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E U R E K A

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* * *

EUREKA!

Martin Gardner, in the *Mathematical Games* section of the April 1975 issue of *Scientific American*, announced two major mathematical discoveries, both made in 1974.

The first provides a counterexample for, and thereby disproves, the conjecture known throughout the mathematical world as *The Four Colour Problem*. In November, 1974, William McGregor, a graph theorist of Wappinger's Falls, N.Y., constructed a map of 110 regions that cannot be coloured with fewer than five colours. Gardner's article contains a sketch of the map. This is the first definitive answer to the problem since it was formulated as early as 1850 by De Morgan.

The second is a proof that

$$e^{\pi\sqrt{163}} \text{ IS AN INTEGER.}$$

This astounding result had been conjectured around 1914 by the Indian mathematician Srinivasa Ramanujan. In May, 1974, John Brillo of the University of Arizona managed to prove that

$$e^{\pi\sqrt{163}} = 262{,}537{,}412{,}640{,}768{,}744.$$

Technical reports on these discoveries have been accepted for publication. The first will be published in 1978 in the *Journal of Combinatorial Theory*, Series B. The second will appear "in a few years" in *Mathematics of Computation*.

Because of the unfortunate delay in the publication of these important discoveries, due to the increasing backlog of articles submitted to these and other scientific journals, Gardner decided to announce them right away, as a public service.

The editor of EUREKA thought that he could do no less, also as a public service, than to announce these results to those of its readers who do not read *Scientific American*.

So now you know. Pass it on.

This is probably a good time to point out that, here at EUREKA, there is at present a backlog of five to six months for problems, but no backlog for articles. So if you make a major mathematical discovery, such as proving the Goldbach Conjecture, disproving the Pythagorean Theorem, or finding an error in some published table of random numbers, submit it to EUREKA for practically instant publication.

The Editor.

* * *

PARADOX

Not truth, nor certainty. These I foreswore
In my novitiate, as young men called
To holy orders must abjure the world.
'If . . . , then . . . ,' this only I assert;
And my successes are but pretty chains
Linking twin doubts, for it is vain to ask
If what I postulate be justified,
Or what I prove possess the stamp of fact.

Yet bridges stand, and men no longer crawl
In two dimensions. And such triumphs stem
In no small measure from the power this game,
Played with the thrice-attenuated shades
Of things, has over their originals.
How frail the wand, but how profound the spell!

CLARENCE R. WYLIE, Jr.

* * *

L'ESPRIT GÉOMÉTRIQUE

Mon grand talent, ce qui me distingue le plus, c'est de voir clair en tout, c'est même mon genre d'éloquence, que de voir sous toutes ses faces le fond de la question. C'est la perpendiculaire plus courte que l'oblique.

NAPOLÉON.

* * *

My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful.

HERMANN WEYL.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompagnied by a solution, should be sent to the editor, whose address appears on page 23.

For the problems given below, solutions, if available, will appear in EUREKA No. 6, to be published around August 15, 1975. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than August 1, 1975.

31. *Proposed by Léo Sauvé, Algonquin College.*

This problem is dedicated, regretfully, as a parting gift, to Charles McCaffray, who is expected to solve it.

A driver cruising on the highway observed that the odometer of his car showed 15951 miles. He noticed that this number is palindromic: it reads the same backward and forward.

"Curious", the driver said to himself. "It will be a long time before that happens again." But exactly two hours later the odometer showed a new palindromic number.

What was the average speed of the car in those two hours?

32. *Proposed by Viktors Linis, University of Ottawa.*

Construct a square given a vertex and a midpoint of one side (consider all cases).

33. *Proposed by Viktors Linis, University of Ottawa.*

On the sides CA and CB of an isosceles right-angled triangle ABC, points D and E are chosen such that $|CD| = |CE|$. The perpendiculars from D and C on AE intersect the hypotenuse AB in K and L respectively. Prove that $|KL| = |LB|$.

34. *Proposed by H.G. Dworschak, Algonquin College.*

Once a bright young lady called Lillian
Summed the numbers from one to a billion.
But it gave her the fidgets
To add up the *digits;*
If you can help her, she'll thank you a million.

35. *Proposed by John Thomas, Digital Methods Ltd.*

Let m denote a positive integer and p a prime. Show that if $p \mid m^4 - m^2 + 1$, then $p \equiv 1 \pmod{12}$.

36. *Proposé par Léo Sauvé, Collège Algonquin.*

Si m et n sont des entiers positifs, montrer que

$$\sin^{2m}\theta\cos^{2n}\theta \le \frac{m^m n^n}{(m+n)^{m+n}},$$

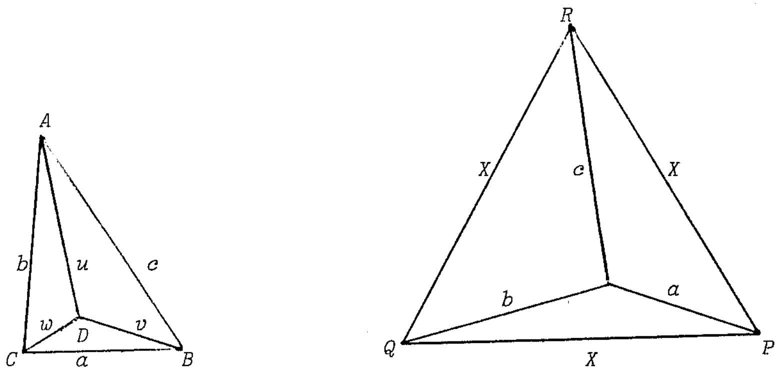
et déterminer les valeurs de θ pour lesquelles il y a égalité.

37. *Proposed by Maurice Poirier, École Secondaire Garneau.*

E, F, G, and H are the midpoints of sides AB, BC, CD, and DA, respectively, of the convex quadrilateral $ABCD$. EX, FY, GZ, and HT are drawn externally perpendicular to AB, BC, CD, and DA, respectively, and $EX = \frac{1}{2}AB$, $FY = \frac{1}{2}BC$, $GZ = \frac{1}{2}CD$, and $HT = \frac{1}{2}DA$. Prove that $XZ = YT$ and $XZ \perp YT$.

38. *Proposed by Léo Sauvé, Algonquin College.*

Consider the two triangles ΔABC and ΔPQR shown below. In ΔABC, $\angle ADB = \angle BDC = \angle CDA = 120^\circ$. Prove that $X = u + v + w$.



(*Third USA Mathematical Olympiad - May 7, 1974*)

39. *Proposé par Maurice Poirier, École Secondaire Garneau.*

On donne un point P à l'intérieur d'un triangle équilatéral ABC tel que les longueurs des segments PA, PB, PC sont 3, 4, et 5 unités respectivement. Calculer l'aire du ΔABC.

40. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $\{a_n\}$ une suite de nombres complexes non nuls pour laquelle il existe un $r > 0$ tel que

$$m \neq n \Longrightarrow |a_m - a_n| \geq r.$$

Si $u_n = \dfrac{1}{|a_n|^\alpha}$, où $\alpha > 2$, montrer que la série $\sum_{n=1}^{\infty} u_n$ converge.

Que peut-on dire si $\alpha = 2$?

* * *

S O L U T I O N S

6. *Late solution: G.D. Kaye, Department of National Defence.*

11. *Proposed by Léo Sauvé, Algonquin College.*

A basket contains exactly *30* apples. The apples are distributed among *10* children, each child receiving n apples, where n is a positive integer. At the end of the distribution, there are n apples left in the basket. Find n.

Solution by the proposer.

It is clear that $n = 1$, $n = 2$, and $n \geq 4$ are all unacceptable. When you have eliminated the impossible, as Sherlock Holmes used to say, what remains, however improbable it may appear, must be true. So the solution, if there is one, must be $n = 3$. Can this solution be reconciled with the statement of the problem? Yes, and in one way only: by assuming that the tenth child received his share of 3 apples *in the basket*.

Also solved by H.G. Dworschak and F.G.B. Maskell, both of Algonquin College.

12. *Proposed by Viktors Linis, University of Ottawa.*

There are about *100* apples in a basket. It is possible to divide the apples equally among *2, 3,* and *5* children but not among *4* children. How many apples are there in the basket?

Solution by H.G. Dworschak, Algonquin College.

The required number must be a multiple of the relatively prime numbers 2, 3, and 5, and hence a multiple of their product 30. The only multiple of 30 that is near 100 and not a multiple of 4 is 90. This is the required number.

Also solved by F.G.B. Maskell and Léo Sauvé, both of Algonquin College; and the proposer. Late solution by André Ladouceur, Ecole Secondaire De La Salle.

13. *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que la somme de p entiers positifs dont le plus grand est q est égale à la somme de q entiers positifs dont le plus grand est p.

Solution du proposeur.

La somme de p rangées d'unités dont la plus longue en contient q est égale à la somme de q colonnes d'unités dont la plus longue en contient p..

Par exemple, le tableau ci-dessous illustre le fait que la somme des cinq entiers 2, 4, 1, 6, 1, dont le plus grand est 6, est égale à la somme des six entiers 5, 3, 2, 2, 1, 1, dont le plus grand est 5.

p rangées	1	1				
	1	1	1	1		
	1					
	1	1	1	1	1	1
	1					

q colonnes

One incorrect solution was received.

14. *Proposed by Viktors Linis, University of Ottawa.*

If a, b, c are lengths of three segments which can form a triangle, show that the same is true for $\frac{1}{a+c}$, $\frac{1}{b+c}$, $\frac{1}{a+b}$.

Solution by F.G.B. Maskell, Algonquin College.

We can assume $a \le b \le c$ so that $\frac{1}{b+c} \le \frac{1}{c+a} \le \frac{1}{a+b}$, and it is sufficient to prove that

$$\frac{1}{b+c} + \frac{1}{c+a} > \frac{1}{a+b}. \qquad (1)$$

We have

$$\frac{1}{b+c} + \frac{1}{c+a} - \frac{1}{a+b} = \frac{a^2+ab+b^2+c(a+b-c)}{(b+c)(c+a)(a+b)} > 0,$$

since $a + b > c$, and (1) follows.

Also solved by G.D. Kaye, Department of National Defence; H.G. Dworschak, Algonquin College; and the proposer. Late solution by André Ladouceur.

15. *Proposed by H.G. Dworschak, Algonquin College.*

Let A, B, C be three distinct points on a rectangular hyperbola. Prove that the orthocentre of ΔABC lies on the hyperbola.

Solution by Léo Sauvé, Algonquin College.

In a suitably selected co-ordinate system, the equation of the hyperbola will be $y = \frac{k}{x}$, where k is a constant. Let a, b, c be the abscissas of A, B, C, respectively. The slope of AB is

$$m_{AB} = \frac{\frac{k}{b} - \frac{k}{a}}{b - a} = -\frac{k}{ab} .$$

Let $H(h, \frac{k}{h})$ be the point where the altitude from C meets the hyperbola. The slope of CH is

$$m_{CH} = \frac{\frac{k}{h} - \frac{k}{c}}{h - c} = -\frac{k}{ch} ,$$

and the orthogonality of AB and CH gives $\frac{k^2}{abch} = -1$, whence $h = -\frac{k^2}{abc}$. The symmetry of this result shows that the remaining two altitudes would intersect the hyperbola in the same point H, which is therefore the orthocentre of ΔABC. Its co-ordinates are $(-\frac{k^2}{abc}, -\frac{abc}{k})$.

Also solved by G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and the proposer. Late solution by André Ladouceur.

16. *Proposed by Léo Sauvé, Algonquin College.*

For $n = 1, 2, 3, \ldots$, the finite sequence S_n is a permutation of $1, 2, 3, \ldots, n$, formed according to a law to be determined. According to this law, we have

$$S_1 = (1)$$
$$S_2 = (1,2)$$
$$S_3 = (1,3,2)$$
$$S_4 = (4,1,3,2)$$
$$\cdots\cdots\cdots$$
$$S_9 = (8,5,4,9,1,7,6,3,2)$$

Discover a law of formation which is satisfied by the above sequences, and then give S_{10}.

Solution by the proposer.

In accordance with the law described below we would have

$$S_{10} = (8,\ 5,\ 4,\ 9,\ 1,\ 7,\ 6,\ 10,\ 3,\ 2)$$

The law is that, for each n, the numbers in S_n are *in alphabetical order.*

WHOOOSHHH! (That blur you've just seen is the proposer running for cover.)

Also solved by André Bourbeau, École Secondaire Garneau; H.G. Dworschak, Algonquin College; Maurice Poirier, Ecole Secondaire Garneau; Richard J. Semple, Carleton University; and John Thomas, Digital Methods Ltd.

17. *Proposed by Viktors Linis, University of Ottawa.*

Prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{999999}{1000000} < \frac{1}{1000}.$$

I. *Solution by the proposer.*

Let P_1 represent the product of 500000 factors on the left of the given inequality, and let

$$P_2 = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{1000000}{1000001},$$

so that $P_1P_2 = \frac{1}{1000001}$. We note that $P_1 < P_2$ since, for $1 \le k \le 500000$, each factor $\frac{2k-1}{2k}$ of P_1 is less than the corresponding factor $\frac{2k}{2k+1}$ of P_2. Hence

$$P_1^{\,2} < P_1P_2 = \frac{1}{1000001} \quad \text{and} \quad P_1 < \frac{1}{\sqrt{1000001}} < \frac{1}{1000}.$$

In general, one can prove the inequality

$$\prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{2n}},$$

for all positive integers n. For large n, using Stirling's approximation formula for factorials, one can obtain the stronger inequality

$$\prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{\pi n}},$$

which happens to be true also for *all* positive integers n.

II. *Solution by Léo Sauvé, Algonquin College.*

The double inequality

$$\frac{\sqrt{r-1}}{\sqrt{r}} < \frac{2r-1}{2r} < \frac{\sqrt{3r-2}}{\sqrt{3r+1}}$$

can easily be shown to hold for all integers $r \geq 2$ by squaring each member. If we substitute successively $r =$ 2, 3,..., n and multiply corresponding members, we get

$$\frac{1}{\sqrt{n}} < \frac{3\cdot 5\cdots(2n-1)}{4\cdot 6\cdots 2n} < \frac{2}{\sqrt{3n+1}},$$

and multiplication by $\frac{1}{2}$ yields

$$\frac{1}{2\sqrt{n}} < Q_n < \frac{1}{\sqrt{3n+1}}, \qquad (1)$$

where Q_n is the factorial quotient $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$.

An improvement on the proposed inequality results if we set n = 500,000 in (1), namely,

$$0.000707 < \frac{1\cdot 3\cdot 5\cdots 999999}{2\cdot 4\cdot 6\cdots 1000000} < 0.000817.$$

Editor's comment.

Here are a few interesting properties of the factorial quotient Q_n. Can any reader supply more?

(i) For $-1 \leq x < 1$, the binomial series gives

$$1 + \sum_{n=1}^{\infty} Q_n x^n = (1-x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x}}.$$

(ii) It follows from the left inequality in (1) that

$$\sum_{n=1}^{\infty} Q_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} Q_n^2$$

both diverge.

(iii) It follows from the right inequality in (1) that $\sum_{n=1}^{\infty} Q_n^3$ converges. In fact (*see Mathematics Magazine*, 1968, p.166)

$$1 + \sum_{n=1}^{\infty} Q_n^3 = \frac{[\Gamma(\frac{1}{4})]^4}{4\pi^3} \doteq 1.393203929685.$$

Also solved by F.G.B. Maskell, Algonquin College.

18. *Proposé par Jacques Marion, Université d'Ottawa.*

Montrer que, dans un triangle rectangle dont les côtés ont *3, 4,* et *5* unités de longueur, aucun des angles aigus n'est un multiple rationnel de π.

I. *Solution de Léo Sauvé, Collège Algonquin.*

Nous allons montrer plus généralement que si, dans un triangle rectangle, les angles aigus ont des tangentes *rationelles,* ces angles aigus ne sont pas des multiples rationnels de π. Puisque la somme des deux angle aigus est $\frac{\pi}{2}$, il suffit évidemment de faire la démonstration pour un seul des deux angles.

Soit donc θ un angle aigu d'un triangle rectangle tel que $\tan\theta = \frac{b}{a}$, rationnel irréductible. On peut supposer que l'angle choisi est celui pour lequel le dénominateur a de la tangente est impair et le numérateur b pair. Puisque ce n'est que la parité de a et b qui entrera dans la démonstration, on peut aussi bien supposer que $a = 3$ et $b = 4$, tel que dans le problème proposé.

Nous utiliserons l'identité classique

$$\frac{\tan n\theta}{\tan\theta} = \frac{\binom{n}{1} - \binom{n}{3}\tan^2\theta + \binom{n}{5}\tan^4\theta \;- \cdots}{1 \;- \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \cdots}.$$

Substituons dans cette formule $\tan\theta = \frac{4}{3}$ et multiplions les deux membres de la fraction de droite par 3^n; il résulte

$$\frac{3}{4}\tan n\theta = \frac{n\cdot 3^n - a_3\cdot 4^2 + a_5\cdot 4^4 - \cdots}{3^n - a_2\cdot 4^2 + a_4\cdot 4^4 - \cdots}, \qquad (1)$$

où les coefficients $a_2, a_3, \ldots, a_n$ sont des entiers.

On constate que le dénominateur dans (1) ne peut s'annuler, car son premier terme est impair et tous les autres termes sont pairs; et le numérateur dans (1) ne peut s'annuler pour la même raison si n est impair.

Cela dit, supposons que $\theta = r\pi$, où r est un rationnel irréductible. Si $r = \frac{m}{n}$, avec n impair, alors $n\theta = m\pi$ et $\tan n\theta = 0$, ce qui est impossible car le numérateur dans (1) ne peut s'annuler. Si $r = \frac{m}{2n}$, alors m est impair, $n\theta = m \cdot \frac{\pi}{2}$ et $\cot n\theta = 0$, ce qui est impossible car le dénominateur dans (1) ne peut s'annuler.

On conclut qu'il n'existe pas de rationnel irréductible r tel que $\theta = r\pi$.

II. *Solution du proposeur.*

Le problème proposé est une conséquence immédiate du théorème suivant de J.M.H. Olmsted (*American Mathematical Monthly*, Vol. 52, 1945, pp.507-508), qui élargit le problème encore plus que la solution I donnée plus haut.

THÉORÈME. Si θ est un multiple rationnel de π, les seules valeurs rationnelles possibles de $\cos\theta$ sont 0, $\pm\frac{1}{2}$, et ± 1.

Also solved by F.G.B. Maskell, Algonquin College.

19. *Proposed by H.G. Dworschak, Algonquin College.*

How many different triangles can be formed from n straight rods of lengths $1, 2, 3, \ldots, n$ units?

Solution by Léo Sauvé, Algonquin College.

Let $T(n)$ be the required number. If a rod of length $n + 1$ is added, the number of new triangles which can be formed is $T(n + 1) - T(n)$. For each of these new triangles the sum of the remaining sides is one of the numbers

$$2n - 1,\ 2n - 2, \ldots,\ n + 3,\ n + 2;$$

and the numbers of triangles with these sums are

$$1, 1, 2, 2, 3, 3, \ldots, \text{ to } n - 2 \text{ terms.}$$

Hence

$$\begin{aligned} T(n + 1) - T(n) &= 1+1+2+2+3+3+ \ldots \text{ to } n - 2 \text{ terms} \\ &= \frac{1}{4}\, n(n - 2), \text{ if } n \text{ is even,} \\ &= \frac{1}{4}\,(n - 1)^2, \text{ if } n \text{ is odd.} \end{aligned}$$

Now

$$\begin{aligned} T(2r) - T(2r - 2) &= \{T(2r) - T(2r - 1)\} + \{T(2r - 1) - T(2r - 2)\} \\ &= (r - 1)^2 + (r - 1)(r - 2) \\ &= 2r^2 - 5r + 3. \end{aligned}$$

If in this relation we replace r by $r - 1, r - 2, \ldots, 4, 3$ and add, noting that $T(4) = 1$, we obtain

$$T(2r) = \frac{1}{6}r(r-1)(4r-5),$$

and we can now also find

$$T(2r+1) = T(2r) + r(r-1)$$
$$= \frac{1}{6}r(r-1)(4r+1).$$

In terms of n,

$$T(n) = \frac{1}{24}n(n-2)(2n-5), \qquad \text{if } n \text{ is even,}$$
$$= \frac{1}{24}(n-1)(n-3)(2n-1), \quad \text{if } n \text{ is odd.}$$

Also solved by G.D. Kaye, Department of National Defence; and the proposer.

20. *Proposé par Jacques Marion, Université d'Ottawa.*

La fonction $f:R \to R$ est définie par

$$f(x) = \begin{cases} x, & \text{si } x \text{ est irrationnel,} \\ p \sin \frac{1}{q}, & \text{si } x = \frac{p}{q} \text{ (rationnel irréductible).} \end{cases}$$

En quel points f est-elle continue?

I. *Solution du proposeur.*

Je montrerai que f est continue aux irrationnels et à 0 seulement.

Soit α un irrationnel quelconque. Ont sait que f est continue à α si et seulement si, pour toute suite $\{\alpha_n\}$,

$$\alpha_n \to \alpha \Longrightarrow f(\alpha_n) \to f(\alpha) = \alpha.$$

Si α_{n_0} est irrationnel, on a $f(\alpha_{n_0}) = \alpha_{n_0}$. Il suffit donc de considérer une suite quelconque de rationnels irréductibles $\frac{p_n}{q_n} \to \alpha$ et montrer que $f\left(\frac{p_n}{q_n}\right) \to \alpha$. De plus, la suite $\left\{\frac{p_n}{q_n}\right\}$ étant infinie, l'ensemble $\{q_n: n = 1, 2, \ldots\}$ ne peut être borné. On peut donc supposer sans perte de généralité que $\frac{1}{q_n} \to 0$. (On n'a qu'à prendre une suite partielle au besoin.)

Vu que $\lim_{x\to 0} \frac{\sin x}{x} = 1$, il vient, en posant

$$R(x) = 1 - \frac{\sin x}{x},$$

que $\sin x = x(1 - R(x))$, avec $\lim_{x\to 0} R(x) = 0$.

Donc

$$f\left(\frac{p_n}{q_n}\right) = p_n \sin\left(\frac{1}{q_n}\right) = \frac{p_n}{q_n}\left(1 - R\left(\frac{1}{q_n}\right)\right) \to \alpha,$$

puisque

$$\frac{p_n}{q_n} \to \alpha \qquad \text{et} \qquad R\left(\frac{1}{q_n}\right) \to 0,$$

d'où la continuité de f aux irrationnels.

Supposons maintenant que $\frac{p_n}{q_n} \to \frac{p}{q}$, rationnel irréductible. Alors

$$f\left(\frac{p_n}{q_n}\right) = \frac{p_n}{q_n}\left(1 - R\left(\frac{1}{q_n}\right)\right) \to \frac{p}{q};$$

mais

$$\frac{p}{q} = f\left(\frac{p}{q}\right) \Longleftrightarrow \frac{p}{q} = p \sin \frac{1}{q} \Longleftrightarrow p = 0.$$

Donc f est discontinue à tous les rationnels sauf à $x = 0$.

II. *Solution by Léo Sauvé, Algonquin College.*

Since

$$f\left(\frac{p}{-q}\right) = p \sin\left(\frac{1}{-q}\right) = -p \sin \frac{1}{q} = f\left(\frac{-p}{q}\right),$$

the function f is well-defined at the rationals, and we can assume $q > 0$ for each irreducible rational $\frac{p}{q}$.

Consider the function $g\colon R \to R$ defined by

$$g(x) = x - f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ p\left(\frac{1}{q} - \sin \frac{1}{q}\right), & \text{if } x = -\frac{p}{q}. \end{cases}$$

Since the function $x \mapsto x$ is everywhere continuous, f is continuous if and only if g is continuous. It is therefore sufficient to study the latter.

We will show that, for all $x_0 \in R$,

$$\lim_{x \to x_0} g(x) = 0,$$

and it will immediately follow that g, and hence f, is continuous at x_0 if and only if $g(x_0) = 0$, that is, if and only if x_0 is 0 or irrational.

Let $\epsilon > 0$ be given, and select arbitrarily $x_0 \in R$.

We define

$$I_0 = \{x \in R : 0 < |x - x_0| < 1\}$$

and

$$P_0 = \{x \in I_0 : |g(x)| \geq \epsilon\}.$$

The elements of P_0 are the irreducible rationals $\frac{p}{q} \neq 0$ such that

$$0 < \left|\frac{p}{q} - x_0\right| < 1 \qquad (1)$$

and

$$\left|g\left(\frac{p}{q}\right)\right| = |p|\left(\frac{1}{q} - \sin\frac{1}{q}\right) \geq \epsilon. \qquad (2)$$

We will show that P_0 contains a finite number of elements.

Condition (1) implies

$$1 > \left|\frac{p}{q} - x_0\right| \geq \left|\frac{p}{q}\right| - |x_0|,$$

so that

$$\left|\frac{p}{q}\right| < 1 + |x_0|; \qquad (3)$$

and conditions (2) and (3) require

$$\begin{aligned}(1 + |x_0|) \cdot \frac{1}{3!q^2} &> (1 + |x_0|)\left[\frac{1}{3!q^2} - \frac{1}{5!q^4} + \frac{1}{7!q^6} - \cdots\right] \\ &> \left|\frac{p}{q}\right|\left[\frac{1}{3!q^2} - \frac{1}{5!q^4} + \frac{1}{7!q^6} - \cdots\right] \\ &= |p|\left[\frac{1}{3!q^3} - \frac{1}{5!q^5} + \frac{1}{7!q^7} - \cdots\right] \\ &= |p|\left(\frac{1}{q} - \sin\frac{1}{q}\right) \\ &\geq \epsilon,\end{aligned}$$

which is satisfied only when

$$q < \sqrt{\frac{1 + |x_0|}{3!\,\epsilon}}. \qquad (4)$$

Finally, for each value of q which satisfies (4), there is only a finite number of values of p which satisfy (3). It is thus clear that P_0 is a finite set.

Now we select $\delta > 0$ sufficiently small so that the punctured interval $0 < |x - x_0| < \delta$ contains no element of P_0. We then have, for all $x \in R$,

$$0 < |x - x_0| < \delta \implies |g(x)| < \epsilon,$$

and so $\lim_{x \to x_0} g(x) = 0$, which completes the proof.

* * *

PROFESSOR BRILLO, MEET PROFESSOR STARK

JOHN THOMAS, Digital Methods Ltd.

As reported on page 23 of this issue of EUREKA, Professor John Brillo of the University of Arizona has proved that $e^{\pi\sqrt{163}}$ is an integer. Unfortunately, his proof will only be published "in a few years" in *Mathematics of Computation*.

While we are waiting for his proof to appear, let us take a look into *An Introduction to Number Theory*, by Harold M. Stark (Markham, 1970). On page 179 appears the following problem:

Correct to ten decimal places,

$$e^{\pi\sqrt{163}} = 262537412640768744.0000000000;$$

show that, nevertheless, $e^{\pi\sqrt{163}}$ *is not an integer.*

If Professor Brillo will just stand back for a few minutes, I will attempt to prove Professor Stark's problem, with the help of the Gelfond-Schneider Theorem:

If α *and* β *are algebraic numbers with* $\alpha \neq 0$, $\alpha \neq 1$, *and if* β *is not a real rational number, then any value of* α^β *is transcendental.*

It is well-known that e^π is one of the values of i^{-2i}; hence $e^{\pi\sqrt{163}}$ is one of the values of $i^{-2i\sqrt{163}}$. Since $\alpha = i$ and $\beta = -2i\sqrt{163}$ satisfy the requirements of the Gelfond-Schneider Theorem, it follows that $e^{\pi\sqrt{163}}$ is transcendental and hence not an integer.

Editor's comment.

The last word has evidently not been said about this matter. EUREKA will try to keep its readers informed of further developments.

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LAST-MINUTE NEWS FLASH: The nearest approximation to John Brillo at the University of Arizona is Dr. John Brillhart. The latest conjecture is that Brillo is, if not imaginary, at least irrational.

Will someone please ask William Rubin, of Wappingers Falls N.Y., whether William McGregor of Wappingers Falls N.Y., has any existence outside the pages of the *April* 1975 issue of *Scientific American*? The latest conjecture here is that Martin Gardner will, perhaps in the June or July issue of *Scientific American*, come out with a *four-colour* representation of McGregor's map.