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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first problems we present are from the 7th annual American Invitational Mathematics Examination (A.I.M.E.) written March 21, 1989. The time allowed was three hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
Tuesday, March 21, 1989


2. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

3. Suppose $n$ is a positive integer and $d$ is a single digit in base 10. Find $n$ if

$$\frac{n}{810} = 0.d25d25d25...$$

4. If $a < b < c < d < e$ are consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube, what is the smallest possible value of $c$?

5. When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $i/j$, in lowest terms, be the probability that the coin comes up heads exactly 3 times out of 5. Find $i + j$. 

6. Two skaters, Allie and Billie, are at points $A$ and $B$, respectively, on a flat, frozen lake. The distance between $A$ and $B$ is 100 meters.

Allie leaves $A$ and skates at a speed of 8 meters per second along a straight line that makes an angle of $60^\circ$ with $AB$, as shown. At the same time that Allie leaves $A$, Billie leaves $B$ at a speed of 7 meters per second and follows the straight line path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?

7. If the integer $k$ is added to each of the numbers 36, 300 and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find $k$.

8. Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

- $x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 = 1$
- $4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 = 12$
- $9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 = 123$.

Find the value of $16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$.

9. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed that there is a positive integer $n$ such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$ 

Find the value of $n$.

10. Let $a$, $b$, $c$ be the three sides of a triangle, and let $\alpha$, $\beta$, $\gamma$, respectively, be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find $\cot \frac{\gamma}{\cot \alpha + \cot \beta}$.

11. A sample of 121 integers is given, each between 1 and 1000 inclusive, with repetitions allowed. The sample has a unique mode (most frequent value). Let $D$ be the difference between the mode and the arithmetic mean of the sample. If $D$ is as large as possible, what is $[D]$? (For real $x$, $[x]$ is the greatest integer less than or equal to $x$.)
12. Let $ABCD$ be a tetrahedron with $AB = 41$, $AC = 7$, $AD = 18$, $BC = 36$, $BD = 27$, and $CD = 13$, as shown in the figure. Let $d$ be the distance between the midpoints of edges $AB$ and $CD$. Find $d^2$.

13. Let $S$ be a subset of $\{1,2,3,...,1989\}$ such that no two members of $S$ differ by 4 or 7. What is the largest number of elements $S$ can have?

14. Given a positive integer $n$, it can be shown that every complex number of the form $r + si$, where $r$ and $s$ are integers, can be uniquely expressed in the base $-n + i$ using the integers $0,1,2,...,n^2$ as "digits". That is, the equation

$$r + si = a_m(-n + i)^m + a_{m-1}(-n + i)^{m-1} + \cdots + a_0(-n + i) + a_0$$

is true for a unique choice of non-negative integer $m$ and digits $a_0, a_1, ..., a_m$ chosen from the set $\{0,1,2,...,n^2\}$, with $a_m \neq 0$. We then write

$$r + si = (a_m a_{m-1} \cdots a_0)_{-n+i}$$

to denote the base $-n + i$ expansion of $r + si$. There are only finitely many integers $k + 0i$ that have four-digit expansions

$$k = (a_3 a_2 a_1 a_0)_{-3+i} \quad a_3 \neq 0.$$ 

Find the sum of all such $k$.

15. Point $P$ is inside $\triangle ABC$. Line segments $APD$, $BPE$ and $CPF$ are drawn with $D$ on $BC$, $E$ on $CA$, and $F$ on $AB$ (see the figure at the right). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$ and $CF = 20$, find the area of $\triangle ABC$.

This month’s Olympiad is the 1987 Hungarian National Olympiad. The questions were collected and translated by Gy. Karolyi and J. Pataki of the János Bolyai Mathematical Society. They were forwarded to me by Bruce Shawyer, of the Mathematics Department of Memorial University of Newfoundland.
1. The surface area and the volume of a cylinder are equal to each other. Determine the radius and the altitude of the cylinder if both values are even integers.

2. Cut the regular (equilateral) triangle $AXY$ from rectangle $ABCD$ in such a way that the vertex $X$ is on side $BC$ and the vertex $Y$ is on side $CD$. Prove that among the three remaining right triangles there are two, the sum of whose areas equals the area of the third.

3. Determine the minimum of the function
\[ f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2} \]
where $a$, $b$, $c$ are positive numbers.

4. Consider points $A$ and $B$ on given rays (semilines) starting from $C$, such that the sum $CA + CB$ is a given constant. Show that there is a point $D \neq C$ such that for each position of $A$ and $B$ the circumcircle of triangle $ABC$ passes through $D$.

5. Show that
   (a) there exist 1324 different positive integers less than 1987 such that there are no three pairwise relatively prime ones among them;
   (b) any set of 1325 different positive integers less than 1987 has three elements which are pairwise relatively prime.

2nd Round, Basic Level

6. $N$ is a 4-digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find $N$.

7. Let $a$, $b$, $c$ be the sides and $\alpha$, $\beta$, $\gamma$ be the opposite angles of a triangle. Show that if
\[ ab^2 \cos \alpha = bc^2 \cos \beta = ca^2 \cos \gamma \]
then the triangle is equilateral.

8. Let $u$ and $v$ be two real numbers such that $u$, $v$ and $uv$ are roots of a cubic polynomial with rational coefficients. Prove or disprove that $uv$ is rational.
2nd Round, Advanced Level

9. The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

10. Let us build congruent regular pyramids on the faces of a $2 \times 2 \times 2$ cube, and call the solid obtained a "thorny cube". Determine the maximum of $V/S$ where $V$ and $S$ denote the volume and the surface area of the thorny cube, respectively. Is it possible to tesselate 3-space with copies of the thorny cube realizing this maximum ratio?

11. The domain of function $f$ is $[0,1]$, and for any $x_1 \neq x_2$

$$|f(x_1) - f(x_2)| < |x_1 - x_2|. \quad (1)$$

Moreover, $f(0) = f(1) = 0$. Prove that for any $x_1, x_2$ in $[0,1]$,

$$|f(x_1) - f(x_2)| < 1/2.$$

2nd Round, Superior Level

12. We call a natural number unary if each digit of its decimal representation is 1. For what natural numbers $m$ do there exist $m$ unary numbers such that any two of them give different remainders on division by $m$?

13. Let $f$ be a continuous function on the interval $[0,1]$ such that for each $x$ with $0 < x < 1$ there is $h$ such that

$$0 \leq x - h < x + h \leq 1$$

and

$$f(x) = \frac{f(x-h) + f(x+h)}{2}.$$

Show that $f(x) = x$ for all $x$ in $[0,1]$.

14. The centroid of triangle $A_1A_2A_3$ is $S$, and its circumcircle is $c$. The second intersection of $A_iS$ and $c$ is $B_i$ for $i = 1,2,3$. Show that

$$SB_1 + SB_2 + SB_3 \geq SA_1 + SA_2 + SA_3.$$

As promised we now begin the solutions to problems proposed for the 1987 I.M.O. in Havana, Cuba, and given in the November 1987 column. We have not received as complete a set of solutions as for the October 1987 problems, and readers
are encouraged to send in nice solutions to fill the gaps.

**Belgium 1.** [1987: 277]

Determine the least possible value of the natural number \( n \) such that \( n! \) ends in exactly 1987 zeros.

*Independent solutions by Curtis Cooper, Central Missouri State University; George Evagelopoulos, Law student, Athens, Greece; Douglass L. Grant, University College of Cape Breton, Sydney, N.S.; John Morvay, Dallas, Texas; and M.A. Selby, Department of Mathematics, University of Windsor, Ontario.*

The value of \( n \) is such that in the prime factorization of \( n! \), the factor 5 appears 1987 times, since the even numbers between multiples of 5 give ample powers of 2 to give a factor of \( 10^{1987} \). Let \([x]\) denote the greatest integer in \( x \). The largest power of 5 which divides \( n! \) is well-known to be given by

\[
h(n) = \left[ \frac{n}{5} \right] + \left[ \frac{n}{5^2} \right] + \cdots + \left[ \frac{n}{5^k} \right],
\]

where \( k \) is largest such that \( 5^k \leq n \).

The least \( n \) for which \( h(n) = 1987 \) is clearly a multiple of 5, and so it may be written in the form

\[
n = \sum_{i=1}^{k} a_i 5^i
\]

where each \( a_i \in \{0,1,2,3,4\} \). Now

\[
h(n) = \sum_{i=1}^{k} a_i h(5^i)
\]

where

\[
h(5^m) = 1 + 5 + 5^2 + \cdots + 5^{m-1} = \frac{1}{4}(5^m - 1).
\]

Thus \( h(5) = 1, h(5^2) = 6, h(5^3) = 31, h(5^4) = 156, h(5^5) = 781 \). Since

\[
1987 = 2 \cdot 781 + 2 \cdot 156 + 3 \cdot 31 + 3 \cdot 6 + 2 \cdot 1
\]

we conclude that

\[
n = 2 \cdot 5^5 + 2 \cdot 5^4 + 3 \cdot 5^3 + 3 \cdot 5^2 + 2 \cdot 5 = 7960
\]

satisfies \( h(n) = 1987 \). Clearly 7960 is the least such \( n \).

**France 2.** [1987: 277]

Let \( ABCDA'C'D' \) be an arbitrary parallelepiped as shown.
Establish the inequality

\[ AB' + AD' + AC \leq AB + AD + AA' + AC'. \]

When is there equality?

Solution by George Evagelopoulos, Law student, Athens, Greece and Murray S. Klamkin, Mathematics Department, The University of Alberta.

We set \( X_1 = \overrightarrow{AB}' \), \( X_2 = \overrightarrow{AD}' \), and \( X_3 = \overrightarrow{AA'} \), giving three vectors in 3-space. We must prove that

\[ \sum \left| X_i \right|^2 + \sum \left| X_i \right| \leq \sum \left| X_i \right|^2 + \sum \left| X_i \right| \leq \sum \left( \sum \left| X_i \right| \right)^2. \]

Let \( \langle u \mid v \rangle \) denote the usual inner (dot) product. Then, with summations cyclic over \( X_1, X_2, X_3, \)

\[
\left( \sum |X_i| \right)^2 - \left| \sum X_i \right|^2 = 2 \sum (|X_i| |X_j| - \langle X_i | X_j \rangle) = \sum \{(|X_i| + |X_j|)^2 - |X_i + X_j|^2\} = \sum \left( |X_i| + |X_j| + |X_i + X_j| \right) \left( |X_i| + |X_j| - |X_i + X_j| \right). \]

Now

\[ |X_i| + |X_j| - |X_i + X_j| \geq 0 \quad (3) \]

and

\[ |X_i| + |X_j| + |X_i + X_j| \leq \sum |X_i| + \left| \sum X_i \right|. \]

Thus, substituting into (2) we obtain

\[
\left( \sum |X_i| \right)^2 - \left| \sum X_i \right|^2 \leq \left( \sum |X_i| + \left| \sum X_i \right| \right) \cdot \sum \left( |X_i| + |X_j| - |X_i + X_j| \right).
\]

Cancelling

\[ \sum |X_i| + \left| \sum X_i \right| \]

from both sides gives
\[
\sum |x_i| - \sum x_i \leq \sum (|x_i| + |x_j| - |x_i + x_j|)
\]
\[
= 2 \sum |x_i| - \sum |x_i + x_j|
\]
which immediately gives the desired inequality (1).

In the case that some \( x_i = 0 \) or the sum \( x_1 + x_2 + x_3 = 0 \), we have obvious equality in (1). We thus assume below that \( x_1 + x_2 + x_3 \neq 0 \) and that no \( x_i = 0 \).

From the above proof, inequality (1) is strict unless equality holds in (3) for any distinct pair \( x_i, x_j \) such that strict inequality holds in (4). Now equality in (3) is equivalent to \( x_i \) and \( x_j \) being collinear and in the same direction. Equality in (4) is equivalent to the third vector \( x_k \) being collinear with, and in the opposite direction as, \( x_1 + x_2 + x_3 \).

Notice that equality in (3) for two pairs of vectors from \( \{x_i, x_2, x_3\} \) implies, since no \( x_i = 0 \), that all three vectors are collinear and in the same direction. In this case equality in (1) is again clear. Thus we now assume equality fails in (3) for at least two pairs of vectors, say for \( (x_i, x_2) \) and \( (x_i, x_3) \). Then for equality in (1) to hold, equality in (4) must hold for these same pairs of vectors. This gives that \( x_2 \) and \( x_3 \) are both collinear with, and in the opposite direction as, \( x_1 + x_2 + x_3 \), which entails (since \( x_1 + x_2 + x_3 \neq 0 \)) that \( x_i \) is collinear with \( x_2 \) and \( x_3 \), in the opposite direction as them, and \( |x_i| \geq |x_2 + x_3| \). Moreover, since equality holds in (3) for the pair \( (x_2, x_3) \), equality in (1) holds in this case.

In summary, equality holds in (1) precisely in the following cases:

(i) some \( x_i = 0 \);
(ii) \( x_1 + x_2 + x_3 = 0 \);
(iii) \( x_1, x_2, x_3 \) all lie in the same direction;
(iv) \( x_i \) and \( x_j \) lie in the same direction, \( x_k \) in the opposite direction, and \( |x_k| \geq |x_i + x_j| \) \((i, j, k\) being some permutation of 1, 2, 3).

Comments by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Alberta.

We use the notation from the above solution. The equivalent inequality (1) is known as Hlawka's inequality [1]. In [2] it is shown that an equivalent form for (1) is

\[ |u+v+w| + |v+w-u| + |w+u-v| + |u+v-w| \geq 2(|u| + |v| + |w|). \]

Also, the generalization (due to Levi)
\[ \sum |\pm A_1 \pm A_2 \pm \cdots \pm A_r| \geq 2 \left( \frac{r-1}{t} \right) \sum_{i=1}^{r} |A_i| , \]

where the \( A_i \) are vectors in \( E^m \), \( t = \lfloor (r-1)/2 \rfloor \), and the sum on the left hand side is taken over all combinations of + and − signs, is proven by an application of Levi's theorem [1]. Levi's theorem applied here states that to establish this inequality, it is sufficient to prove the result for all vectors in \( E^1 \) (i.e. for all real numbers).

A much wider generalization follows by applying Levi's theorem to the following result of Stankovic [3]:

\[ \sum_{1 \leq i_1 < \cdots < i_k \leq n} (P_{i_1} + \cdots + P_{i_k}) f \left( \frac{P_{i_1} x_{i_1} + \cdots + P_{i_k} x_{i_k}}{P_{i_1} + \cdots + P_{i_k}} \right) \leq \left( \frac{n-2}{k-2} \right) \left[ \frac{n-k}{k-1} \sum_{i=1}^{n} P_i f(x_i) + \sum_{i=1}^{n} P_i f \left( \frac{P_1 x_1 + \cdots + P_n x_n}{P_1 + \cdots + P_n} \right) \right] . \]

Here \( 2 \leq k \leq n-1 \), \( n \geq 3 \); \( P_1, P_2, \ldots, P_n \) are positive numbers; \( f \) is a convex function from \([a,b]\) to \( \mathbb{R} \); and the \( x_i \)'s are positive numbers in \([a,b]\).

For Levi's theorem to apply we also assume that \( f \) is convex over all real numbers, as for example if \( f(x) = |x^s| \), \( s \geq 1 \). Then the Stankovic result extends to the \( x_i \) being vectors in \( E^m \).

References:


**Great Britain 1.** [1987: 277]

Find, with proof, the point \( P \) in the interior of an acute-angled triangle \( ABC \) for which \( (BL)^2 + (CM)^2 + (AN)^2 \) is a minimum, where \( L, M, N \) are the feet of the perpendiculars from \( P \) onto \( BC, CA, AB \), respectively.

*Solution by George Evagelopoulos, Law student, Athens, Greece and Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Alberta.*

Since \( (BL)^2 - (LC)^2 = (PB)^2 - (PC)^2 \), etc.,
\[(BL)^2 + (CM)^2 + (AN)^2 = (LC)^2 + (MA)^2 + (NB)^2 = s,\]
say. Now
\[
2s = ((BL)^2 + (LC)^2) + ((CM)^2 + (MA)^2) + ((AN)^2 + (NB)^2)
\geq 2 \left[ \left( \frac{BL + LC}{2} \right)^2 + \left( \frac{CM + MA}{2} \right)^2 + \left( \frac{AN + NB}{2} \right)^2 \right].
\]
Thus
\[
s \geq \left( \frac{BC}{2} \right)^2 + \left( \frac{CA}{2} \right)^2 + \left( \frac{AB}{2} \right)^2
\]
and equality obtains just if \(BL = LC, CM = MA,\) and \(AN = NB.\) As the perpendicular bisectors meet at the circumcentre, the desired point \(P\) is the circumcentre of the triangle.

**Great Britain 2.** [1987: 278]

Prove that if \(x, y, z\) are real numbers such that \(x^2 + y^2 + z^2 = 2\) then \(x + y + z \leq xyz + 2.\)

**Solutions by George Evagelopoulos, Law student, Athens, Greece and by Dr. David Monk (forwarded by Robert Lyness of Suffolk, England).**

Write \(p\) for \(x + y + z\) and \(r\) for \(xyz.\) It is enough to show that 
\[
E = 4 - (p - r)^2 \geq 0.
\]

Now using \(x^2 + y^2 + z^2 = 2,\)
\[
4E = 2^3 - 2^2(p^2 - 2) + 2(4pr) - 4r^2
\]
\[
= 2^3 - 2^2(2yz + 2zx + 2xy) + 2(4x^2yz + 4xy^2z + 4xyz^2) - 8x^2y^2z^2 + 4r^2
\]
\[
= (2 - 2yz)(2 - 2zx)(2 - 2xy) + 4r^2.
\]

Since
\[
2 - 2yz = x^2 + (y - z)^2,
2 - 2zx = y^2 + (z - x)^2,
2 - 2xy = z^2 + (x - y)^2,
\]
the above quantities are non-negative. Thus, so also is \(E,\) completing the proof.

**Alternate solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Canada.**

Although not elementary, Lagrange multipliers provide a straightforward solution. Here the Lagrangian is
\[
\mathcal{L} = x + y + z - xyz - \lambda \left( \frac{x^2 + y^2 + z^2}{2} \right).
\]

Then we set
\[
\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial z} = 0
\]
to obtain
\[
1 - yz = \lambda x \\
1 - zx = \lambda y \\
1 - xy = \lambda z.
\]
On subtraction, we get
\[(x - y)(z - \lambda) = 0 = (y - z)(x - \lambda).
\]
Thus the critical points are \(x = y = z\) and \(x = y, \ z = (1 - x^2)/x\) and permutations thereof. The maximum value corresponds to the critical point \(x = y, \ z = (1 - x^2)/x\). Since \(x^2 + y^2 + z^2 = 2\) this leads to \((3x^2 - 1)(x^2 - 1) = 0\). Finally, the critical point \((1,1,0)\) and permutations of it give the maximum value of \(x + y + z - xyz\) to be 2.

[Editor's note. A third (and elementary) solution which involved consideration of cases was submitted by Zun Shan and Edward T.H. Wang of the Department of Mathematics, Wilfrid Laurier University.]

**Greece 2. [1987: 278]**

Show that if \(a, b, c\) are the lengths of the sides of a triangle and if \(2s = a + b + c\), then
\[
\frac{a^n}{b + c} + \frac{b^n}{a + c} + \frac{c^n}{a + b} \geq \left(\frac{2}{3}\right)^{n-2} s^{n-1}, \quad n \geq 1.
\]

*First solution by George Evagelopoulos, Law student, Athens, Greece, and by Bob Prielipp, University of Wisconsin–Oshkosh.*

This solution only uses that \(a, b, c\) are positive. Without loss of generality we may assume that \(a \leq b \leq c\), then
\[
\frac{1}{b + c} \leq \frac{1}{a + c} \leq \frac{1}{a + b}
\]
and we may apply the following known inequalities.

(i) **Chebyshev's inequality:** If \(a_1 \leq a_2 \leq \cdots \leq a_m\) and \(b_1 \leq b_2 \leq \cdots \leq b_m\), then
\[
\sum_{i=1}^{n} a_i b_i \geq \frac{1}{m} \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right).
\]

(ii) If \(a, b, c\) are positive numbers and \(n\) is a positive integer, then
\[(a + b + c)^n \leq 3^{n-1}(a^n + b^n + c^n).
\]

(iii) If \(a, b, c\) are positive numbers, then
\[
\frac{1}{b + c} + \frac{1}{a + c} + \frac{1}{a + b} \geq \frac{9}{2(a + b + c)}.
\]

[For a proof of (i), see pp. 16–17 of *Elementary Inequalities* by Mitrinovic et al,
Let \( a_1 = a^n, a_2 = b^n, a_3 = c^n \) and
\[
\frac{1}{b + c}, \frac{1}{a + c}, \frac{1}{a + b}.
\]
By (i)-(iii),
\[
\frac{a^n}{b + c} + \frac{b^n}{a + c} + \frac{c^n}{a + b} \geq \frac{1}{3} \left( \frac{a^n}{b + c} + \frac{b^n}{a + c} + \frac{c^n}{a + b} \right) = \frac{1}{3} \left( a + b + c \right)^n \cdot 2 \frac{9}{2(a + b + c)} = \left( \frac{2}{3} \right)^{n-2} s^{n-1}.
\]

Second solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

We prove more generally that if \( x_1, \ldots, x_m \) are positive then
\[
\frac{x_1^2}{S - x_1} + \frac{x_2^2}{S - x_2} + \cdots + \frac{x_m^2}{S - x_m} \geq \frac{m - 1}{m} \left( \frac{S}{m} \right)^{n-1}
\]
where \( S = x_1 + x_2 + \cdots + x_m \).

This rests on the power mean inequality,
\[
\sum \frac{x_i^{n-1} \cdot x_i / (S - x_i)}{x_i / (S - x_i)} \geq \left( \frac{\sum x_i \cdot x_i / (S - x_i)}{\sum x_i / (S - x_i)} \right)^{n-1}.
\]
(The sums here and subsequently are all from \( i = 1 \) to \( i = m \).) We also show that
\[
\sum \frac{x_i}{S - x_i} \geq \frac{m}{m - 1}
\]
and
\[
\sum \frac{x_i^2 / (S - x_i)}{x_i / (S - x_i)} \geq \frac{S}{m}.
\]
Note that (1) is immediate from (2), (3) and (4).

Let \( S - x_i = a_i \) so that \( x_i = S - a_i \) and \( a_1 + a_2 + \cdots + a_m = S(m - 1) \).
Rewrite (3) and (4) as
\[
\sum \frac{S - a_i}{a_i} = S \sum \frac{1}{a_i} - m \geq \frac{m}{m - 1}
\]
and
\[
\sum \frac{(S - a_i)^2 / a_i}{(S - a_i) / a_i} \geq \frac{S}{m}
\]
respectively. On expanding (6), we get
\[ S^2 \sum \frac{1}{a_i} - 2mS + (m - 1)S \geq \frac{S}{m} \left( S \sum \frac{1}{a_i} - m \right) \]
or
\[ (m - 1)S \sum \frac{1}{a_i} \geq m^2, \]
which is also equivalent to (5). By Cauchy's inequality,
\[ S \sum \frac{1}{a_i} = \frac{1}{m - 1} \sum a_i \cdot \sum \frac{1}{a_i} \geq \frac{m^2}{m - 1}. \]
This gives (5) and (6) and the result follows.

Next issue we continue the November 1987 I.M.O. solutions. Keep sending in your national contests and your nice solutions.

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1989, although solutions received after that date will also be considered until the time when a solution is published.

1431. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
   (a) Show that, for prime \( p \equiv 1, 3 \) or \( 5 \) mod \( 8 \),
   \[
   \sum_{k=1}^{(p-1)/2} \tan \frac{k^2 \pi}{p} = 3 \cdot \sum_{k=1}^{(p-1)/2} \cot \frac{k^2 \pi}{p}. \]  
   \hspace{1cm} (1)
   (b) * Prove that (1) fails for all primes \( p \equiv 7 \) mod \( 8 \).

If the Nagel point of a triangle lies on the incircle, prove that the sum of two of the sides of the triangle equals three times the third side.


Let $ABC$ be a triangle with sides $a > b > c$ and circumcircle $\Omega$. Its internal angle–biseectors meet $BC$, $CA$ and $AB$ at $D$, $E$ and $F$ respectively. The line through $B$ parallel to $EF$ meets $\Omega$ again at $Q$, and $P$ is on $\Omega$ such that $QP \parallel AC$. Prove that $PC = PA + PB$.

1434. Proposed by Harvey Abbott and Murray S. Klamkin, University of Alberta.

It is known that

$$\frac{(3m)! (3n)!}{m! n! (m+n)! (n+m)!}, \quad \frac{(4m)! (4n)!}{m! n! (2m+n)! (2n+m)!}, \quad \frac{(5m)! (5n)!}{m! n! (3m+n)! (3n+m)!}$$

are all integers for positive integers $m$, $n$.

(i) Find positive integers $m$, $n$ such that

$$I(m,n) = \frac{(6m)! (6n)!}{m! n! (4m+n)! (4n+m)!}$$

is not an integer.

(ii) Let $A$ be the set of pairs $(m,n)$, with $n \leq m$, for which $I(m,n)$ is not an integer, and let $A(x)$ be the number of pairs in $A$ satisfying $1 \leq n \leq m \leq x$. Show that $A$ has positive lower density in the sense that

$$\lim_{x \to \infty} \inf_{x^2} \frac{A(x)}{x^2} > 0.$$

1435. Proposed by J.B. Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Find all pairs of integers $x$, $y$ such that

$$(xy - 1)^2 = (x + 1)^2 + (y + 1)^2.$$ 


A point $P$ lies on the circumcircle $\Gamma$ of a triangle $ABC$, $P$ not coinciding with one of the vertices. Circles $\Gamma_1$ and $\Gamma_2$ pass through $P$ and are tangent to $AB$ at $B$, and to $AC$ at $C$, respectively. $\Gamma_1$ and $\Gamma_2$ intersect at $P$ and at $Q$.

(a) Show that $Q$ lies on the line $BC$.

(b) Show that as $P$ varies over $\Gamma$ the line $PQ$ passes through a fixed point on $\Gamma$. 

1437. Proposed by George Tsintsifas, Thessaloniki, Greece.
Let $A'B'C'$ be an equilateral triangle inscribed in a triangle $ABC$, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. If
\[
\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B},
\]
prove that $ABC$ is equilateral.

1438. Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.
$AB$ is a fixed chord in a given circle, and $C$ is a variable point on the circle, other than $A$ and $B$. Prove that
\[
\frac{\sin A + \cos B}{\sin\left(\frac{A-B}{2}\right) + \cos\left(\frac{A-B}{2}\right)}
\]
is independent of $C$, where the angles $A$ and $B$ are the angles of $\triangle ABC$.

Prove that
\[
\frac{1}{6}[(2 + \sqrt{3})^{2n-3} + (2 - \sqrt{3})^{2n-3} + 2]
\]
is a perfect square for all positive integers $n$.

Prove or disprove that if $A$, $B$, $C$ are the angles of a triangle,
\[
\frac{\sin A}{\sqrt{\sin A + \sin B}} + \frac{\sin B}{\sqrt{\sin B + \sin C}} + \frac{\sin C}{\sqrt{\sin C + \sin A}} \leq \frac{3}{2} \sqrt{3}.
\]

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

Find, without calculus, the largest possible value of
\[
\frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x}.
\]

Let

\[ T = \sin 5x + \cos 3x = \frac{\sin 5x + \sin(90^\circ + 3z)}{\sin 4x + \sin(90^\circ + 4x)} \]

\[ = \frac{2 \sin \left(\frac{8x + 90^\circ}{2}\right) \cos \left(\frac{90^\circ - 2z}{2}\right)}{2 \sin \left(\frac{8x + 90^\circ}{2}\right) \cos \left(\frac{90^\circ}{2}\right)} \]

\[ = \cos(45^\circ - z) = \sqrt{2} \cos(45^\circ - z). \]

Then \( \cos(45^\circ - z) \) is maximal (and equals 1) when \( z = 45^\circ \). Thus \( T_{\text{max}} = \sqrt{2} \).

With calculus it takes much longer!

II. Solution by J. Suck, Essen, Federal Republic of Germany.

Par les théorèmes d'addition et le théorème de Pythagore,

\[ \left(\sin 5x + \cos 3x\right)^2 = \left(\sin 4x \cos x + \cos 4x \sin x + \cos 4x \cos x + \sin 4x \sin x\right)^2 \]

\[ = (\sin x + \cos x)^2 \]

\[ = 1 + 2 \sin x \cos x = 1 + \sin 2x \leq 2. \]

On a l'égalité pour \( x = \pi/4 \), par exemple. Ainsi, la valeur cherchée est \( \sqrt{2} \).

Also solved by SEUNG–JIN BANG, Seoul, Korea; DUANE M. BROLINE, Eastern Illinois University; CURTIS COOPER, Central Missouri State University; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; H. FUKAGAWA, Aichi, Japan; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALThER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; SIDNEY KRAVITZ, Dover, New Jersey; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Penn State University at Harrisburg; M.A. SELBY, University of Windsor; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Janous and Klamkin generalized to the maximum value of

\[ \frac{\sin(a + b)x + \cos(a - b)x}{\sin ax + \cos ax} \]
for $a, b$ real.


Let $A, B, C$ be fixed points in the plane. An ant starts from a certain point $P_0$ of the plane and crawls in a straight line through $A$ to a point $P_1$ such that $P_0A = AP_1$. It then crawls from $P_1$ through $B$ to $P_2$ such that $P_1B = BP_2$, then through $C$ to $P_3$, and so on. Altogether it repeats the same action 1986 times successively through the points $A, B, C, A, B, C, A, ..., $ finally stopping in exhaustion at $P_{1986}$. Where is the ant now? (Compare with problem 2 of the 1986 I.M.O. [1986: 173].)

I. Solution by Jordi Dou, Barcelona, Spain.

The result of two successive symmetries with respect to the points $A$ and $B$ is a translation equal to $2 \cdot \overrightarrow{AB}$. The result of the six symmetries with respect to $A, B, C, A, B, C$ will be the translation

$$2 \cdot \overrightarrow{AB} + 2 \cdot \overrightarrow{CA} + 2 \cdot \overrightarrow{BC},$$

which is equal to the identity. Thus the result of the $1986 = 6 \cdot 331$ symmetries will be the identity. The ant finishes at $P_0$.

Note: The same result holds for $2k(2n + 1)$ successive symmetries through $2n + 1$ ordered points. If there are $2n$ points the result of $k \cdot 2n$ symmetries is $kT$ where

$$T = 2(\overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{2n-1}A_{2n}}).$$

II. Solution by Eric Holleman, student, Memorial University of Newfoundland.

Set up a Cartesian coordinate system with $P_0$ at the origin. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$.

If the ant is at a point $(x_0, y_0)$ and crawls in a straight line through another point $(x_1, y_1)$ to a final point $(x_2, y_2)$, such that the point $(x_1, y_1)$ is equidistant from the other two, then

$$x_2 = x_1 - (x_0 - x_1) = 2x_1 - x_0,$$
$$y_2 = y_1 - (y_0 - y_1) = 2y_1 - y_0.$$ 

Thus the ant’s journey starts as follows:

$(0,0)$ through $A$ to $(2a_1, 2a_2)$;
through $B$ to $(2b_1 - 2a_1, 2b_2 - 2a_2)$;
through $C$ to $(2a_1 - 2b_1 + 2c_1, 2a_2 - 2b_2 + 2c_2)$;
through $A$ again to $(2b_1 - 2c_1, 2b_2 - 2c_2)$;
through $B$ again to $(2c_1, 2c_2)$;
through $C$ again to $(0, 0)$.

After six legs of the journey the ant returns to his starting point. Since
$1986 \equiv 0 \mod 6$, after $1986$ legs the ant will be at $P_0$.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg,
Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T.
GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes,
California; WALThER Janous, Ursulinengymnasium, Innsbruck, Austria; M.S.
KLAMKIN, University of Alberta; Z.F. LI, University of Regina; P. PENNING, Delft,
The Netherlands; colIn SPRInGER, student, University of Waterloo; C.
WILDHAGEN, Breda, The Netherlands; and the proposer.

Festraets-Hamoir's proof was the same as Dou's. The other proofs were
similar to solution II. Janous and Penning (like Dou) generalized to any odd number
of points in $n$-dimensional space. Penning also considered the situation when a crawl
after passing through a point extends beyond only a fraction $\lambda(< 1)$ of the distance to
that point.

* * *

Assume that $a_1, a_2, a_3, \ldots$ are real numbers satisfying the inequality
\[ |a_{m+n} - a_m - a_n| \leq C \]
for all $m, n \geq 1$ and for some constant $C$. Prove that there exists a constant $k$ such that
\[ |a_n - nk| \leq C \]
for all $n \geq 1$.

Solution by Jim Braselton, student, The Ohio State University.
For $n, m \in \mathbb{N}$ we have that
\[
|a_{n+m} - n a_m| = \left| \sum_{i=0}^{n-2} (a_{(n-i)m} - a_{(n-i-1)m} - a_m) \right|
\leq \sum_{i=0}^{n-2} |a_{(n-i)m} - a_{(n-i-1)m} - a_m|
\]
Therefore
\[ \left| \frac{a_{nm}}{nm} - \frac{a_n}{m} \right| \leq \frac{(n - 1)C}{nm} \leq C \]
so that
\[ \left| \frac{a_n}{n} - \frac{a_m}{m} \right| = \left| \frac{a_n}{n} - \frac{a_{nm}}{nm} + \frac{a_{nm}}{nm} - \frac{a_m}{m} \right| \]
\[ \leq \left| \frac{a_n}{n} - \frac{a_{nm}}{nm} \right| + \left| \frac{a_{nm}}{nm} - \frac{a_m}{m} \right| \]
\[ \leq C \left( \frac{1}{n} + \frac{1}{m} \right). \tag{1} \]
Consequently \( \{a_n/n\} \) is a Cauchy sequence in \( \mathbb{R} \) and hence converges. Let
\[ k = \lim_{n \to \infty} \frac{a_n}{n}. \]
Fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \), and choose \( M \) such that
\[ \left| \frac{a_m}{m} - k \right| < \frac{\epsilon}{2n} \text{ for } m \geq M \]
and also
\[ \frac{nC}{M} < \frac{\epsilon}{2}. \]
Then for \( m \geq M \),
\[ |a_n - nk| \leq n \left| \frac{a_n}{n} - \frac{a_m}{m} \right| + n \left| \frac{a_m}{m} - k \right| \]
\[ < nC \left( \frac{1}{n} + \frac{1}{m} \right) + n \cdot \frac{\epsilon}{2n} \]
\[ < C + \frac{\epsilon}{2} + \frac{\epsilon}{2} = C + \epsilon. \]
Since \( \epsilon \) was arbitrary, this establishes that \( |a_n - nk| \leq C. \)

Also solved by WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; C. WILDHAGEN, Breda, The Netherlands; and the proposer. Two other readers sent in incorrect solutions.

Janous and Wildhagen noted that inequality (1) (with \( C = 1 \)) occurs as problem 6 of the 1980 Austrian–Polish Mathematics Competition [1986: 5], with a proof different from the above. Janous and the proposer also gave the above proof.

1321. [1988: 76] Proposed by Jordi Dou, Barcelona, Spain. (Dedicated in memoriam to Léo Sauvé.)

The circumcircle of a triangle is orthogonal to an excircle. Find the ratio of their radii.


Let $ABC$ be the triangle, with circumcircle $(O,R)$ and excircle $(I_c,r_c)$ meeting at a point $S$, where $\angle OSI_c = 90^\circ$. Then

$$R^2 + r_c^2 = OI_c^2.$$ (1)

It is known (p.187 of R.A. Johnson, Advanced Euclidean Geometry) that

$$OI_c^2 = R^2 + 2Rr_c.$$ (2)

Hence from (1) and (2)

$$r_c^2 = 2Rr_c,$$

so

$$R/r_c = 1/2.$$

Also solved by SVETOSLAV JOR. BILCHEV, Technical University, Russe, Bulgaria; CLAYTON W. DODGE, University of Maine, Orono; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Most solvers used equation (2), Janous referring to p. 43 of E. Donath, Die Merkwürdigen Punkte und Linien des Ebenen Dreiecks, Berlin, 1969.

* * *


Show that closed form expressions exist for the two integrals

$$\int t^{p-1}(1 - t)^{q-1}dt$$ and $$\int t^{p-1}(1 + t)^{q-1}dt$$

for the special case $p + q = 1$, $p$ rational.

I. Solution by Colin Springer, student, University of Waterloo.

Let $p = a/b$, $a$, $b$ integers, so that $q - 1 = -a/b$. Then
\[
\int t^{p-1}(1 - t)^{q-1} dt = \int \frac{1}{t} \left( \frac{t}{1 - t} \right)^{a \cdot b} dt.
\]

Substitute

\[ u^b = \frac{t}{1 - t} ; \]

then

\[ t = \frac{u^b}{u^b + 1}, \quad dt = \frac{b u^{b-1}}{(u^b + 1)^2} du, \]

so we have

\[ \int \frac{1}{t} \left( \frac{t}{1 - t} \right)^{a \cdot b} dt = \int \frac{b u^{a-1}}{u^b + 1} du. \]

It is well known that a closed-form expression exists for the integral of any rational function (see any calculus text). Thus a closed-form expression exists for the first given integral.

We proceed similarly for the second integral:

\[
\int t^{p-1}(1 + t)^{q-1} dt = \int \frac{1}{t} \left( \frac{1}{1 + t} \right)^{a \cdot b} dt, \quad \text{where} \quad p = a/b;
\]

\[ u^b = \frac{t}{1 + t} \Rightarrow t = \frac{u^b}{1 - u^b} \quad \text{and} \quad dt = \frac{b u^{b-1}}{(1 - u^b)^2} du; \]

then

\[ \int \frac{1}{t} \left( \frac{1}{1 + t} \right)^{a \cdot b} dt = \int \frac{b u^{a-1}}{1 - u^b} du, \]

and again a closed-form expression exists for this integral.

II. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

These two integrals are special cases of the following

**Theorem.** ([1], p.25; or [2], 2.202, p.71) Integrals of type

\[ \int x^r(ax^t + b)^s dx, \quad r, s, t, \text{ rational,} \]

can be represented as a combination of finitely many elementary functions if and only if at least one of

\[ s, \quad \frac{r + 1}{t}, \quad \frac{r + 1}{t} + s \]

is an integer.

**References:**


Also solved by KEE-WAI LAU, Hong Kong; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

Wildhagen remarked that the problem is a special case of a result of Tschebyscheff and referred to G.M. Fichtenholz, Differential- und Integralrechnung, II, §279.

* * *


Let $ABC$ be the right triangle with $\angle A = 1$ radian and $\angle C = \pi/2$. Let $I$ be the incenter, $O$ the midpoint of $AB$, and $N$ the midpoint of $OC$. Is $\triangle NIO$ an acute, obtuse, or right triangle?

Solution by Jordi Dou, Barcelona, Spain.

Suppose $\overline{AB} = 4$. Then $\overline{OA} = 2$, so

$$\overline{ON} = \overline{CN} = 1.$$  Also

$$\angle ICN = \frac{\pi}{4} - \angle B = 1 - \frac{\pi}{4},$$

and the inradius $r$ of $\triangle ABC$ is

$$r = \overline{IE} = \overline{CE} = s - \overline{AB}$$

$$= 2(\cos 1 + \sin 1 - 1)$$

where $s$ is the semiperimeter. Hence

$$\overline{IN}^2 = \overline{CN}^2 + \overline{CI}^2 - 2 \overline{CN} \cdot \overline{CI} \cdot \cos \angle ICN$$

$$= 1 + 2r^2 - 2r\sqrt{2}\cos(1 - \pi/4)$$

$$\approx 0.055910219$$

and

$$\overline{IO}^2 = \overline{AI}^2 + \overline{AO}^2 - 2 \overline{AI} \cdot \overline{AO} \cdot \cos \angle OAI$$

$$= \left(\frac{r}{\sin 1/2}\right)^2 + 4 - \frac{4r}{\tan 1/2}$$

$$\approx 0.945813674,$$

so

$$\overline{IN}^2 + \overline{IO}^2 \approx 1.001723894 > 1 = \overline{NO}^2.$$  Therefore $\triangle NIO$ is an acute triangle. In fact $\angle NIO \approx 89^\circ 47'7''$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; and the proposer. There was one incorrect solution submitted.

* * *

Proposed by Len Bos, University of Calgary.

Let $S_i(x_1, x_2, \ldots, x_n)$ be the $i$th elementary symmetric function, i.e. $S_0 = 1$, $S_1 = x_1 + x_2 + \cdots + x_n$, $S_2 = \sum_{i<j} x_ix_j$, etc. Let $J$ be the $n \times n$ Jacobian matrix of $S_1, \ldots, S_n$ with respect to $x_1, \ldots, x_n$; i.e.

$$J_{ij} = \frac{\partial S_i}{\partial x_j}.$$ 

Show that

$$\det J = \prod_{i<j} (x_j - x_i),$$

the Vandermonde determinant.

Solution by Colin Springer, student, University of Waterloo.

We see that if $x_i = x_j$ for $i \neq j$ then column $i$ and column $j$ of $J$ are identical, so $\det J = 0$. Therefore (since $\det J$ is a polynomial) $x_i - x_j$ is a factor of $\det J$ for all $i \neq j$. Thus (since the degree of $\det J$ is clearly $1 + 2 + \cdots + (n - 1)$)

$$\det J = k \prod_{i<j} (x_i - x_j)$$

for some constant $k$.

Consider the term

$$kx_1^{n-1}x_2^{n-2} \cdots x_{n-1}$$

arising from multiplying the first term of each $x_i - x_j$ in the above product. In $\det J$ it is necessarily the result of multiplying terms

$$1 \cdot x_1(x_1x_2)(x_1x_2x_3) \cdots (x_1x_2 \cdots x_{n-1}).$$

Now, since no term involving $x_k$ exists in column $k$ of $J$,

term $x_1x_2 \cdots x_{n-1}$ comes only from column $n$, row $n$,

thus

term $x_1x_2 \cdots x_{n-2}$ comes only from column $n - 1$, row $n - 1$,

$$\vdots$$

term $x_1$ comes only from column 2, row 2,

and finally

term 1 comes only from column 1, row 1;

i.e. $x_1^{n-1}x_2^{n-2} \cdots x_{n-1}$ arises only in multiplying together the main diagonal. Therefore it has coefficient 1, so $k = 1$ and

$$\det J = \prod_{i<j} (x_i - x_j).$$

Let \( P \) be any point inside a unit circle. Perpendicular chords \( AB \) and \( CD \) pass through \( P \). Two other chords passing through \( P \) form four angles of \( \theta \) radians each as shown in the figure. Prove that the area of the shaded region is \( 2\theta \).

I. Solution by Jörg Härterich, student, University of Stuttgart.

I'm first going to prove

Lemma. If \( AB \) and \( CD \) are two perpendicular chords of a circle of radius \( r \) and the intersection of \( AB \) and \( CD \) is called \( P \), then

\[
AP^2 + BP^2 + CP^2 + DP^2 = 4r^2.
\]

Proof. Referring to the diagram, we have

\[
AP^2 + BP^2 + CP^2 + DP^2 = AC^2 + BD^2
\]

\[
= 2r^2 - 2r^2 \cos \alpha + 2r^2 - 2r^2 \cos \beta
\]

\[
= 4r^2 - 2r^2 (\cos \alpha + \cos \beta).
\]

The lemma is proved if we can show that \( \alpha + \beta = 180^\circ \). But this follows from

\[
\frac{\alpha}{2} + \frac{\beta}{2} + 90^\circ = \angle ABC + \angle DBC + 90^\circ = 180^\circ.
\]

We now use this lemma to solve the problem. Choose polar coordinates with centre \( P \) and denote by \( s \) the distance from \( P \) to the circle. Then the area of the shaded region is

\[
A = \frac{1}{2} \left( \int_0^\theta s^2 d\varphi + \int_{\pi/2}^{\pi/2} s^2 d\varphi + \int_\pi^{3\pi/2} s^2 d\varphi \right).
\]

After substitution, and by using the lemma, we get
\[ A = \frac{1}{2} \int_0^\theta (s^2(\varphi) + s^2(\varphi + \frac{\pi}{2}) + s^2(\varphi + \pi) + s^2(\varphi + 3\pi/2))d\varphi \]
\[ = \frac{1}{2} \int_0^\theta 4d\varphi = 2\theta. \]

II. Solution by Shiko Iwata, Gifu, Japan. (Translated by Hidetosi Fukagawa.)

Let \( P \) be any point inside a unit circle with centre \( O \) and let \( AB \) and \( A'B' \) be two chords through \( P \). We first show that the area of the shaded region is
\[ S_{ab}(\theta) = \theta + d^2\sin \theta \cos(2\omega + \theta), \]
where \( OP = d, \omega = \angle APO, \) and \( \theta = \angle APA' \).

Let \([X]\) denote the area of the region \( X \). Then
\[ S_{ab}(\theta) = [OAA'P] + [OB'BP] - ([OAP] + [OPB']). \]

\[ = [OAA'] + [OBB'] + ([OPA'] - [OB'P]) + ([OBP] - [OAP]) \]
\[ = \frac{1}{2} \angle AOA' + \frac{1}{2} \angle BOB' + \frac{1}{2} d \cdot \sin(\theta + \omega) \cdot (A'M' - PB') + \frac{1}{2} d \cdot \sin \omega \cdot (PB - AP). \]

Now, letting \( M \) and \( M' \) be the midpoints of \( AB \) and \( A'B' \) respectively,
\[ \overline{AP} - \overline{PB}' = (\overline{AM'} + d \cdot \cos(\omega + \theta)) - (\overline{B'M} - d \cdot \cos(\omega + \theta)) \]
\[ = 2d \cdot \cos(\omega + \theta) \]

and
\[ \overline{PB} - \overline{AP} = (BM - d \cdot \cos \omega) - (AM + d \cdot \cos \omega) = -2d \cdot \cos \omega. \]

Thus
\[ S_{ab}(\theta) = \angle AAB'A' + \angle BABB' + d^2\sin(\omega + \theta)\cos(\omega + \theta) - d^2\sin \omega \cos \omega \]
\[ = \theta + \frac{d^2}{2} (\sin 2(\omega + \theta) - \sin 2\omega) \]
\[ = \theta + d^2\sin \omega \cos(2\omega + \theta). \]

The same argument works for the chord \( CD \) with \( \omega \) replaced by \( \omega + \pi/2, \) i.e.
\[ S_{cd}(\theta) = \theta + d^2\sin \theta \cos(2\omega + \pi + \theta) = \theta - d^2\sin \theta \cos(2\omega + \theta). \]

Therefore the required area is
\[ S_{ab}(\theta) + S_{cd}(\theta) = 2\theta. \]

Also solved by JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; TADASI KAMIYA, Yokosuka High School, Aichi, Japan; M.S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; BRUCE SHAWYER, Memorial University of Newfoundland; DAN SOKOLOWSKY, Williamsburg, Virginia; COLIN SPRINGER,
student, University of Waterloo; and the proposer. One incorrect solution was sent in. Iwata's solution was the only one received which didn't use calculus. Several solutions, including the proposer's, were similar to solution I.

Klamkin generalized the problem to n chords through P with equal angles of $\pi/n$ between successive chords. The area swept out when these chords are rotated through an angle $\theta$ about P then comes out to be $n \theta$. (This can also be proved using the result in solution II. Can it be proved as in solution I?)

* * *

1328. [1988: 77] Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Solve

$$\cos^4 2\theta + 2 \sin^2 2\theta = 17(\sin \theta + \cos \theta)^8$$

for $0 \leq \theta \leq 360^\circ$.


The given equation can be written

$$(1 - \sin^2 2\theta)^2 + 2 \sin^2 2\theta = 17(1 + \sin 2\theta)^4,$$

which with the substitution $x = \sin 2\theta$ becomes

$$1 + x^4 = 17(1 + x)^4$$

or, dividing by $x^2$,

$$\frac{1}{x^2} + x^2 = 17\left(\frac{1}{x} + 2 + x\right)^2.$$

The substitution $z = x + 1/x$ yields

$$z^2 - 2 = 17(z + 2)^2,$$

or

$$16z^2 + 68z + 70 = 2(4z + 7)(2z + 5) = 0,$$

so

$$z = -1.75 \text{ or } z = -2.5.$$

Case (i): $x + 1/x = z = -1.75$. There is no solution.

Case (ii): $x + 1/x = z = -2.5$. Then $x = -1/2$ or $-2$. When $x = -2$ there is no solution for $\theta$, so

$$\sin 2\theta = x = -1/2.$$

Thus

$$2\theta = 210^\circ + k \cdot 360^\circ \text{ or } 2\theta = 330^\circ + k \cdot 360^\circ$$

so the solutions for $\theta$ are

$$105^\circ, \ 285^\circ, \ 165^\circ, \ 345^\circ.$$
Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \( x_1, x_2, x_3 \) be the distances of the vertices of a triangle from a point \( P \) in the same plane. Let \( r \) be the inradius of the triangle, and \( p \) be the power of the point \( P \) with respect to the circumcircle of the triangle. Prove that

\[
x_1 x_2 x_3 > 2rp.
\]

Solution by Murray S. Klamkin, University of Alberta.

It is a known result \([1]\) that if one takes a point on each of the three sides (extended or not) of a given triangle, then the circumradius of the triangle determined by the three points is greater than or equal to the inradius of the given triangle. We now show that the proposed inequality is equivalent to this result.

With the more usual notation of \( x_i = R_i \), the given inequality is

\[
R_1 R_2 R_3 \geq 2r(R^2 - OP^2). \tag{1}
\]

Letting \( A_1A_2A_3 \) be the triangle, we now consider the pedal triangle of \( P \), whose vertices are the feet of the perpendiculars from \( P \) to the sides of \( A_1A_2A_3 \). It is known that the sides of this triangle are

\[
R_1 \sin A_1, \quad R_2 \sin A_2, \quad R_3 \sin A_3,
\]

and its area is

\[
\frac{1}{2} |R^2 - OP^2| \sin A_1 \sin A_2 \sin A_3
\]

[e.g. see Theorems 190 and 198, pp. 136 and 139 of R.A. Johnson, *Advanced Euclidean Geometry*]. If \( P \) is on the circumcircle, the area of the pedal triangle is zero (the three vertices lie on a line called the pedal or Simson line), while if \( P \) is
outside the circumcircle, the signed area is negative. We now use the result that the
product of the sides of a triangle is four times the product of its circumradius and
area, so that (1) reduces to \( R' \geq r \), where \( R' \) is the circumradius of the pedal
triangle, as was noted in the first paragraph.

References:
simplex, Mathematics Magazine 52 (1979) 20–22.

[Editor’s note: The proposer’s original question had \(|p|\) in the inequality rather than
\( p \) (the absolute value was inadvertently left off by the editor). The above solution
handles this stronger inequality.]

II. Solution by the proposer.

We consider the inversion \( I \) with pole \( P \) and power 1. Let \( ABC \) be
the given triangle, with sides \( a, b, c \) and circumcircle \((O,R)\), and let
\[ A'B'C' = I(ABC), \quad (O',R') = I((O,R)). \]

Let \( a', b', c' \) be the sides of \( A'B'C' \), \( x'_1, x'_2, x'_3 \) the distances of \( A', B', C' \) from
\( P \), and \( p' \) the power of \( P \) with respect to \((O',R')\). Then it is well–known from
inversion theory that
\[
\begin{align*}
x_1 = \frac{1}{x'_1}, & \quad x_2 = \frac{1}{x'_2}, & \quad x_3 = \frac{1}{x'_3}, \\
a = \frac{a'}{x'_2 x'_3}, & \quad b = \frac{b'}{x'_3 x'_1}, & \quad c = \frac{c'}{x'_1 x'_2},
\end{align*}
\]
and
\[ R' = |p'| \cdot R \]

[e.g. pp. 48–51 of R.A. Johnson, Advanced Euclidean Geometry]. Also, a well–known
inequality for the triangle \( ABC \) is
\[ ax_1 + bx_2 + cx_3 \geq 4F, \]
where \( F \) is the area of \( ABC \) (e.g. see Crux 866 [1984: 327]). From (1)–(4) follows
\[
a' + b' + c' = ax'_2 x'_3 + bx'_3 x'_1 + cx'_1 x'_2 = \frac{ax_1 + bx_2 + cx_3}{x_1 x_2 x_3}
\]
\[
\geq \frac{4F}{x_1 x_2 x_3} = \frac{abc}{R x_1 x_2 x_3} = \frac{a'b'c'}{R' x'_1 x'_2 x'_3} \cdot |p'|
\]
\[
= \frac{4F'}{x'_1 x'_2 x'_3} \cdot |p'|,
\]
\( F' \) being the area of \( A'B'C' \). Hence for the triangle \( ABC \) we have
\[
2s = a + b + c \geq \frac{4F}{x_1 x_2 x_3} \cdot |p| = \frac{4sr}{x_1 x_2 x_3} \cdot |p|,
\]
or

\[ x_1 x_2 x_3 \geq 2r \cdot |p| . \]

There was one incorrect solution submitted.

---


Use a combinatorial argument to establish the (known) identity

\[
\binom{n}{0} \binom{n}{m} + \binom{n}{1} \binom{n-1}{m-1} + \cdots + \binom{n}{m} \binom{n-m}{0} = 2^n \binom{n}{m} .
\]

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

From a jar containing \( n \) different pieces of candy, Jimmy will choose \( m \) pieces, \( j \) of them to be placed in his left hand and \( m-j \) of them to be placed in his right hand. How many different such selections are possible for all \( j = 0, 1, \ldots, m \)?

(i) For \( j = 0, 1, \ldots, m \), Jimmy selects \( j \) pieces from the \( n \) in the jar with his left hand, and then selects \( m-j \) pieces from the \( n-j \) remaining with his right hand. The number of such selections is given by the expression on the left.

(ii) Jimmy chooses \( m \) pieces from the \( n \) in the jar with his right hand. He then selects a subset of these \( m \) pieces with his left hand. The number of all such selections is given by the expression on the right.

Also solved by FRANCISCO BELLOT ROSADO, Emilio Ferrari High School, Valladolid, Spain; JORG HARTERICH, student, University of Stuttgart; G.P. HENDERSON, Campbellcroft, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; PETER ROSS, Santa Clara University, Santa Clara, California; COLIN SPRINGER, student, University of Waterloo; J. SUCK, Essen, Federal Republic of Germany; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

All solutions were of course equivalent, and the editor chose the one that appealed to him the most. Here are two others. Suck counted the number of ways a professor and his student can choose \( m \) problems from the \( n \) that appear each month in an eminent periodical (actually Suck used \( n = 10 \), and then divide the \( m \) problems up between them. The proposers counted the number of ways of choosing \( m \) shoes from \( n \) distinct pairs so that no two shoes chosen form a pair.
Let $ABC$ be a triangle, and let congruent circles $C_1$, $C_2$, $C_3$ be tangent to half lines $AB$ and $AC$, $BA$ and $BC$, $CA$ and $CB$, respectively.

(a) Determine the locus of the circumcentre $P$ of $\triangle DEF$, where $D$, $E$, $F$ are the centres of $C_1$, $C_2$, $C_3$.

(b) If $C_1$, $C_2$, $C_3$ all pass through the same point, show that their radius $\rho$ satisfies

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{R}$$

where $r$ and $R$ are the inradius and circumradius of $\triangle ABC$.

**Solution by J.T. Groenman, Arnhem, The Netherlands.**

(a) The centers $D$, $E$, $F$ have to lie on the angle bisectors of angles $A$, $B$, $C$ respectively. As the radii of the three circles are equal, $DE \parallel AB$, $EF \parallel BC$ and $FD \parallel CA$. Thus triangles $ABC$ and $DEF$ are similar and homothetic, with center $I$, the common point of the bisectors. Thus the locus of the circumcenter is the half-line $OI$.

(b) If there is a point $Q$ lying on the three congruent circles, then $QD = QE = QF$ as the radii are equal. Therefore $Q$ is the circumcenter of $\triangle DEF$ and so lies on $IO$. The circumradius of $\triangle DEF$ is therefore $\rho$. $I$ is the incenter of $\triangle DEF$, so the inradius of $\triangle DEF$ is $r - \rho$. Thus

$$R : \rho = r : r - \rho,$$

$$Rr - R \rho = \rho r,$$

and so, dividing by $R \rho$,

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{R}.$$  

Also solved by JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Penning and Springer apparently assumed that the circles lie inside the triangle and so obtained only the segment $OI$ as the answer. On the other hand, there seems to be another solution for (b), namely

$$\frac{1}{\rho} = \frac{1}{r} - \frac{1}{R},$$

for circles which extend outside the triangle, which none of the other solvers noticed!
Janous remarks that the problem extends problem 5 of the 1981 I.M.O.


Find all positive integers \( j \) such that \( \lfloor \sqrt{j} \rfloor \) divides into \( j \), where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \).

Solution by Ginger Bolton, Swainsboro, Georgia.

Let \( n, j \) be positive integers. Then

\[
\lfloor \sqrt{j} \rfloor = n \iff n \leq \sqrt{j} < n + 1 \\
\iff n^2 \leq j < (n + 1)^2.
\]

The only positive integers \( j \) satisfying \( n^2 \leq j < (n + 1)^2 \) which are divisible by \( n = \lfloor \sqrt{j} \rfloor \) are

\[
n^2, \ n^2 + n, \ n^2 + 2n,
\]

so these for \( n = 1,2,... \) are the answers to the problem.

Also solved by SEUNG–JIN BANG, Seoul, Korea; CLAYTON W. DODGE, University of Maine, Orono; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; SAM MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; BRUCE SHAWYER, Memorial University of Newfoundland; COLIN SPRINGER, student, University of Waterloo; J. SUCK, Essen, Federal Republic of Germany; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; JÜRGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer.

Priélipp remembered that the problem appeared as E2491 of the American Math. Monthly, with solution, and generalization to \( \lfloor \sqrt{j} \rfloor \), on pp. 854–855 of the October 1975 issue. (Solvers Janous and Klamkin of the current problem also gave this generalization.) Priélipp included a solution that he and N.J. Kuenzi had submitted back then.
Find the smallest positive integer \( a \) so that
\[
13 | 11^{3n+1} + a \cdot 5^n \quad \text{and} \quad 31 | 23^{2n+1} + 2^{n+a}
\]
both hold for all positive integers \( n \).

Solution by Bob Priess, University of Wisconsin-Oshkosh.
Let \( n \) be an arbitrary positive integer.
Since \( 11 \equiv -2 \mod 13 \) and \((-2)^3 = -8 = 5 \mod 13\),
\[
11^{3n+1} + a \cdot 5^n \equiv -2 \cdot 5^n + a \cdot 5^n \mod 13
\]
\[
\equiv (-2 + a)5^n \mod 13.
\]
It follows that \( a \equiv 2 \mod 13 \).
Because \( 23 \equiv -8 \mod 31 \) and \((-8)^2 = 64 \equiv 2 \mod 31\),
\[
23^{2n+1} + 2^{n+a} \equiv -8 \cdot 2^n + 2^{n+a} \mod 31
\]
\[
\equiv (-8 + 2^a)2^n \mod 31.
\]
Hence \( 2^a \equiv 8 \mod 31 \). Since 5 is the smallest positive integer solution of the congruence \( 2^k \equiv 1 \mod 31 \), \( a = 5k + 3 \) for some non-negative integer \( k \).

We now have the following collection of equivalent congruences:
\[
5k + 3 \equiv 2 \mod 13,
\]
\[
5k \equiv -1 \equiv 25 \mod 13,
\]
\[
k \equiv 5 \mod 13.
\]
When \( k = 5 \), \( a = 5k + 3 = 28 \), so 28 is our answer.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALThER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; SAM MALTBYY, student, Calgary; P. PENNING, Delft, The Netherlands; M.A. SELBY, University of Windsor; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; KENNETH M. WILKE, Topeka, Kansas; and the proposer. There was one incorrect solution submitted.

Many solvers remarked that \( a \) satisfies the two given properties precisely if \( a \equiv 28 \mod 65 \).

Janous wondered if the number of this problem was assigned because of the appearance of 13 and 31 in the statement. The answer is yes! (However, it was perhaps a lucky coincidence that the proposer sent in such a problem just at the time Crux 1331 was to be created.) The editor invites readers to send in more problems designed for a specific problem number.
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