sacred rites of the Muses are less important than constructing squares? Here is aid brought to you, though it be but slight, by poetry, and poetry couched in the language of Sophocles—there is a double thrust at your vanity!"[7]

[7] The author, by request, made this translation many years later.

[8] ἰοῦ διέλιπον πέντε γραμμάτων ἔπος.