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SOME SUMS ARE NOT RATIONAL FUNCTIONS OF $R$, $r$, AND $s$

Stanley Rabinowitz

Let $R$, $r$, and $s$ denote the circumradius, inradius, and semiperimeter of a triangle with angles $A$, $B$, and $C$. In problem 652 of this journal [2], W.J. Blundon pointed out the well-known formulae

$$
\sum \sin A = \frac{s}{R} \\
\sum \cos A = \frac{R + r}{R} \\
\sum \tan A = \frac{2rs}{s^2 - 4R^2 - 4Rr - r^2} \\
\sum \tan \frac{A}{2} = \frac{4R + r}{s}
$$

where the sums are cyclic over the angles of the triangle. He asked if there were similar formulae for $\sum \sin A/2$ and $\sum \cos A/2$. All the solutions received were very complicated. Murray Klamkin [3] pointed out that Anders Bager in [1] tacitly implied that there were no known simple $R - r - s$ representations for the following symmetric triangle functions:

$$
\sum \sin \frac{A}{2}, \quad \sum \sin \frac{B}{2} \sin \frac{C}{2}, \quad \sum \csc \frac{A}{2}, \quad \sum \csc \frac{B}{2} \csc \frac{C}{2}, \\
\sum \cos \frac{A}{2}, \quad \sum \cos \frac{B}{2} \cos \frac{C}{2}, \quad \sum \sec \frac{A}{2}, \quad \sum \sec \frac{B}{2} \sec \frac{C}{2} \quad (1)
$$

Klamkin went on to conjecture that these sums cannot be expressed as rational functions of $R$, $r$, and $s$. (A rational function is the quotient of two polynomials.) This conjecture is made plausible by the fact that compendiums of such formulae (such as chapter 4 of [4]) do not include values for these particular sums. In this note, we will prove Klamkin's conjecture.

**THEOREM.** $\sum \sin A/2$ and $\sum \cos A/2$ can not be expressed as rational functions of $R$, $r$, and $s$.

**Proof.** Consider a triangle $ABC$ with sides $BC = 13$, $CA = 14$, and $AB = 15$. This triangle has area 84, semiperimeter 21, inradius 4, and circumradius $65/8$. 
From the Law of Cosines, we can easily compute the cosines of the angles, finding

\[
\begin{align*}
\cos A &= \frac{3}{5} & \sin A &= \frac{4}{5} \\
\cos B &= \frac{33}{65} & \sin B &= \frac{56}{65} \\
\cos C &= \frac{5}{13} & \sin C &= \frac{12}{13}
\end{align*}
\]

From the half-angle formulae, we find that

\[
\begin{align*}
\sin \frac{A}{2} &= \frac{1}{\sqrt{5}} & \sin \frac{B}{2} &= \frac{4}{\sqrt{65}} & \sin \frac{C}{2} &= \frac{2}{\sqrt{13}} \\
\cos \frac{A}{2} &= \frac{2}{\sqrt{5}} & \cos \frac{B}{2} &= \frac{7}{\sqrt{65}} & \cos \frac{C}{2} &= \frac{3}{\sqrt{13}} \\
\sec \frac{A}{2} &= \frac{1}{\sqrt{5}} & \sec \frac{B}{2} &= \frac{1}{\sqrt[4]{65}} & \sec \frac{C}{2} &= \frac{1}{\sqrt[3]{13}} \\
\csc \frac{A}{2} &= \sqrt{5} & \csc \frac{B}{2} &= \frac{1}{\sqrt[4]{65}} & \csc \frac{C}{2} &= \frac{1}{\sqrt[2]{13}}
\end{align*}
\]

We thus see that

\[
\sum \sin \frac{A}{2} = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{65}} + \frac{2}{\sqrt{13}}
\]

which is irrational. This shows that \(\sum \sin A/2\) cannot be a rational function of \(R\), \(r\), and \(s\), for if it were, then in this particular case, its numerical value would be rational (since \(R\), \(r\), and \(s\) are rational in this case), a contradiction. Similarly, \(\sum \cos A/2\) cannot be a rational function of \(R\), \(r\), and \(s\), because that would be contradicted by this particular case, in which

\[
\sum \cos \frac{A}{2} = \frac{2}{\sqrt{5}} + \frac{7}{\sqrt{65}} + \frac{3}{\sqrt{13}}
\]

is also irrational. \(\Box\)

A similar calculation and argument shows that none of the expressions in display (1) can be expressed as rational functions of \(R\), \(r\), and \(s\). In fact, the same argument shows further than none of these expressions can be expressed as rational functions of \(R\), \(r\), \(s\), \(a\), \(b\), \(c\), and \(K\), where \(a\), \(b\), and \(c\) are the lengths of the sides of the triangle and \(K\) is its area.

In many cases, similar results can be shown using simpler examples. For example, let \(m_a\), \(m_b\), \(m_c\) denote the lengths of the medians of a triangle. Using a 3–4–5 right triangle, I showed in [5] that there is no rational function, \(M\), of \(a\), \(b\), and \(c\) such that each of \(m_a\), \(m_b\), \(m_c\) can be expressed as rational functions of \(a\), \(b\), \(c\), and \(\sqrt{M}\).
As an exercise, the reader can prove that \( \sin \frac{x}{2} \) and \( \cos \frac{x}{2} \) can not be expressed as rational functions of \( \sin x \) and \( \cos x \). It is well known that \( \tan \frac{x}{2} \) can be so expressed, namely

\[
\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}.
\]

References:


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**THE OLYMPIAD CORNER**

No. 111

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year has passed and it is time to thank the readership for its continued interest and participation. One of the projects for this year will be to present the results of the challenges thrown out in 1988 to fill in the gaps with solutions to problems posed in the Corner. The response was very good and I'm sure some of you are beginning to wonder what happened to the solutions you sent me. But I also want to continue with solutions to more current problems so as to ensure that our new solvers' efforts are recognized. Beginning next month the
solutions will be split in two parts, "archival" and "current" problem solutions. I would particularly like to thank those who have sent in problem sets, solutions, and comments. Among those whose efforts have helped this column appear during the year are Seung–Jin Bang, Francisco Bellot, Len Bos, Curtis Cooper, Nicos Diamantis, George Evagelopoulos, Douglass L. Grant, the late J.T. Groenman, H.N. Gupta, R.K. Guy, Denis Hanson, Walther Janous, Gy. Karoly, Murray S. Klamkin, Andy Liu, Robert Lyness, Stewart Metchette, Dave McDonald, David Monk, John Morvay, Gillian Nonay, Richard Nowakowski, J. Pataki, Bob Prielipp, M.A. Selby, Zun Shan, Bruce Shawyer, Dimitris Vathis, and Edward T.H. Wang.

The problem sets that we present this month all come to us from Andy Liu, The University of Alberta, who also translated the Chinese I.M.O. Selection Test questions for us.

SINGAPORE MATHEMATICAL SOCIETY
INTERSCHOOL MATHEMATICAL COMPETITION 1988
Part B (2 hours)

[Editor's note: Part A consisted of 10 multiple-choice questions, to be answered in one hour.]

1. Let \( f(x) \) be a polynomial of degree \( n \) such that \( f(k) = \frac{k}{k+1} \) for each \( k = 0,1,2,...,n \). Find \( f(n+1) \).

2. Suppose \( \triangle ABC \) and \( \triangle DEF \) in the figure are congruent. Prove that the perpendicular bisectors of \( AD, BE, \) and \( CF \) intersect at the same point.

3. Find all positive integers \( n \) such that \( P_n \) is divisible by 5, where \( P_n = 1 + 2^n + 3^n + 4^n \). Justify your answer.

4. Prove that for any positive integer \( n \), any set of \( n+1 \) distinct integers chosen from the integers \( 1,2,...,2n \) always contains 2 distinct integers such that one of them is a multiple of the other.
5. Find all positive integers \( x, y, z \) satisfying the equation
\[
5(xy + yz + zx) = 4xyz.
\]

The next two sets were forwarded to Andy Liu by Professor Zun Shan from Hefei in Anhui province of the People’s Republic of China.

**FIRST SELECTION TEST OF THE CHINESE I.M.O. TEAM**
May 3, 1988 (4½ hours)

1. What necessary and sufficient conditions must real numbers \( A, B, C \) satisfy in order that
\[
A(x - y)(x - z) + B(y - z)(y - x) + C(z - x)(z - y)
\]
is non-negative for all real numbers \( x, y \) and \( z \)?

2. Determine all functions \( f \) from the rational numbers to the complex numbers such that
(i) \( f(x_1 + x_2 + \cdots + x_{1988}) = f(x_1)f(x_2)\cdots f(x_{1988}) \)
for all rational numbers \( x_1, x_2, \ldots, x_{1988} \), and
(ii) \( f(1988)f(x) = f(1988)f(x) \) for all rational numbers \( x \), where \( \overline{z} \) denotes the complex conjugate of \( z \).

3. In triangle \( ABC \), angle \( C \) is 30°. \( D \) is a point on \( AC \) and \( E \) is a point on \( BC \) such that \( AD = BE = AB \). Prove that \( OI = DE \) and \( OI \) is perpendicular to \( DE \), where \( O \) and \( I \) are respectively the circumcentre and incentre of triangle \( ABC \).

4. Let \( k \) be a positive integer. Let \( S_k = \{ (a, b): a, b = 1, 2, \ldots, k \} \). Two elements \((a, b)\) and \((c, d)\) of \( S_k \) are said to be indistinguishable if and only if \( a - c \equiv 0 \) or \( \pm 1 \mod k \) and \( b - d \equiv 0 \) or \( \pm 1 \mod k \). Let \( r_k \) denote the maximum number of pairwise indistinguishable elements in \( S_k \).
   (i) Determine \( r_5 \), with justification.
   (ii) Determine \( r_7 \), with justification.
   (iii) Determine \( r_k \) in general. (Justification is not required.)

**SECOND SELECTION TEST OF THE CHINESE I.M.O. TEAM**
May 4, 1988 (4½ hours)

1. Define \( x_n = 3x_{n-1} + 2 \) for all positive integers \( n \). Prove that an integer value can be chosen for \( x_0 \) such that 1988 divides \( x_{100} \).
2. ABCD is a trapezoid with AB parallel to DC. M and N are fixed points on AB with M closer to A than N is. P is a variable point on CD. Let DN cut AP at E and CM at F, and let CM cut BP at G. For which point P is the area of PEFG maximized?

3. A polygon in the xy-plane has area greater than n. Prove that it contains points \((x_i, y_i), 1 \leq i \leq n + 1\) such that \(x_i - x_j\) and \(y_i - y_j\) are integers for all \(i\) and \(j\).

4. With numbers \(u\) and \(v\) as input, a machine generates the number \(uv + v\) as output. In the first operation, the only numbers that can be used for input are 1, \(-1\) and a fixed number \(c\). In later operations, numbers generated by the machine as output in preceding operations may also be used. Prove that for any polynomial \(f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n\) with integer coefficients, the machine can generate \(f(c)\) as output after a finite number of operations.

We now continue with solutions for problems posed in the 1988 numbers of the Corner. The first is a solution to a problem from the January number.

**1985.3 [1988: 3] Kürschak Competition.**

Let each vertex of a triangle be reflected across the opposite side. Prove that the area of the triangle determined by the three points of reflection is less than 5 times the area of the original triangle.

**Solution by Murray S. Klamkin, Mathematics Department, University of Alberta, Edmonton.**

Let \(F\) and \(F'\) denote the respective areas of the original triangle \(ABC\) and the new triangle \(A'B'C'\). We extend the problem by finding both upper and lower bounds for \(F'/F\) for the case when \(ABC\) is non-obtuse, and also for several cases when \(ABC\) is obtuse. The five cases to be considered are

(i). \(ABC\) is non-obtuse;
(ii). \(ABC\) is obtuse with \(A > 90^\circ, A - C < 90^\circ, A - B < 90^\circ;\)
(iii). \(ABC\) is obtuse with \(A > 90^\circ, A - C > 90^\circ, A - B > 90^\circ;\)
(iv). \(ABC\) is obtuse with \(120^\circ \geq A > 90^\circ, A - C < 90^\circ, A - B > 90^\circ;\)
(v). \(ABC\) is obtuse with \(135^\circ > A > 120^\circ, A - C < 90^\circ, A - B > 90^\circ.\)

As indicated by the five associated figures, these cases lead to different representations of \(F'/F\). Actually, this difference is one of sign only, but it affects the upper and lower bounds.
Case (i)

Case (ii)

Case (iii)

Case (iv)

Case (v)

For all cases

\[ h_b = c \sin A, \quad h_c = b \sin A, \quad h_a = c \sin B, \]

\[ BH = 2R \cos B, \quad CH = 2R \cos C, \quad AH = \pm 2R \cos A \]

(according to whether \( A \) is non-obtuse or obtuse). Here \( h_a \) is the altitude to side \( a \), etc., \( R \) is the circumradius, and \( H \) is the orthocenter. Also,

\[ F = 2R^2 \sum \cos B \cos C \sin A. \quad (1) \]

The summations here and subsequently are symmetric over the elements of \( ABC \).

We also write \([X]\) for the area of figure \( X \).

Case (i). \[ F' = [B'HC'] + [C'HA'] + [A'HB'] \]

\[ = \sum \frac{HB' \cdot HC'}{2} \sin B'HC' \]

\[ = 2 \sum (h_b - R \cos B)(h_c - R \cos C) \sin A. \quad (2) \]

Since

\[ \sum h_bh_c \sin A = \sum bc \sin^3 A = 2F \sum \sin^2 A \]

and

\[ 2R \sum (h_c \cos B + h_b \cos C) \sin A = 2R \sum (\sin B \cos C + \sin C \cos B)h_a \]

\[ = 2R \sum h_a \sin A \]

\[ = \sum ah_a = 6F, \]

we have from (1) and (2)
\[ \frac{F'}{F} = 4 \sum \sin^2 A - 5. \]

It is a known result (see for example [1]) that for non-obtuse triangles

\[ 2 \leq \sum \sin^2 A \leq \frac{9}{4}. \]

The upper bound is obtained only for the equilateral triangle and the lower bound for right triangles. Thus for the non-obtuse case,

\[ 3 \leq \frac{F'}{F} \leq 4. \]

Case (ii). For this case we have

\[ 2h_b > 2R \cos B \quad \text{and} \quad 2h_c > 2R \cos C. \]

To see this note that

\[ 2h_b = 2c \sin A = 4R \sin C \sin A \]
\[ = 2R \cos(A - C) - 2R \cos(A + C) \]
\[ > -2R \cos(A + C) = 2R \cos B \]

since \( \cos(A - C) > 0. \) Similarly \( 2h_c > 2R \cos C \) follows from \( \cos(A - B) > 0. \)

Proceeding as before,

\[ \frac{F'}{F} = 4 \sum \sin^2 A - 5. \]

Now \( A > 90^\circ \) and since \( 2A - B - C < 180^\circ, A \) is also less than \( 120^\circ. \) Here, due to restrictions on the angles,

\[ 0 < \frac{F'}{F} < 3. \]

The upper bound of 3, for example, corresponds to \( \sum \sin^2 A < 2. \) To see the latter inequality, we keep \( A \) fixed and minimize

\[ \cos 2B + \cos 2C = 2 \cos(\pi - A)\cos(C - B). \]

This gives \( B = 0 \) and \( C = \pi - A. \) These bounds are best possible since we can get arbitrarily close to them by choosing triangles with angles \((120^\circ - 2\epsilon, 30^\circ + \epsilon, 30^\circ + \epsilon)\) for the lower bound and angles \((90^\circ + \epsilon, 2\epsilon, 90^\circ - 3\epsilon)\) for the upper bound. Note that for the case where the angles are \((120^\circ,30^\circ,30^\circ), F' = 0 \) since \( B', C', \) and \( H \) coincide.

Case (iii). Here \( A - C > 90^\circ, B - C > 90^\circ \) so that \( A > 120^\circ \) and

\[ F' = [C'H_A'] + [A'H_B'] - [B'H_C'] . \]

Proceeding as before we get the same expression except for a change of sign, i.e.

\[ \frac{F'}{F} = 5 - 4 \sum \sin^2 A. \]

Hence

\[ 0 < \frac{F'}{F} < 5. \]

Again these bounds are best possible since we can get arbitrarily close to them by choosing triangles with angles \((120^\circ + 2\epsilon, 30^\circ - \epsilon, 30^\circ - \epsilon)\) for the lower bound and
angles \((180^\circ - 2\epsilon, \epsilon, \epsilon)\) for the upper bound.

**Case (iv).** In this case we have \(A - C < 90^\circ, A - B > 90^\circ\) and
\[
F' = [A'HB'] - [B'HC'] - [C'HA'].
\]
As before this leads to the same expression and bounds for \(F'/F\) as in case (iii). To
get arbitrarily close to the bounds we use triangles of the slightly different angles
\((120^\circ, 30^\circ - \epsilon, 30^\circ + \epsilon)\) and \((90^\circ + 2\epsilon, \epsilon, 90^\circ - 3\epsilon)\) (to satisfy the angle
hypothesis).

**Case (v).** In this case \(A - B > 90^\circ > A - C\), and \(3A/2 > 180^\circ\) so that
\[
F' = [B'HC'] + [C'HA'] - [A'HB'].
\]
This leads to the same expression as in case (iii), i.e.
\[
\frac{F'}{F} = 5 - 4 \sum \sin^2 A.
\]
We can get arbitrarily close to the lower bound \(F'/F > 0\) by choosing angles
\((120^\circ + \epsilon, 30^\circ - 3\epsilon, 30^\circ + 2\epsilon)\). The upper bound here is 1 and is obtained for
angles \((135^\circ - \epsilon, \epsilon, 45^\circ)\). To see this, holding \(A\) fixed we want to maximize
\(\cos 2B + \cos 2C\). Here we can take
\[
C = 90^\circ - A/2 + x \quad \text{and} \quad B = 90^\circ - A/2 - x
\]
where
\[
90^\circ - A/2 > x > 3A/2 - 180^\circ.
\]
Since
\[
\cos 2B + \cos 2C = 2 \cos(180^\circ - A) \cos 2x,
\]
\(x\) must be as small as possible, i.e. \(x = 3A/2 - 180^\circ\). (Subsequently we will add an
\(\epsilon\) to it.) Then \(C = A - 90^\circ\) and \(B = 270^\circ - 2A\). Then
\[
\sum \sin^2 A = \sin^2 A + \cos^2 A + \cos^2 2A
\]
is a minimum for \(A = 135^\circ\).

In summary, for all triangles
\[
0 \leq \frac{F'}{F} < 5.
\]

Another way of proceeding is to reflect the triangle across each side as in the figure. Although the case
corresponding to \(B, C \leq 60^\circ, A \leq 120^\circ\) is rather simple
and leads easily to
\[
\frac{F'}{F} = 4 \sum \sin^2 A - 5,
\]
there are different, less simple, cases to be considered when
\(A > 120^\circ\).
Consider the set $M$ of all points in the plane whose coordinates $(x,y)$ are both whole numbers that satisfy $1 < x < 12, 1 < y < 13$.

(i) Show that every subset of $M$ containing at least 49 points must contain the four vertices of a rectangle having its sides parallel to the coordinate axes.

(ii) Construct a counterexample to (i) if the subset is allowed to consist of only 48 elements.

Solution by Curtis Cooper, Central Missouri State University.

(i) Let $S \subseteq M$ be such that $|S| = 49$. Let $y_i = |\{x: (x,i) \in S\}|$ for $i = 1,2,...,13$. Let $T = \{(t_1,t_2,\cdots,t_{13}): t_i \in \mathbb{N}, 0 \leq t_i \leq 12 \text{ for } i = 1,2,...,13 \text{ and } \sum_{i=1}^{13} t_i = 49\}$. It follows that $(y_1,y_2,\cdots,y_{13}) \in T$. Finally, let $P_i = \{\{r,s\}: r \neq s \text{ and } (r,i),(s,i) \in S\}$ for $i = 1,2,...,13$. Now
\[
\sum_{i=1}^{13} |P_i| = \sum_{i=1}^{13} \binom{y_i}{2} \geq \min_T \sum_{i=1}^{13} \binom{t_i}{2} = 3 \cdot \binom{3}{2} + 10 \cdot \binom{4}{2} = 69
\]
(see the claim below). However, there are $\binom{12}{2} = 66$ ways to pick integers $r$ and $s$ such that $1 \leq r, s \leq 12$ and $r \neq s$. Hence, there exist $j \neq k$ such that $\{a,b\} \in P_j$ and $\{a,b\} \in P_k$.

Thus $(a,j), (b,j), (a,k), (b,k) \in S$ form a rectangle having its sides parallel to the coordinate axes, establishing (i).

Claim. $\min_T \sum_{i=1}^{13} \binom{t_i}{2} = 3 \cdot \binom{3}{2} + 10 \cdot \binom{4}{2}$. 

*
To see the claim, notice first that \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \) and \( \begin{pmatrix} k+1 \\ 2 \end{pmatrix} - \begin{pmatrix} k \\ 2 \end{pmatrix} = k \) for \( k \geq 0 \). Now assume that \((z_0, \ldots, z_{13})\) achieves the minimum. In this case each \( z_i \leq 4 \), for if say \( z_i \geq 5 \) then, as the remaining twelve terms sum to at most 44, one must have \( z_j \leq 3 \) for some \( j \). Decreasing \( z_i \) by 1 and augmenting \( z_j \) by 1 would decrease the sum. Next we argue that in fact \( z_i \geq 3 \) for all \( i \). Otherwise suppose some \( z_i \leq 2 \). The remaining terms must sum to at least 47, but with the observation above this gives \( z_i = 2 \), eleven 4's and one 3, or \( z_i = 1 \) and twelve 4's. The values of the corresponding expressions are 70 and 72, respectively, which are not the minimum. Consequently \( 3 \leq z_i \leq 4 \) for \( i = 1, 2, \ldots, 13 \). The only possibility is to use three 3's and ten 4's.

(ii) 

```
13  
12  
11  
10  
 9  
 8  
 7  
 6  
 5  
 4  
 3  
 2  
 1  
 1  2  3  4  5  6  7  8  9  10 11 12
*   *   *
```

The next solutions that we give are to problems posed in the March 1988 number of the Corner. The first are for the three problems from the Entrance Examination of the Polytechnic of Athens (1962).


Prove that

\[
\left( x - \frac{1}{2} \right) \left( x + \frac{1}{2} \right) < x^2 < \left( x - \frac{23}{48} \right) \left( x + \frac{25}{48} \right)
\]

for all real \( x \geq 6 \). Then prove that
and calculate $\sum_{i=1}^{\infty} \frac{1}{i^2}$ accurate to three decimal places.

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Noting that
\[
\left( x - \frac{1}{2} \right) \left( x + \frac{1}{2} \right) < x^2 < \left( x - \frac{23}{48} \right) \left( x + \frac{25}{48} \right)
\]
is equivalent to
\[
x^2 - \frac{1}{4} < x^2 < x^2 + \frac{x}{24} - \frac{575}{48^2} ,
\]
the first inequality is evident, and for $x \geq 6$ the second follows since
\[
\frac{x}{24} - \frac{575}{48^2} \geq \frac{1}{4} - \frac{575}{48^2} > 0.
\]

Taking reciprocals
\[
\frac{1}{(x - 1/2)(x + 1/2)} > \frac{1}{x^2} > \frac{1}{(x - 23/48)(x + 25/48)} .
\]
Equivalently
\[
\frac{1}{x - 1/2} - \frac{1}{x + 1/2} > \frac{1}{x^2} > \frac{1}{x - 23/48} - \frac{1}{x + 25/48} .
\]

Now, summing all three quantities over all $x \geq 6$, the outside series telescope with
\[
\sum_{i=6}^{\infty} \left( \frac{1}{i - 1/2} - \frac{1}{i + 1/2} \right) = \lim_{N \to \infty} \left( \frac{2}{\Pi} - \frac{1}{N + 1/2} \right) = \frac{2}{\Pi}
\]
and
\[
\sum_{i=6}^{\infty} \left( \frac{1}{i - 23/48} - \frac{1}{i + 25/48} \right) = \lim_{N \to \infty} \left( \frac{48}{265} - \frac{1}{N + 25/48} \right) = \frac{48}{265} .
\]

As each inequality is strict,
\[
\frac{2}{\Pi} > \sum_{i=6}^{\infty} \frac{1}{i^2} > \frac{48}{265} .
\]

Now we have
\[
\sum_{i=1}^{5} \frac{1}{i^2} \approx 1.4636 , \quad \frac{2}{\Pi} \approx 0.1818 \quad \text{and} \quad \frac{48}{265} \approx 0.1811 .
\]
This gives $1.6447 < \sum_{i=1}^{\infty} \frac{1}{i^2} < 1.6454$. Therefore, to three decimal places $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is 1.645.

The well known exact value of $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is $\frac{\pi^2}{6}$ which agrees with this approximation.


Determine those odd natural numbers $n$ such that the common roots of
\[ f(x) = (x + 1)^n - x^n - 1 \]
and
\[ h(x) = (x + 1)^{n-1} - x^{n-1} \]
contain the roots of $x^2 + x + 1$.

Solution by George Evagelopoulos, Law student, Athens, Greece.

The roots of the polynomial $x^2 + x + 1$ are the complex cube roots of 1, namely
\[ \omega_1 = \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \omega_2 = \frac{-1 - i\sqrt{3}}{2} \]
where
\[ \omega_1^3 = \omega_2^3 = 1 \]
\[ \omega_1 = \omega_2^2, \quad \omega_2 = \omega_1^2 \]
\[ \omega_1 + \omega_2 = -1, \quad \omega_1\omega_2 = 1. \]

In order to find which odd natural numbers $n$ have the property that $\omega_1$ and $\omega_2$ are roots of both $f(x)$ and $h(x)$, first observe that
\[ f(\omega_1) = (\omega_1 + 1)^n - \omega_1^n - 1 = (-\omega_1^n - \omega_1^n - 1 = (-1)^n\omega_1^n - \omega_1^n - 1 \]
\[ h(\omega_1) = (\omega_1 + 1)^{n-1} - \omega_1^{n-1} = (-\omega_1^n - \omega_1^n - \omega_1^{n-1} = (-1)^n\omega_1^n - \omega_1^{n-1} - \omega_1^{n-2} - \omega_1^{n-1}. \]

Next distinguish the following cases for values of $n$:

(i) $n = 6k + 1$, $k \in \mathbb{N}$. Then
\[ f(\omega_1) = -\omega_1^{2k+2} - \omega_1^{6k+1} - 1 = -\omega_1^2 - \omega_1 - 1 = -\omega_1^2 - \omega_1 + 1 = 0, \]
\[ h(\omega_1) = \omega_1^{2k} - \omega_1^{6k} = 1 - 1 = 0. \]

(ii) $n = 6k + 3$, $k \in \mathbb{N}$. Then
\[ f(\omega_1) = -\omega_1^{2k+6} - \omega_1^{6k+3} - 1 = -1 - 1 - 1 = -3 \neq 0, \]
\[ h(\omega_1) = \omega_1^{2k+4} - \omega_1^{6k+2} = \omega_1 - \omega_1^2 \neq 0. \]

(iii) $n = 6k + 5$, $k \in \mathbb{N}$. Then
\[ f(\omega_1) = -\omega_1^{2k+10} - \omega_1^{6k+5} - 1 = -\omega_1 - \omega_1^2 - 1 = 0, \]

\[ h(\omega_1) = \omega_1^{2k+8} - \omega_1^{6k+4} = \omega_1 - \omega_1^2 - 1 = 0. \]
\[ h(\omega_1) = \omega_1^{2k+8} - \omega_1^{6k+4} = \omega_1^4 - \omega_1 \neq 0. \]

We have the same results for \( \omega_2 \), and consequently the roots of \( x^2 + x + 1 \) are also roots of both \( f(x) \) and \( h(x) \) if and only if \( n = 6k + 1 \) for some integer \( k \).


Prove that the polynomial
\[ f_n(x) = x \sin a - x \sin(na) + \sin(n - 1)a \]
is exactly divisible by
\[ h(x) = x^2 - 2x \cos a + 1 \]
where \( a \) is a real number and \( n \) is a natural number \( \geq 2 \).

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There is an obvious misprint in the statement of the problem: the function \( f_n(x) \) should be \( x^a \sin a - x \sin(na) + \sin(n - 1)a \). By the quadratic formula the roots of \( h(x) \) are easily found to be \( \cos a \pm i \sin a \). Thus to show \( h(x) \) divides \( f_n(x) \) we must show that \( f_n(\cos a \pm i \sin a) = 0 \). Now
\[ f_n(\cos a \pm i \sin a) = (\cos(na) \pm i \sin(na)) \sin a - (\cos a \pm i \sin a) \sin(na) \]
\[ + (\sin(na) \cos a - \cos(na) \sin a) = 0. \]

A similar calculation gives
\[ f(\cos a - i \sin a) = 0. \]

[Editor's note: E.T.H. Wang continues with more on exact division.]

To see that \( h(x) \) divides \( f_n(x) \) exactly consider
\[ f_n'(x) = nx^{n-4} \sin a - \sin(na). \]

Now
\[ f_n'(\cos a \pm i \sin a) = n[\cos(n - 1)a \pm i \sin(n - 1)a] \sin a - \sin(na) \neq 0 \]
for \( n > 1 \), whence \( h(x) \) divides \( f_n(x) \) but \( (h(x))^2 \) does not.

Since \( f_1(x) = 0 \), all powers of \( h(x) \) divide it.

* 

The next solutions are from the 1986 Spanish Olympiad in Valladolid, Spain.


Find all \( x, y, z \) (real numbers) such that
\[ xyz = \frac{x^3 + y^3 + z^3}{3}. \]

Solutions by Seung-Jin Bang, Seoul, Korea; George Evagelopoulos, Law student, Athens, Greece; Bob Prielipp, University of Wisconsin, Oshkosh; M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario; and
From
\[ x^3 + y^3 + z^3 - 3xyz = \left(\frac{x + y + z}{3}\right)^2 \left( (x - y)^2 + (y - z)^2 + (z - x)^2 \right) , \]
the given equality holds if and only if \( x + y + z = 0 \) or \( x = y = z \).

[Editor's note: At the bottom of the page of Bang's solution was a solution to Kürschak Competition problem 1983.1 [1988: 2] which should have been mentioned among the alternate solutions found in [1989: 229]. My apologies.]


Consider the equations
\[ x^2 + bx + c = 0 \quad \text{and} \quad x^2 + b'x + c' = 0 \]
where \( b, c, b' \) and \( c' \) are integers such that
\[ (b - b')^2 + (c - c')^2 > 0 . \]
Show that if the equations have a common root, then the second roots are distinct integers.

Solutions by Bob Prielipp, University of Wisconsin, Oshkosh, and by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Suppose that \( r \) is a common root, and let \( s \) and \( s' \) denote the other roots of \( x^2 + bx + c = 0 \) and \( x^2 + b'x + c' = 0 \), respectively. It follows that
\[ r + s = -b, \quad r + s' = -b', \quad rs = c \quad \text{and} \quad rs' = c' . \]
If \( s = s' \) then \( b = b' \) and \( c = c' \) contradicting \( (b - b')^2 + (c - c')^2 > 0 \). Thus \( s \neq s' \). Thus \( b \neq b' \). Because \( r \) is a common root,
\[ r^2 + br + c = 0 \quad \text{and} \quad r^2 + b'r + c' = 0 \]
so
\[ (b - b')r = c' - c , \]
and
\[ r = \frac{c' - c}{b - b'} \]
is rational. As \( x^2 + bx + c \) is a monic polynomial \( r \) must be an integer. It follows that \( s = -b - r \) and \( s' = -b' - r \) are both integers as well.


Let \( a_1, a_2, \ldots, a_n \) be \( n \) distinct real numbers. Calculate those points \( x \) on the real line which minimize \( \sum_{i=1}^{n} |x - a_i| \), the sum of the distances from \( x \) to the \( a_i \), in the cases \( n = 3 \) and \( n = 4 \).
Comment by Seung-Jin Bang, Seoul, Korea.

This problem is a special known case of a problem I posed on p. 95, vol. 16, 1987 of the *Mathematical Chronicle*.

Solution by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Assume $a_i < a_j$ for $i < j$. Define

$$f(x) = \sum_{i=1}^{n} |x - a_i|.$$ 

Then $f(x)$ is continuous on $(-\infty, \infty)$ and differentiable everywhere except at $x = a_i$. Clearly $f(x)$ has an absolute minimum since $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. This minimum occurs on $[a_1, a_n]$. The only critical points are the $a_i$, and hence the minimum must occur at one of them (or an end point).

For $n = 3$,

$$f(a_1) = |a_2 - a_1| + |a_3 - a_1| = a_2 - a_1 + a_3 - a_1,$$
$$f(a_2) = a_2 - a_1 + a_3 - a_2,$$
$$f(a_3) = a_3 - a_1 + a_3 - a_2.$$ 

Now

$$a_2 - a_1 + a_3 - a_1 > a_2 - a_1 + a_3 - a_2$$

and

$$a_3 - a_1 + a_3 - a_2 > a_2 - a_1 + a_3 - a_2.$$ 

Thus the minimum occurs at $x = a_2$ and $f(a_2) = a_3 - a_1$.

For $n = 4$,

$$f(a_1) = a_2 - a_1 + a_3 - a_1 + a_4 - a_1,$$
$$f(a_2) = a_2 - a_1 + a_3 - a_2 + a_4 - a_2,$$
$$f(a_3) = a_3 - a_1 + a_3 - a_2 + a_4 - a_3,$$
$$f(a_4) = a_4 - a_1 + a_4 - a_2 + a_4 - a_3.$$ 

Now

$$f(a_1) > f(a_2) = a_4 - a_1 + a_3 - a_2$$

and

$$f(a_4) > f(a_3) = a_4 - a_1 + a_3 - a_2.$$ 

Hence the minimum occurs at $x = a_2$ or $x = a_3$. Actually $f(x) = a_4 - a_1 + a_3 - a_2$ for $a_2 \leq x \leq a_3$.

For $n = 3$,

$$f(x) = \begin{cases} 
  a_2 - a_1 + a_3 - x, & a_1 \leq x \leq a_2, \\
  x - a_1 + a_3 - a_2, & a_2 \leq x \leq a_3,
\end{cases}$$

and for $n = 4$, 

We can now conclude that for $n = 3$ the minimum occurs at $x = a_2$, while for $n = 4$ it occurs on the interval $a_2 \leq x \leq a_3$.

In a triangle $ABC$ with opposite sides $a, b, c$ respectively, show that if
$$a + b = (a \tan A + b \tan B)\tan(C/2),$$
then the triangle is isosceles.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh.
Let $ABC$ be a triangle such that neither angle $A$ nor angle $B$ is a right angle, and let $R$ be the circumradius. Then we have the following equivalent equalities:
$$a + b = (a \tan A + b \tan B)\tan(C/2),$$
$$\tan\left(\frac{A + B}{2}\right) = \frac{a}{a + b} \cdot \tan A + \frac{b}{a + b} \cdot \tan B,$$
$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2R \sin A \tan A}{2R \sin A + 2R \sin B} + \frac{2R \sin B \tan B}{2R \sin A + 2R \sin B} = \frac{\sin A \tan A + \sin B \tan B}{\sin A + \sin B},$$
$$\sin^2 A + 2 \sin A \sin B + \sin^2 B = \sin^2 A + \cos A \sin B \tan B + \cos B \sin A \tan A + \sin^2 B,$$
$$2 \sin A \sin B = \frac{\cos A \sin^2 B}{\cos B} + \frac{\cos B \sin^2 A}{\cos A},$$
$$\sin^2 A \cos^2 B - 2 \sin A \cos B \cos A \sin B + \cos^2 A \sin^2 B = 0,$$
$$\sin^2(A - B) = 0,$$
and thus $A = B$. It follows that triangle $ABC$ is isosceles.

[Editor's note: This problem was also solved by George Evagelopoulos, Law student, Athens, Greece, by a somewhat different sequence of identities.]

In a right triangle $ABC$, with centroid $G$, consider the triangles $AGB$, $GBC$, $CGA$. If the sides of $ABC$ are all natural numbers, show that the areas of $AGB$, $BGC$, $CGA$ are even natural numbers.
Solution by Bob Prielipp, University of Wisconsin, Oshkosh.

By hypothesis triangle $ABC$ is a Pythagorean triangle. It suffices to establish the required result when triangle $ABC$ is a primitive Pythagorean triangle. In the remainder we assume this is the case. Let $a$ and $b$ be the lengths of the legs of triangle $ABC$ where $b$ is an even positive integer, and let $c$ be the length of the hypotenuse. Then there are positive integers $r$ and $s$, $r > s$, such that $a = r^2 - s^2$, and $b = 2rs$, where $r$ and $s$ are of opposite parity and are relatively prime. [See for example, Theorem 5.1, p. 139 of *An Introduction to the Theory of Numbers*, 4th Edition, by Niven and Zuckerman, John Wiley & Sons, New York, 1980.]

**Lemma.** The area of triangle $ABC$ is divisible by 6.

**Proof.** Because $r$ and $s$ are of opposite parity, 4 divides $b$. If 3 divides $r$ or 3 divides $s$, then 3 divides $b$. Otherwise $r^2 \equiv 1 \pmod{3}$ and $s^2 \equiv 1 \pmod{3}$, and so $r^2 \equiv s^2 \pmod{3}$. Thus 3 divides $r^2 - s^2$, and so 3 divides $a$. It now follows that 12 divides $ab$, giving the lemma.

It is known that triangles $AGB$, $BGC$, and $CGA$ all have the same area, $L$, say [see Theorem 23, p. 47 of *Modern College Geometry*, by Davis, Addison–Wesley, Reading, MA, 1949]. Then $3L = K$, the area of triangle $ABC$. But as 6 divides $K$, we have that $L$ is a positive even integer.

* * *

Olympiad Season is fast approaching. Remember to send me your national and regional contests!

* * *

**P R O B L E M S**

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

Two circles $K$ and $K_1$ touch each other externally. The equilateral triangle $ABC$ is inscribed in $K$, and points $A$, $B$, $C$ lie on $K_1$ such that $AA_1$, $BB_1$, $CC_1$ are tangent to $K_1$. Prove that one of the lengths $AA_1$, $BB_1$, $CC_1$ equals the sum of the other two. (The case when the circles are internally tangent was a problem of Florow in Praxis der Mathematik 13, Heft 12, page 327.)


$AB$ is a chord, not a diameter, of a circle with centre $O$. The smaller arc $AB$ is divided into three equal arcs $AC$, $CD$, $DB$. Chord $AB$ is also divided into three equal segments $AC'$, $C'D'$, $D'B$. Let $CC'$ and $DD'$ intersect in $P$. Show that $\angle APB = \frac{1}{3}\angle AOB$.

1503. Proposed by M.S. Klamkin, University of Alberta.

Prove that

$$1 + 2\cos(B+C)\cos(C+A)\cos(A+B) \geq \cos^2(B+C) + \cos^2(C+A) + \cos^2(A+B),$$

where $A$, $B$, $C$ are nonnegative and $A + B + C \leq \pi$.

1504. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $A_1A_2\cdots A_n$ be a circumscribable $n$-gon with incircle of radius 1, and let $F_1, F_2, \ldots, F_n$ be the areas of the $n$ corner regions inside the $n$-gon and outside the incircle. Show that

$$\frac{1}{F_1} + \cdots + \frac{1}{F_n} \geq \frac{n^2}{n \tan(\pi/n) - \pi}.$$ 

Equality holds for the regular $n$-gon.

1505. Proposed by Marcin E. Kuczma, Warszawa, Poland.

Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{x_n} \left( \sqrt{1 + x_n^2} - 1 \right).$$

Show that the sequence $(2^n x_n)$ converges and find its limit.

1506. Proposed by Jordi Dou, Barcelona, Spain.

Let $A$ and $P$ be points on a circle $\Gamma$. Let $l$ be a fixed line through $A$ but not through $P$, and let $x$ be a variable line through $P$ which cuts $l$ at $L_x$ and $\Gamma$ again at $G_x$. Find the locus of the circumcentre of $\triangle AL_xG_x$. 

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before August 1, 1990, although solutions received after that date will also be considered until the time when a solution is published.
1507. Proposed by Nicos D. Diamantis, student, University of Patras, Greece.
Find a real root of
\[ y^5 - 10y^3 + 20y - 12 = 0. \]

Let \( a < b < c \) be the lengths of the sides of a right triangle. Find the largest constant \( K \) such that
\[ a^2(b + c) + b^2(c + a) + c^2(a + b) \geq Kabc \]
holds for all right triangles and determine when equality holds. It is known that the inequality holds when \( K = 6 \) (problem 351 of the College Math. Journal; solution on p. 259 of Volume 20, 1989).

1509. Proposed by Carl Friedrich Sutter, Viking, Alberta.
Professor Chalkdust teaches two sections of a mathematics course, with the same material taught in both sections. Section 1 runs on Mondays, Wednesdays, and Fridays for 1 hour each day, and Section 2 runs on Tuesdays and Thursdays for 1.5 hours each day. Normally Professor Chalkdust covers one unit of material per hour, but if she is teaching some material for the second time she teaches twice as fast. The course began on a Monday. In the long run (i.e. after \( N \) weeks as \( N \to \infty \)) will one section be taught more material than the other? If so, which one, and how much more?

\( P \) is any point inside a triangle \( ABC \). Lines \( PA, PB, PC \) are drawn and angles \( PAC, PBA, PCB \) are denoted by \( \alpha, \beta, \gamma \) respectively. Prove or disprove that
\[ \cot \alpha + \cot \beta + \cot \gamma \geq \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}, \]
with equality when \( P \) is the incenter of \( \triangle ABC \).

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

Show that the sides of the pedal triangle of any interior point \( P \) of an equilateral triangle \( T \) are proportional to the distances from \( P \) to the corresponding
vertices of $T$.

Solution by Wilson da Costa Areias, Rio de Janeiro, Brazil.

$AC_1PB_1$, $BA_1PC_1$ and $CA_1PB_1$ are inscribed quadrilaterals in the circles whose diameters are respectively $PA$, $PB$ and $PC$. Using the law of sines in $\triangle AB_1C_1$, we have

$$\frac{B_1C_1}{\sin 60^\circ} = \frac{B_1A}{\sin ZAC_1B_1} = \frac{B_1A}{\sin ZAPB_1} = PA,$$

so

$$\frac{B_1C_1}{PA} = \frac{\sqrt{3}}{2}.$$

Similarly

$$\frac{A_1C_1}{PB} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{A_1B_1}{PC} = \frac{\sqrt{3}}{2},$$

therefore

$$\frac{B_1C_1}{PA} = \frac{A_1C_1}{PB} = \frac{A_1B_1}{PC}.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Several readers, including the proposer, pointed out that the formulae

$$B_1C_1 = PA \sin A,$$

etc.

for the sides of a pedal triangle are known (e.g. Theorem 190 p. 136 of R.A. Johnson, Advanced Euclidean Geometry), and that the result follows easily, as above. The proposer’s original solution was more involved, though.

* * *


Let $A_1A_2\cdots A_n$ be a polygon inscribed in a circle and containing the centre of the circle. Prove that

$$n - 2 + \frac{4}{\pi} < \sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} \leq \frac{n^2}{\pi} \sin \frac{\pi}{n},$$

where $a_i$ is the side $A_iA_{i+1}$ and $\hat{a}_i$ is the arc $A_iA_{i+1}$. 
Solution by Walther Janous, Ursulengymnasium, Innsbruck, Austria.

Let

$$2\varphi_i = \angle A_i OA_{i+1}, \quad i = 1, \ldots, n \quad (A_{n+1} = A_1),$$

where $O$ is the center of the circle. Then, with $R$ the radius,

$$a_i = 2R \sin \varphi_i, \quad \hat{a}_i = 2R \varphi_i,$$

whence

$$\frac{a_i}{\hat{a}_i} = \frac{\sin \varphi_i}{\varphi_i}, \quad i = 1, \ldots, n.$$

Now it is either known or not hard to check that

$$f(\varphi) = \frac{\sin \varphi}{\varphi}, \quad 0 < \varphi \leq \frac{\pi}{2},$$

is concave. (Indeed,

$$f'(\varphi) = \frac{\varphi \cos \varphi - \sin \varphi}{\varphi^2},$$

$$f''(\varphi) = \frac{-\varphi^2 \sin \varphi - 2\varphi \cos \varphi + 2 \sin \varphi}{\varphi^3},$$

and from

$$d'(\varphi) = -\varphi^2 \cos \varphi < 0$$

and $d(0) = 0$ we infer $f'(\varphi) \leq 0$ for $0 < \varphi \leq \pi/2$.)

Thus

$$\sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} = \sum_{i=1}^{n} f(\varphi_i) \leq nf\left(\sum_{i=1}^{n} \frac{\varphi_i}{n}\right) = nf\left(\frac{\pi}{n}\right) = \frac{n^2}{\pi} \sin \frac{\pi}{n}.$$

Since the polygon contains $O$, all $\varphi_i$'s are $\leq \pi/2$. This and the concavity of $f$ yield (cf. [1], p. 22)

$$\sum_{i=1}^{n} f(\varphi_i) > f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + (n-2)f(0) = \frac{4}{\pi} + n - 2.$$

[Editor's note: Other solvers noted that

$$(\varphi_1, \varphi_2, \ldots, \varphi_n) \prec \left(\frac{\pi}{2}, \frac{\pi}{2}, 0, \ldots, 0\right)$$

and applied the majorization inequality to obtain this last inequality.]

Reference:


Also solved (in the same way) by MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; COLIN SPRINGER, student, University of Waterloo; and the proposer.

The given bounds are best possible, as is evident from the above proof. The proposer conjectures that if the inscribed polygon is not required to contain the centre,
then the best bounds are
\[ n - 1 < \sum_{i=1}^{n} \frac{a_i}{\tilde{a}_i} < \frac{n^2}{\pi} \sin \frac{\pi}{n}. \]

Any comments?

* * *


Show that for all positive integers \( d, a, n \) such that \( 3 < d \leq 2^{n+1} \), \( d \) does not divide into \( a^{2^n} + 1 \).

Solution by Sverrir Thorvaldsson, student, University of Iceland, Reykjavik.

Let \( d \geq 3 \) be a number that divides into \( a^{2^n} + 1 \); we show that \( d > 2^{n+1} \). Since \( a^{2^n} \) is a perfect square, it is clear that 4 does not divide into \( a^{2^n} + 1 \). Therefore \( d \) has an odd prime divisor \( p \), and \( a^{2^n} \equiv -1 \mod p \). Let \( m \) be the smallest natural number such that \( a^m \equiv 1 \mod p \). Now

\[ a^{2^{n+1}} = (a^{2^n})^2 \equiv (-1)^2 = 1 \mod p, \]

so \( m \mid 2^{n+1} \). If \( m < 2^{n+1} \) write \( m = 2^k \) with \( k \leq n \). But then

\[ 1 \equiv (a^{2^k})^{2^{n-k}} = a^{2^n} \equiv -1 \mod p, \]

a contradiction since \( p \) is odd. Thus \( m = 2^{n+1} \). But according to Fermat’s small theorem, \( a^{p-1} \equiv 1 \mod p \), and therefore \( p - 1 \geq 2^{n+1} \), i.e.

\[ d \geq p \geq 2^{n+1} + 1, \]

which is exactly what we set out to prove. As a byproduct we also get that \( p \equiv 1 \mod 2^{n+1} \).

Also solved by ANDREW CHOW, student, Albert Campbell C.I., Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

* * *


Given four lines \( a, b, c, d \) in general position in the plane, show that there is a unique line \( z \) cutting \( a, b, c, d \) in the respective points \( A, B, C, D \) and in that order, such that \( AB = BC = CD \).
I. Solution by J. Chris Fisher, University of Regina.

Let $P_1$ be the midpoint of the segment joining $bc$ (short for $b \cap c$) to $ab$. Let $Q_1$ be chosen so that $bc$ is the midpoint of the segment joining it to $ac$. Then we observe: for any $P \in P_1Q_1$, if $B'$ is the point of $b$ where it meets the parallel to $c$ through $P$, $C'$ is the point of $c$ where it meets the parallel to $b$ through $P$, and $A' = B' \cap a$, then $A'B' = B'C'$. The proof of this claim comes from the figure using the affine coordinates shown there.

Similarly let $Q_2$ be the midpoint of the segment joining $bc$ to $cd$, and let $P_2$ be chosen so that $bc$ is the midpoint of the segment joining it to $bd$. Then given any $P \in P_2Q_2$, the corresponding statement that $B'C' = C'D'$ follows by replacing $a$ by $d$ and interchanging the roles of $b$ and $c$ in the above figure.

Finally, $P_1Q_1$ meets $P_2Q_2$ in a unique point $O$ since we assume the given configuration to be in general position (i.e. no pair of either the given or constructed lines are parallel). Let $B$ be the point of $b$ where it meets the parallel to $c$ through $O$, and let $C$ be the point of $c$ where it meets the parallel to $b$ through $O$. Then $x = BC$ is the line we were to construct.


Introduce the intersections $a \cap c = M$, $b \cap d = N$, $a \cap b = F$, $b \cap c = G$, $c \cap d = H$. The envelope of the lines for which $AB = BC$ is the parabola $P_1$, tangent to $a$, $b$, $c$ in $P$, $Q$, $R$ with $MF = FP$, $FQ = QG$, $MG = GR$. The envelope of the lines for which $BC = CD$ is the parabola $P_2$, tangent to $b$, $c$, $d$ in $S$, $T$, $U$ with $NG = GS$, $GT = TH$, $NH = HU$. $P_1$ and $P_2$ have common tangents $b$ and $c$. The third tangent
that $P_1$ and $P_2$ have in common is the line we are looking for.

Also solved by L.J. HUT, Groningen, The Netherlands; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer (whose solution was the same as Penning's).

Can someone supply the editor with an accessible reference for the "envelope" property of the parabola, used in Solution II?

* * *

1389. [1988: 269] Proposed by Derek Chang, California State University, Los Angeles, and Raymond Killgrove, Indiana State University, Terre Haute.

Find

$$\max_{\pi \in S_n} \sum_{i=1}^{n} |i - \pi(i)| ,$$

where $S_n$ is the set of all permutations of $\{1,2,\ldots,n\}$.

I. Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let

$$T(\pi) = \sum_{i=1}^{n} |i - \pi(i)| .$$

We will show that

$$\max_{\pi \in S_n} T(\pi) = \left\lfloor \frac{n^2}{2} \right\rfloor , \quad (1)$$

where $[\ ]$ is the floor function.

Consider the permutation $\pi = (n,n-1,\ldots,1)$. Then

$$T(\pi) = |1 - n| + |2 - (n - 1)| + \cdots + |n - 1| .$$

By expanding the absolute value functions and collecting like terms we obtain

$$T(\pi) = \begin{cases} 
2 \left\{ n + (n-1) + \cdots + \left( \frac{n+1}{2} \right) - 1 - 2 - \cdots - \frac{n}{2} \right\} , & n \text{ even} , \\
2 \left\{ n + (n-1) + \cdots + \frac{n+3}{2} \right\} + \frac{n+1}{2} - 2 \left\{ 1 + 2 + \cdots + \frac{n-1}{2} \right\} - \frac{n+1}{2} , & n \text{ odd} .
\end{cases} \quad (2)$$

It is clear that this is the highest sum that can be obtained with these numbers, since all of the larger numbers appear preceded by a plus sign whereas all of the smaller numbers are preceded by a minus sign. [And the expansion of the sum $T(\pi)$ for any $\pi$ will always produce each of the numbers $1,\ldots,n$ twice, with $n$ of these $2n$ numbers preceded by a plus sign and $n$ by a minus sign.] There are, of course, other permutations that will produce the same sum. From (2) we obtain
These two results may be replaced by a single expression by use of the floor function as in (1).

**II. Generalization by Murray S. Klamkin, University of Alberta.**

Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be nonincreasing sequences; \( c_1, c_2, \ldots, c_n \) any permutation of the \( b \) sequence; \( F(t) \) a convex function; and

\[
S = \sum_{i=1}^{n} F(a_i - c_i) .
\]

Then

\[
\max S = F(a_1 - b_n) + F(a_2 - b_{n-1}) + \cdots + F(a_n - b_1) ,
\]

the maximum taken over all rearrangements \( c_1, c_2, \ldots, c_n \).

To prove this we use the majorization inequality [1], [2], i.e. the conditions

\[
x_1 \geq x_2 \geq \cdots \geq x_n, \quad y_1 \geq y_2 \geq \cdots \geq y_n,
\]

\[
\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \quad \text{for} \quad k = 1, 2, \ldots, n - 1,
\]

and

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i
\]

are necessary and sufficient in order that for every convex function \( F \),

\[
\sum_{i=1}^{n} F(x_i) \geq \sum_{i=1}^{n} F(y_i) .
\]

(The above conditions are denoted by

\[(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)\]

and we say that the left hand vector majorizes the right hand vector.)

Here it is clear that

\[(a_1 - b_n, a_2 - b_{n-1}, \ldots, a_n - b_1) \succ (a_1 - c_1, a_2 - c_2, \ldots, a_n - c_n)\]

after the components of the latter vector are arranged in nonincreasing order, and the result follows.

The given problem corresponds to the special case \( F(x) = |x| \), and \( a_i = b_i = i \) for \( i = 1, 2, \ldots, n \). Here the maximum sum is \( 2m^2 \) if \( n = 2m \) and \( 2m^2 + 2m \) if
\[ n = 2m + 1. \]

References:

Also solved by WALThER JANous, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; PAUL YIU, University of Hong Kong; and the proposers. Two further readers sent in the correct value for the maximum, one with no proof and one with an incorrect proof.


\[ 1390. \text{[1988: 269]} \quad \text{Proposed by H. Fukagawa, Aichi, Japan.} \]

A, B, C are points on a circle \( \Gamma \) such that \( CM \) is the perpendicular bisector of \( AB \). \( P \) is a point on \( CM \) and \( AP \) meets \( \Gamma \) again at \( D \). As \( P \) varies over segment \( CM \), find the largest radius of the inscribed circle tangent to segments \( PD, PB \), and arc \( DB \) of \( \Gamma \), in terms of the length of \( CM \).

Solution by Marcin E. Kuczma, Warszawa, Poland.

The looked-for maximum equals \( \frac{CM}{4} \), attained when \( P \) is the midpoint of \( CM \). This is amazing! In fact, for any \( P \) on \( CM \), the diameter of the incircle under consideration is the harmonic mean of the lengths of \( CP \) and \( PM \); the claim hence follows immediately.

There must exist some nice geometric proof of this fact, but I was not lucky enough to find it. The only solution I was able to work out was a lengthy one, via unpleasant calculations. Here is an outline.

We let \((O,r)\) be the center and radius of \( \Gamma \), \((Q,y)\) the center and radius of the incircle of the curved triangle \( PBD \) (\( BD \) the arc of the circle), and put \( CM = h \), \( MP = x \). Then our claim is
\[ y = \frac{2y}{h} (h - x) \]  

\[ [= \frac{1}{2}\text{-harmonic mean of } x \text{ and } h - x]. \]

First, \( PQ \) bisects \( \angle BPD \), hence \( PQ \parallel AB \) and so

\[ \frac{1}{2}PQ \cdot x = \text{Area}(PQM) = \text{Area}(PQB) = \frac{1}{2}PB \cdot y. \]  \hspace{1cm} (1)

Other relevant identities are

\[ PQ^2 = OQ^2 - OP^2 = (r - y)^2 - (r + x - h)^2 \]  \hspace{1cm} (2)

and

\[ PB^2 = MP^2 + OB^2 - OM^2 = x^2 + r^2 - (h - r)^2. \]  \hspace{1cm} (3)

Putting (2) and (3) into the square of (1) leads, after some manipulation, to the quadratic equation

\[ ax^2 + bx + c = 0 \]

where

\[ a = h(2r - h), \quad b = 2rx^2, \quad c = x^2(h - x)(h - x - 2r). \]

The positive root is

\[ y = \frac{a(h - x)}{h} \]

(verification by substitution); the other root is negative since \( a > 0 \) and \( c < 0 \). (Of course, these roots can be found without guessing, by the standard algorithm.)

[Editor's note: The editor apologizes for an unclear statement of the problem (not in the proposer's original formulation), which misled two other readers into not completely answering the problem. Namely, it should have been said that point \( M \) lies on \( AB \). L.J. HUT, Groningen, The Netherlands, and P. PENNING, Delft, The Netherlands, both took \( M \) to be the centre of the circle \( \Gamma \). Hut ended up maximizing the radius for points \( P \) on the diameter through \( C \) but outside \( CM \), and his result also appears to be interesting. Penning's solution (when properly interpreted) is correct, and may be shorter than the one above. The editor invites Hut and Penning, and other Crux readers, to find a "nice" algebraic, or perhaps purely geometric, solution.]

Also solved by the proposer, whose proof (algebraic) also contained the lovely "harmonic mean" relation given by Kuczma.

The problem was taken from the 1840 Japanese mathematics book Sanpo Senmonsho.

* * *


Let \( ABC \) be a triangle and \( D \) the point on \( BC \) so that the incircle of \( \triangle ABD \) and the excircle (to side \( DC \)) of \( \triangle ADC \) have the same radius \( \rho_1 \). Define \( \rho_2 \),
\[ \rho_3 \text{ analogously. Prove that} \]
\[ \rho_1 + \rho_2 + \rho_3 \geq \frac{9}{4}r, \]

where \( r \) is the inradius of \( \triangle ABC \).

I. Solution by Emilia A. Velikova and Svetoslav J. Bilchev, Technical University, Russe, Bulgaria.

Let the points \( I, I_1 \) and \( I'_1 \) be the centres of the incircles and the excircle in the triangles \( ABC \) and \( ABD \) as shown, and let \( E, E_1 \) be on \( AB \) and \( F, P \) be on \( AC \) so that \( IE \parallel AB, I_1E_1 \parallel AB, IF \parallel AC, \) and \( I'_1P \parallel AC \). Then from the similar triangles \( I'_1CP \) and \( ICF \) we have
\[ \frac{\rho_1}{CP} = \frac{I'_1P}{IP} = \frac{CF}{IF} = \frac{s - c}{r}, \quad (1) \]

where \( a, b, c, s \) are the sides and semiperimeter of \( \triangle ABC \). It is well-known that
\[ CP = AP - AC = \frac{b + AD + CD}{2} - b = \frac{AD + CD - b}{2}, \quad (2) \]
and from (1) and (2) we get
\[ \frac{\rho_1}{s - c} = \frac{AD + CD - b}{2}. \quad (3) \]

Further, from the similarity of triangles \( BIE \) and \( BI_1E_1 \) we obtain
\[ \frac{\rho_1}{r} = \frac{I_1E_1}{IE} = \frac{BE_1}{BE} = \frac{c + BD - AD}{2(s - b)}, \]

i.e.
\[ \frac{(s - b)}{r} \rho_1 = \frac{c + BD - AD}{2}. \quad (4) \]

Adding (3) and (4) we get
\[ \left( \frac{1}{s - c} + \frac{1}{r} \right) \rho_1 = \frac{a + c - b}{2} = s - b, \]
from where
\[ \rho_1 = \frac{r(s - b)(s - c)}{r^2 + (s - b)(s - c)}. \quad (5) \]

From (5) and
\[ \rho^2 = \frac{(s - a)(s - b)(s - c)}{s}, \]
we obtain the beautiful expression
\[ \rho_1 = \frac{rs}{2s - a} = \frac{F}{b + c}. \]
where $F$ is the area of $\triangle ABC$. Analogously we get

$$\rho_2 = \frac{F}{c + a}, \quad \rho_3 = \frac{F}{a + b}.$$  

Then the inequality to be proved is equivalent to

$$F\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq \frac{9}{4}r,$$

i.e.

$$2(a + b + c)\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq 9.$$  

But this is equivalent to the well-known inequality

$$[(b + c) + (c + a) + (a + b)]\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq 3^2,$$

so the proof is complete.

**II. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.**

We show more generally: if $t \neq 0$ is a real number, then

$$\rho_1 + \rho_2 + \rho_3 \begin{cases} \geq 3(3/4)^t r^t & \text{if } t > 0 \text{ or } t < -1, \\ = 4/r & \text{if } t = -1, \\ \leq 3(3/4)^t r^t & \text{if } -1 < t < 0. \end{cases}$$

[Editor's note: At this point Janous derived and used the expression

$$\rho_1 = \frac{r}{1 + \tan(B/2)\tan(C/2)},$$

but in view of solution I it is easier to borrow the formula

$$\rho_1 = \frac{rs}{2s - a}$$

proved there. So adapted, Janous's argument continues ... ]

Thus

$$\rho_1 t + \rho_2 t + \rho_3 t = \frac{2s - a + 2s - b + 2s - c}{rs} = \frac{4}{r},$$

which is the case $t = -1$. The rest follows by the general means inequality. For $t > -1$, $t \neq 0$ we have $M_t \geq M_{-1}$, i.e.

$$\left(\frac{1}{3} \sum_{i=1}^{3} \rho_i^t\right)^{1/t} \geq \left(\frac{1}{3} \sum_{i=1}^{3} \rho_i^{-1}\right)^{-1} = \frac{3r}{4},$$

whence

$$\sum_{i=1}^{3} \rho_i^t \begin{cases} \geq 3(3/4)^t r^t & \text{if } t > 0, \\ \leq 3(3/4)^t r^t & \text{if } -1 < t < 0. \end{cases}$$

For $t < -1$, $M_t \leq M_{-1}$ and so
Thus

\[
\sum_{i=1}^{3} \rho_i^t \geq 3(3/4)^t r^t.
\]

Done!

Also solved by L.J. HUT, Groningen, The Netherlands; T. SEIMIYA, Kawasaki, Japan; and the proposer.

* * *


An immense spherical balloon is being inflated so that it constantly touches the ground at a fixed point \( A \). A boy standing at a point at unit distance from \( A \) fires an arrow at the balloon. The arrow strikes the balloon at its nearest point (to the boy) but does not penetrate it, the balloon absorbing the shock and the arrow falling vertically to the ground. What is the longest distance through which the arrow can fall, and how far from \( A \) will it land in this case?

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

From the diagram we easily obtain

\[
\frac{r}{r + z} = \sin \theta,
\]

\[
D = z \sin \theta
\]

from which we derive

\[
D = \tan \theta (1 - \sin \theta).
\]  

In order to find the maximum \( D \), we set \( dD/d\theta = 0 \):

\[
(1 - \sin \theta) \sec^2 \theta - \tan \theta \cos \theta = 0.
\]

Multiplying by \( 1 + \sin \theta \) gives

\[
1 - \sin \theta (1 + \sin \theta) = 0
\]

or

\[
\sin^2 \theta + \sin \theta - 1 = 0.
\]

Solving for \( \sin \theta \) gives

\[
\sin \theta = \frac{\sqrt{5} - 1}{2}.
\]

(The other root is ignored because it would make \( \sin \theta \) greater than 1.) Substituting this result into (1) gives
\[ D_{\text{max}} = \sqrt{\frac{\sqrt{5} - 1}{2}} \cdot \frac{3 - \sqrt{5}}{2} = 0.300283106\ldots \]

It can easily be shown that \( d^2D/d\theta^2 \) is negative throughout the region \( 0 < \theta < \pi/2 \), so this is indeed a maximum, not a minimum. [Editor's note: or just look at the endpoints \( r \to 0, r \to \infty \).] The distance from \( A \) to the point where the arrow lands is

\[ x = 1 - \frac{D}{\tan \theta} = \sin \theta = \frac{\sqrt{5} - 1}{2} = 0.618033988\ldots \]

Also solved by HAYO AHLBURG, Benidorm, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; M.S. KLAMKIN, University of Alberta; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; and the proposer.

Not all solvers found simple expressions for the required distances. Ahlburg, on the other hand, noted that, with

\[ \phi = \frac{\sqrt{5} + 1}{2}, \]

the maximum falling distance is \( \phi^{-5/2} \), the radius of the balloon when this occurs is \( \phi^{1/2} \), and the distance from \( A \) to where the arrow hits the ground is \( \phi^3 \): in his words, "a delightful ubiquity of the Golden Section in such a simple figure".

* * *

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A package bearing the above label arrived here over the Christmas holidays from J. SUCK of Essen, and was speedily transformed into the empty set by the editor and others. Sadly, all that remains for the *Crux* readership is the label. The editor thanks Mr. Suck for his thoughtful and appropriate gift.

Hmmm. Does there perhaps exist an automobile manufacturer with a line of *Crux* sports cars?
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