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AN EXTENSION OF OPPENHEIM’S AREA INEQUALITY
FOR TRIANGLES

Ji Chen

The area $\Delta$ of a triangle is a well known function of the lengths $a$, $b$, $c$ of its sides:
$$\Delta = [s(s-a)(s-b)(s-c)]^{1/2}$$
(where $s = (a + b + c)/2$), or equivalently
$$16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \quad (1)$$
In [3], A. Oppenheim proved that, if $0 < p < 1$, then
$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^p \leq \frac{16}{3}\Delta^2(a^p,b^p,c^p) \quad (2)$$
(here $\Delta(x,y,z)$ denotes the area of the triangle with sides $x$, $y$, $z$), which from (1) can be written
$$3^{1-p}(16\Delta^2)^p \leq 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^{4p} - b^{4p} - c^{4p} \quad (3)$$
$0 < p < 1$. Equality holds (for $0 < p < 1$) if and only if $a = b = c$.

In this paper, we will prove that the reverse of inequality (3) holds for all other values of $p$.

**Theorem.** For $p < 0$ or $p > 1$,
$$3^{1-p}(16\Delta^2)^p \geq 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^{4p} - b^{4p} - c^{4p}, \quad (4)$$
equivalently, if $a^p$, $b^p$, $c^p$ are the sides of a triangle,
$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^p \geq \frac{16}{3}\Delta^2(a^p,b^p,c^p) \quad (5)$$

**Proof.** If $a^p$, $b^p$, $c^p$ are not the edges of a triangle, we have
$$(a^p + b^p + c^p)(b^p + c^p - a^p)(c^p + a^p - b^p)(a^p + b^p - c^p) \leq 0,$$
and (4) is true obviously. Thus we suppose $a^p$, $b^p$, $c^p$ are the edges of a triangle, and show (5). Also, equality holds in (5) when $a = b = c$, so we suppose that $a$, $b$, $c$ are not all equal.

When $p > 1$, then $0 < 1/p < 1$. Using (2) on the triangle with edges $a^p$, $b^p$, $c^p$, we have
$$\left[\frac{16}{3}\Delta^2(a^p,b^p,c^p)\right]^{1/p} \leq \frac{16}{3}\Delta^2(a,b,c),$$
which is the same as (5).

Assume $p < 0$. We define a function $f$ on $(-\infty, +\infty)$ by
$$f(x) = \left[\frac{b^{2x}c^{2x} + c^{2x}a^{2x} + a^{2x}b^{2x}}{3}\right]^{1/2} - \left[\frac{a^{4x} + b^{4x} + c^{4x} + 3(16\Delta^2/3)^x}{6}\right]^{1/2}, \; x \neq 0,$$
$$f(0) = (abc)^{4/3} - (abc)^{2/3}\left[\frac{16\Delta^2}{3}\right]^{1/2}.$$
Then $f$ is continuous at $x = 0$ (take logarithms and apply L'Hôpital's rule).

Since we are assuming that $a$, $b$, $c$ are not all equal, from the well known inequality

$$\Delta < \frac{\sqrt{3}}{4}(abc)^{2/3}$$

(see item 4.14 of [2]) we know $f(0) > 0$. Thus we can find $q < 0$, $q > p$ such that $f(q) > 0$, i.e.

$$\left(\frac{b^2q + c^2q + a^2q}{3}\right)^{1/q} > \left(\frac{a^4q + b^4q + c^4q + 3(16\Delta^2/3)^q}{6}\right)^{1/q},$$

or

$$2b^2q + 2c^2q + 2a^2q < a^4q + b^4q + c^4q + 3(16\Delta^2/3)^q,$$

or, from (1),

$$\frac{16}{3}\Delta^2(a^q, b^q, c^q) < \left[\frac{16}{3}\Delta^2(a, b, c)\right]^q. \tag{6}$$

For $p < q < 0$, $0 < q/p < 1$. Using (2) on the triangle with edges $a^p$, $b^p$, $c^p$ we obtain

$$\left[\frac{16}{3}\Delta^2(a^p, b^p, c^p)\right]^{q+p} < \frac{16}{3}\Delta^2(a^q, b^q, c^q). \tag{7}$$

From (6) and (7) we have

$$\left[\frac{16}{3}\Delta^2(a^p, b^p, c^p)\right]^{q+p} < \left[\frac{16}{3}\Delta^2(a, b, c)\right]^p,$$

so that, since $q/p > 0$,

$$\left[\frac{16}{3}\Delta^2(a^p, b^p, c^p)\right]^p > \frac{16}{3}\Delta^2(a^q, b^q, c^q),$$

which is (5). □

As a special case, take $p = -1/2$ in (4); then we have the inequality

$$\frac{3\sqrt{3}}{4\Delta} > \frac{2}{bc} + \frac{2}{ca} + \frac{2}{ab} - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2}.$$

This could also be written

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{3\sqrt{3}}{4\Delta} + \left[\frac{1}{b} + \frac{1}{c}\right]^2 + \left[\frac{1}{c} - \frac{1}{a}\right]^2 + \left[\frac{1}{a} - \frac{1}{b}\right]^2.$$

For another application of the theorem, we extend the following inequality of S. Beatty [1]:

$$\Delta^2 > \frac{(K - H)(3K - 5H)}{12}, \tag{8}$$

where

$$H = (a^2 + b^2 + c^2)/2, \ K = bc + ca + ab$$

(see also item 4.18 of [2]). Suppose that $p \geq 1$ or $p \leq 0$ and that the triples $(a, b, c)$ and $(a^p, b^p, c^p)$ are each the sides of a triangle. Then from (8) applied to the triangle with sides $a^p$, $b^p$, $c^p$.\]
\[ \Delta^2(a^p, b^p, c^p) \geq (K_p - H_p)(3K_p - 5H_p), \]

where

\[ H_p = \frac{a^2p + b^2p + c^2p}{2}, \quad K_p = b^p c^p + c^p a^p + a^p b^p. \]

By using (5) we thus get

\[ \Delta^2p \geq 2^{2-4p}3^{p-2}(K_p - H_p)(3K_p - 5H_p). \]

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* * *

THE OLYMPIAD CORNER
No. 101
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This column marks the second anniversary of my taking over the Corner. In spite of the odd error, the occasional misquoted problem, and misunderstanding of solutions, I hope that the resulting column provides a forum for discussion of the Olympiads, and is a source of good problems for those who coach others (or who are preparing themselves) for mathematics contests. For the rest of us the problems provide the pleasure of a good teaser. I’m grateful to all those who have sent in problem sets, comments, and solutions. Those whose contributions were used in Volume 14 include Beno Arbel, Francisco Bellot, Curtis Cooper, Ed Doolittle, George Evagelopoulos, Richard Gibbs, Douglass Grant, J.T. Groenman, Branko Grunbaum, R.K. Guy, F.D. Hammer, Walther Janous, Murray Klamkin, A.H. Lachlan, Andy Liu, Alan Mekler, John Morvay, V.N. Murty, Richard Nowakowski,
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* * *

This month's Olympiad problems begin with the final rounds of the Flanders Mathematics Olympiad, 1985–86 and 1986–87. Thanks go to Bruce Shawyer who forwarded them to me. In each exam, students were allowed three hours to solve the problems. Some of these problems will be familiar to readers, and there are obvious possibilities for generalization. Please send in your "nice" solutions.

FLANDERS MATHEMATICS OLYMPIAD 1985–86 (final round)

1. A circle with radius \( R \) is divided into twelve equal parts. The twelve dividing points are connected with the centre of the circle, producing twelve rays. Starting from one of the dividing points a segment is drawn perpendicular to the next ray in the clockwise sense; from the foot of this perpendicular another perpendicular segment is drawn to the next ray, and the process is continued \textit{ad infinitum}. What is the limit of the sum of these segments (in terms of \( R \))?

2. Prove that, for every natural number \( n \), we have

\[
n! \leq \left( \frac{n + 1}{2} \right)^n.
\]

3. A sequence of numbers \( \{a_k\} \) is defined as follows:

\[
a_0 = 0,
\]

\[
a_{k+1} = 3a_k + 1, \quad k \geq 0.
\]

Show that \( a_{155} \) is divisible by 11.

4. To put a marble with radius 1 cm in a cube it is obvious that the cube must have an edge with at least a length of 2 cm. What is the minimum length of the edge of a cube which can contain two marbles of radius 1 cm? (Prove your answer.)

*
1. A rectangle $ABCD$ is given. On the side $AB$, $n$ different points are chosen strictly between $A$ and $B$. Similarly, $m$ different points are chosen on the side $AD$. Lines are drawn from the points parallel to the sides. How many rectangles are formed in this way? (One possibility is shown in the figure.)

2. Two parallel lines $a$ and $b$ meet two other lines $c$ and $d$. Let $A$ and $A'$ be the points of intersection of $a$ with $c$ and $d$, respectively. Let $B$ and $B'$ be the points of intersection of $b$ with $c$ and $d$, respectively. If $X$ is the midpoint of the line segment $AA'$ and $Y$ is the midpoint of the segment $BB'$, prove that

$$|XY| \leq \frac{|AB| + |A'B'|}{2}.$$  

($|XY|$ represents the length of the line segment $XY$.)

3. Determine all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^2 = -\frac{x^2}{12}x^2 + 7xf(x) + 16(f(x))^2.$$  

4. Prove that, for every $r \in \mathbb{R}$ with $r > 1$

$$\lim_{n \to \infty} \left( \frac{1^r + 2^r + \cdots + (n-1)^r + n^r + (n-1)^r + \cdots + 2^r + 1^r}{n^2} \right) = +\infty.$$  

What is the value of the limit when $r = 1$?

*  

*  

*  

Next we give four problems from the *Arany Daniel Competition 1987, Junior Level* (age 15), from Hungary. These were collected by Gy. Karolyi and J. Pataki, and forwarded to me by Bruce Shawyer.

1. The real numbers $x$, $y$, $z$ satisfy the following equation:

$$\frac{y^2 + z^2}{2yz} + \frac{z^2 + x^2}{2zx} - \frac{x^2 + y^2}{2xy} + \frac{x^2 + y^2 - z^2}{2xy} = 1.$$  

Prove that two of the three fractions have the value 1.

2. The median lines of a convex quadrilateral divide it into four smaller ones. Prove that the sum of the areas of two of the quadrilaterals with no common side equals the sum of the areas of the other pair of such quadrilaterals.

3. Choose $n$ points on a circle and label them with the numbers $1, 2, \ldots, n$. Say that two non-neighbouring points $A$ and $B$ are *connectable* if the points on at least one of the two arcs containing $A$ and $B$ are all labelled with numbers that are less than
those for \( A \) and \( B \). Prove that the number of connectable pairs of points is \( n - 3 \).

4. Let four distinct points be given in space. Determine all the planes having the same distance from each of the points.

Now we return to problems from the 1987 I.M.O. (Havana) given in the October 1987 number of the Corner.

France 2. [1987: 246]

Let \( ABC \) be a triangle. For each point \( M \) of the segment \( BC \) denote by \( B' \) and \( C' \) the orthogonal projections of \( M \) on the lines \( AC \) and \( AB \), respectively. Determine those points \( M \) for which the length of \( B'C' \) is minimum.

Solutions independently by George Evangelopoulos, law student, Athens, Greece, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The four points \( A, C, M \) and \( B' \) lie on a circle with \( AM \) as a diameter. Using the law of sines it follows from this that

\[
|B'C'| = |AM| \sin A.
\]

[Let \( I \) be the center of the circle. In triangle \( IC'B' \) we have

\[
\angle C'B' = 2A, \quad \angle IC'B' = \angle IB'C' = 90^\circ - A
\]

and so

\[
|B'C'| = \frac{|IB'|}{\sin(90^\circ - A)} \sin 2A = \frac{|AM|}{2 \cos A} \cdot 2 \sin A \cos A.
\]

But this means that the length of \( B'C' \) is minimized where \( AM \) is as short as possible. This occurs when \( AM \) is an altitude, if angles \( B \) and \( C \) are both acute. If \( B \) (or \( C \)) is obtuse the point \( M \) should be chosen to be \( B \) (\( C \), respectively).

Great Britain 1. [1987: 246]

Prove that if the equation

\[
x^4 + ax^3 + bx + c = 0
\]

has all its roots real then \( ab \leq 0 \).


Let \( p(x) = x^4 + ax^3 + bx + c = 0 \) and let \( V(p(x)) \) denote the number of sign changes (variations) among the coefficients of \( p(x) \). We use Descartes' rule of signs to prove the
assertion by contradiction. We denote by \( r^+(p) \), \( r^0(p) \), and \( r^-(p) \) the number of positive, zero, and negative real roots of \( p(x) \), respectively. Suppose \( ab > 0 \). Then there are two cases:

**Case (i).** \( a > 0 \) and \( b > 0 \). There are three subcases.
If \( c > 0 \), then \( V(p(x)) = 0 \) and \( V(p(-x)) = 2 \). Thus
\[
r^+(p) = 0, \quad r^0(p) = 0 \quad \text{and} \quad r^-(p) = 2 \quad \text{or} \quad 0.
\]
If \( c = 0 \), then \( V(p(x)) = 0 \), and \( V(p(-x)) = 1 \). Thus
\[
r^+(p) = 0, \quad r^0(p) = 1 \quad \text{and} \quad r^-(p) = 1.
\]
If \( c < 0 \), then \( V(p(x)) = 1 \) and \( V(p(-x)) = 1 \). Thus
\[
r^+(p) = 1, \quad r^0(p) = 0 \quad \text{and} \quad r^-(p) = 1.
\]

**Case (ii).** \( a < 0 \) and \( b < 0 \). We simply apply the conclusion of case (i) to
\[
q(x) = p(-x) = x^4 + Ax^3 + Bx + C
\]
where \( A = -a, \ B = -b, \ C = c \).
In both cases \( p(x) \) can have at most two real roots, a contradiction.

**Solution II [using Rolle's Theorem] by Murray S. Klamkin, University of Alberta.**

More generally, assume that the equation
\[
x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \quad \text{(1)}
\]
has all of its roots real. We show that \( a_1a_3 \leq 0 \).

It follows from the above assumption that all the derivatives of the polynomial (1) have all real zeros. In particular, the equation
\[
x^3 + a'_1x^2 + a'_3 = 0,
\]
where \( a'_1 \) and \( a'_3 \) are positive (real) multiples of \( a_1 \) and \( a_3 \), respectively, has real roots, \( r_1, r_2, r_3 \), say. Then
\[
-a'_1 = r_1 + r_2 + r_3
\]
\[
0 = r_2r_3 + r_3r_1 + r_1r_2
\]
\[
-a'_3 = r_1r_2r_3.
\]

If the three roots are 0, then trivially \( a'_1a'_3 = 0 \). If not, there is some pair of roots whose sum is not zero. Suppose it is \( r_2 + r_3 \). Then
\[
r_1 = \frac{-r_2r_3}{r_2 + r_3}.
\]
Hence
\[
a'_1a'_3 = (r_1 + r_2 + r_3)r_1r_2r_3 = \left( r_2 + r_3 - \frac{r_2r_3}{r_2 + r_3} \right) \left( -\frac{(r_2r_3)^2}{r_2 + r_3} \right)
\]
\[
= -\left( \frac{r_2r_3}{r_2 + r_3} \right)^2 \left( r_2^2 + r_2r_3 + r_3^2 \right) \leq 0.
\]
Hence \( a_1a_3 \leq 0 \).
Great Britain 2. [1987: 246]

Numbers \( d(n,m) \), where \( n, m \) are integers and \( 0 \leq m \leq n \), are defined by

\[
d(n,0) = d(n,n) = 1 \quad \text{for all} \quad n \geq 0
\]

and

\[
m \cdot d(n,m) = m \cdot d(n-1,m) + (2n-m) \cdot d(n-1,m-1)
\]

for \( 0 < m < n \). Prove that all the \( d(n,m) \) are integers.


The binomial coefficients \( \binom{n}{m} = \frac{n!}{m!(n-m)!} \) are integers, since they are the number of ways of choosing \( m \) things from \( n \). Thus \( \binom{n}{m}^2 \) is an integer. We show by induction that

\[
d(n,m) = \binom{n}{m}^2.
\]

We assume \( 0 \leq m \leq n \), and recall that \( 0! = 1 \). Now

\[
d(1,0) = 1 = \binom{1}{0}^2 = \binom{1}{1}^2 = d(1,1).
\]

So assume \( 0 < k \), and

\[
d(k-1,m) = \binom{k-1}{m}^2
\]

for \( 0 \leq m \leq k-1 \). Then for \( 0 < m < k \),

\[
m \cdot d(k,m) = m \binom{k-1}{m}^2 + (2k-m) \binom{k-1}{m-1}^2
\]

\[
= \left[ \frac{(k-1)!}{m!(k-m)!} \right]^2 \left[ m(k-m)^2 + (2k-m)m^2 \right]
\]

\[
= m \binom{k}{m}^2.
\]

Thus

\[
d(k,m) = \binom{k}{m}^2.
\]

As

\[
d(k,0) = 1 = \binom{k}{0}^2 = \binom{k}{k}^2 = d(k,k)
\]

this completes the induction step and we conclude

\[
d(n,m) = \binom{n}{m}^2 \quad \text{for} \quad 0 \leq m \leq n.
\]

Editor's note. R.K. Guy asks "Is there a nicer combinatorial proof?"
Great Britain 3. [1987: 247]

Find, with proof, the smallest real number $c$ with the following property: for every sequence $\{X_i\}$ of positive real numbers such that

$$X_1 + X_2 + \cdots + X_n \leq X_{n+1}\quad \text{for } n = 1, 2, 3, \ldots$$

we have

$$\sqrt{X_1} + \sqrt{X_2} + \cdots + \sqrt{X_n} \leq c\sqrt{X_1 + X_2 + \cdots + X_n}$$

for $n = 1, 2, 3, \ldots$ [c is to be independent of the $X_i$ and independent of $n$.]

Solutions by George Evagelopoulos, law student, Athens, Greece, by Murray S. Klamkin, Department of Mathematics, The University of Alberta, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The sequence $X_i = 2^{i-1}$ satisfies

$$X_1 + X_2 + \cdots + X_n \leq X_{n+1}$$

since

$$1 + 2 + \cdots + 2^{n-1} = 2^n - 1.$$ 

Thus we must have

$$1 + 2^{1/2} + 2^{2/2} + \cdots + 2^{(n-1)/2} = \frac{2^{n/2} - 1}{2^{1/2} - 1} \leq c(2^{n-1})^{1/2}.$$ 

This gives

$$c \geq \frac{1 \cdot \frac{2^{n/2} - 1}{\sqrt{2} - 1}}{(2^n - 1)^{1/2}}$$

for all $n$. Since

$$\lim_{n \to \infty} \frac{2^{n/2} - 1}{(2^n - 1)^{1/2}} = \lim_{n \to \infty} \frac{1 - \frac{2^{-n/2}}{2^{1/2}}}{\sqrt{1 - 2^{-n}}} = 1$$

we therefore require

$$c \geq \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1.$$ 

We show $c = \sqrt{2} + 1$ does have the required property by induction on $n$.

Let $\{X_i\}$ satisfy (1) for all $n$. Certainly $\sqrt{X_1} \leq (\sqrt{2} + 1)\sqrt{X_1}$ holds, so the required inequality holds with $n = 1$. Now suppose

$$\sqrt{X_1} + \cdots + \sqrt{X_n} \leq (\sqrt{2} + 1)\sqrt{X_1 + \cdots + X_n}.$$ 

Then

$$\sqrt{X_1} + \cdots + \sqrt{X_{n+1}} \leq (\sqrt{2} + 1)(\sqrt{Y_n} + \sqrt{X_{n+1}},$$

where

$$Y_n = X_1 + \cdots + X_n.$$ 

We shall show

$$(\sqrt{2} + 1)\sqrt{Y_n} + \sqrt{X_{n+1}} \leq (\sqrt{2} + 1)\sqrt{Y_n} + \sqrt{X_{n+1}},$$

by arguing that, more generally, if $0 < A \leq B$ we have
\((\sqrt{2} + 1)\sqrt{A} + \sqrt{B} \leq (\sqrt{2} + 1)\sqrt{A} + B\).

Squaring, we get the equivalent inequality

\((3 + 2\sqrt{2})A + 2(\sqrt{2} + 1)\sqrt{A}B + B \leq (3 + 2\sqrt{2})(A + B),\)

that is,

\(2(\sqrt{2} + 1)\sqrt{A}B \leq 2(\sqrt{2} + 1)B,\)

which is clear for \(0 < A \leq B\) (and strict unless \(A = B\)). This completes the induction step and the proof.

**Greece 1.** [1987: 247]

Consider the regular 1987-gon \(A_1A_2\ldots A_{1987}\) with center \(O\). Show that the sum of vectors belonging to any proper subset of \(M = \{OA_j: j = 1, 2, \ldots, 1987\}\) is nonzero.

**Solutions by George Evagelopoulos, law student, Athens, Greece, by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.**

The number 1987 can be replaced by any prime \(p\) with the same result. We can represent the given set of vectors by the \(p\) complex numbers

\[1, \omega, \omega^2, \ldots, \omega^{p-1}\]

where \(\omega = e^{2\pi i/p}\) satisfies the equation

\[1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0.\]

It is also a known result (which follows by using Eisenstein’s criterion) that the cyclotomic polynomial \(f(x) = 1 + x + \cdots + x^{p-1}\) is irreducible over the reals, i.e. there is no real polynomial of degree less than \(p - 1\) which is satisfied by \(\omega\).

Our proof is indirect. Assume that there is a proper subset of the \(\omega^j\)’s whose sum is zero. It follows from (2) that the sum of the remaining \(\omega^j\)’s is also zero. This gives two real polynomial equations in \(\omega\) (after dividing out the lowest power of \(\omega\) in one of them)

\[1 + \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_r} = 0\
\[1 + \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_s} = 0\]

where the \(\alpha\)’s and \(\beta\)’s are natural numbers less than \(p\). One of these equations must be of degree less than \(p - 1\) which contradicts the fact that \(f\) is irreducible. Thus there is no proper subset of the \(\omega^j\)’s whose sum is zero.

**Editor’s note.** Murray Klamkin adds the following comment.

There is an extension of this result to \(n\) dimensions but it is less sophisticated. Consider the \(n + 1\) vectors from the centroid of a regular \(n\)-dimensional simplex to its vertices \(A_0, \ldots, A_n\). The sum of these \(n + 1\) vectors is zero. However, the sum of any subset of these vectors

\[GA_0^\ast + GA_1^\ast + \cdots + GA_{r-1}^\ast = rGG_r^\ast \neq 0\]  \((r < n + 1)\)

where \(G\) and \(G_r\) are the centroids of the given \(n\)-dimensional simplex and the \((r - 1)\)-
dimensional simplex with vertices $A_0, A_1, \ldots, A_{r-1}$, respectively.

**Greece 2.** [1987: 247]

Solve the equation

$$28^x = 19^y + 87^z$$

where $x, y, z$ are integers.

*Solution by Dave McDonald, Crimson Elk, Alberta.*

Suppose $(x, y, z)$ is a solution.

(i) None of $x, y, z$ are negative, since otherwise there would be a prime (respectively 2, 19, 3) in the denominator of one term which could not be matched by another.

(ii) From (i), $y \geq 0, z \geq 0$ imply $x > 0$ and it is easy to see that $x \neq 1$, so $x \geq 2$.

(iii) From (ii), $0 \equiv 3^y + 7^z \mod 16$. Now $7^z \equiv 7$ or 1 mod 16 according as $z$ is odd or even, respectively, while $3^y \equiv 1, 3, 9$ or 11 mod 16, according as $y \equiv 0, 1, 2$ or 3 mod 4. So $z$ is odd and $y \equiv 2 \mod 4$.

(iv) Working modulo 9, $1 \equiv 1 + (-3)^z$, so $z \geq 2$, and since $z$ is odd, $z \geq 3$.

(v) By (iv), $1 \equiv (-8)^y \mod 27$, so 3 divides $y$, and by (iii), $y \equiv 6 \mod 12$.

(vi) Working modulo 7, $0 \equiv (-2)^y + 3^z$. By (v), $y$ is a multiple of 6, so $3^z \equiv -1 \mod 7$. This gives $z \equiv 3 \mod 6$.

(vii) Now, modulo 13, $2^x \equiv 6^y + (-4)^z$. By (v), $y$ is a multiple of 6, so $6^y \equiv -1 \mod 13$. By (vi), $z$ is a multiple of 3, so $(-4)^z \equiv 1 \mod 13$. This gives $2^x \equiv 0 \mod 13$, a contradiction!

Thus there are no integer solutions of the equation.

*Editor's note.* A second correct, but somewhat more involved, solution was submitted by George Evagelopoulos, law student, Athens, Greece.

**Hungary 1.** [1987: 247]

Does there exist a set $M$ in the usual Euclidean space such that for any plane $\sigma$, the intersection $M \cap \sigma$ is finite and non-empty?

*Solution by George Evagelopoulos, law student, Athens, Greece, and also by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

The answer is yes. Consider the curve

$$M = \{(t, t^3, t^5): t \in \mathbb{R}\}.$$  

If the plane $\sigma$ has equation $Ax + By + Cz + D = 0$ (not all of $A, B, C$ zero) then the points of intersection are given by the solutions of $At + Bt^3 + Ct^5 + D = 0$ which is a polynomial of odd degree. Thus there is at least one and there are only finitely many solutions so $M \cap \sigma$ is
finite and non-empty.

Next month we continue with solutions to these I.M.O. problems. The Olympiad season is fast approaching. Please remember to collect the Olympiads available to you and send them in to me for use in the Corner.

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before August 1, 1989, although solutions received after that date will also be considered until the time when a solution is published.


Given are a circle \( C \) and two straight lines \( l \) and \( m \) in the plane of \( C \) that intersect in a point \( S \) inside \( C \). Find the tangent(s) to \( C \) intersecting \( l \) and \( m \) in points \( P \) and \( Q \) so that the perimeter of \( \triangle ASPQ \) is a minimum.

1402. Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \( M \) be an interior point of the triangle \( A_1A_2A_3 \) and \( B_1, B_2, B_3 \) the feet of the perpendiculars from \( M \) to sides \( A_2A_3, A_3A_1, A_1A_2 \) respectively. Put \( r_i = B_iM, i = 1,2,3 \). \( R' \) is the circumradius of \( \triangle B_1B_2B_3 \), and \( R, r \) the circumradius and inradius of \( \triangle A_1A_2A_3 \). Prove that

\[
R' R r > 2rr_1r_2r_3.
\]

1403* Proposed by Walther Janous, Ursulengymnasium, Innsbruck, Austria.

For \( n \geq 2 \), prove or disprove that

\[
1 < \frac{x_1 + \cdots + x_n}{n} < 2
\]

for all natural numbers \( x_1, x_2, \ldots, x_n \) satisfying

\[
x_1 + x_2 + \cdots + x_n = x_1 \cdot x_2 \cdots x_n.
\]

Let $ABC$ be a triangle with circumradius $R$ and inradius $\rho$. A theorem of Poncelet states that there are an infinity of triangles having the same circumcircle and the same incircle as $\Delta ABC$.

(a) Show that the orthocenters of these triangles lie on a circle.

(b) If $R = 4\rho$, what can be said about the locus of the centers of the nine-point circles of these triangles?

1405. Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.

Two distinct congruent $n$—gons $P$ and $P'$ are inscribed in a noncircular ellipse $E$. Prove or disprove that if $n > 4$, $P'$ must be obtainable from $P$ by a reflection across the axes or center of $E$. (For the cases $n = 3$ and 4 see [1988: 131, 139].)

1406. Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.

If $0 < \theta < \pi$, prove without calculus that

$$\cot \frac{\theta}{4} - \cot \theta > 2.$$ 


Given a rectangle $ABCD$ with $AB = CD > AD = BC$, construct points $X, Y$ on $CD$ between $C$ and $D$ such that $AX = XY = YB$.

1408. Proposed by Jordi Dou, Barcelona, Spain.

Given the equilateral triangle $ABC$, find all positive real numbers $r$ for which there is a point $P(r)$ such that

$$\frac{PA}{r} = \frac{PB}{r} = \frac{PC}{r^2},$$

and describe the locus of $P(r)$.

1409. Proposed by Shailesh Shirali, Rishi Valley School, India.

Show that

$$\frac{n}{n + 1} + \frac{2n(n - 1)}{(n + 1)(n + 2)} + \frac{3n(n - 1)(n - 2)}{(n + 1)(n + 2)(n + 3)} + \cdots = \frac{n}{2}$$

for all positive integers $n$. What if $n \geq 0$ is not an integer?

1410. Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.

Given is a triangle with circumcentre $O$ and circumradius $R$. Interior points $P, P'$ are isogonal conjugates, and $r_1, r_2, r_3$ are the distances from $P$ to the sides of the triangle. Prove that

$$(R^2 - OP^2)^2(R^2 - OP'^2) = 8r_1r_2r_3R^3.$$
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1199* [1986: 283; 1988: 87] Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia. (Dedicated to Léo Sauvé.)

Prove that for acute triangles,

\[
s^2 \leq \frac{27R^2}{27R^2 - 8r^2}(2R + r)^2,
\]

where \(s, r, R\) are the semiperimeter, inradius, and circumradius, respectively.

II. Comment by Walther Janous, Ursulinen-Gymnasium, Innsbruck, Austria.

Let \(h_1, h_2, h_3\) be the altitudes of an acute triangle and \(d_1, d_2, d_3\) the distances from the circumcenter to the sides. Then since

\[
\prod_{i=1}^{3} h_i = \frac{2s^2r^2}{R} \quad \text{and} \quad \prod_{i=1}^{3} d_i = \frac{R}{4}[s^2 - (2R + r)^2],
\]

the above inequality has a remarkable interpretation, namely that

\[
\prod_{i=1}^{3} \frac{h_i}{3} \geq \prod_{i=1}^{3} d_i,
\]

i.e.

geometric mean \(\left\{ \frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3} \right\} \geq \text{geometric mean} \left\{ d_1, d_2, d_3 \right\}\)

On the other hand it is known that

arithmetic mean \(\left\{ \frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3} \right\} \leq \text{arithmetic mean} \left\{ d_1, d_2, d_3 \right\}\)

(see e.g. item 12.3 of Bottema et al, Geometric Inequalities). Thus one could pose the problem: find all exponents \(t \neq 0\) such that

\[M_t\left(\frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3}\right) \geq M_t\left(d_1, d_2, d_3\right)\]

holds for all acute triangles, where

\[M_t(u, v, w) = \left(\frac{u^t + v^t + w^t}{3}\right)^{1/t}.
\]

1292* [1987: 320] Proposed by Jack Garfunkel, Flushing, N.Y.

It has been shown (see Crux 1083 [1987: 96]) that if \(A, B, C\) are the angles of a triangle,
\[ \frac{2}{\sqrt{3}} \sum \sin A \leq \sum \cos \left( \frac{B - C}{2} \right) \leq \frac{2}{\sqrt{3}} \sum \cos \frac{A}{2}, \]

where the sums are cyclic. Prove that

\[ \sum \cos \left( \frac{B - C}{2} \right) \leq \frac{1}{\sqrt{3}} \left[ \sum \sin A + \sum \cos \frac{A}{2} \right], \]

which if true would imply the right hand inequality above.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Proceeding as in the solution of Crux 1083 [1987: 96] we put

\[ \alpha = \frac{\pi - A}{2}, \]

so that \( \alpha, \beta, \gamma \) are the angles of an acute triangle. Then, using the relations given in Crux [1982: 67–68],

\[ \sum \cos \frac{B - C}{2} = \sum \cos(\beta - \gamma) = \frac{x^2 + y^2 + 2x - 2}{2}, \]

\[ \sum \cos \frac{A}{2} = \sum \sin \alpha = y, \]

\[ \sum \sin A = \sum \sin 2\alpha = 2xy, \]

where \( x = r/R, y = s/R \) (\( r, R, s \) being the inradius, circumradius, and semiperimeter of the \( \alpha - \beta - \gamma \) triangle). The claimed inequality now reads

\[ \frac{x^2 + y^2 + 2x - 2}{2} \leq \frac{y + 2xy}{\sqrt{3}}, \]

i.e.

\[ y^2 - \frac{2}{\sqrt{3}} y(1 + 2x) + x^2 + 2x - 2 \leq 0, \tag{1} \]

where \( 0 < x \leq 1/2 \) and \( 0 < y \leq 3\sqrt{3}/2 \). In order to show (1) we have to prove that \( y_1 \leq y \leq y_2 \), where

\[ y_1 = \frac{1}{\sqrt{3}}(1 + 2x - \sqrt{x^2 - 2x + 7}) \]

and

\[ y_2 = \frac{1}{\sqrt{3}}(1 + 2x + \sqrt{x^2 - 2x + 7}). \]

We first show \( y_1 < 0 \). Indeed,

\[ (1 + 2x)^2 = 4x^2 + 4x + 1 < x^2 - 2x + 7 \]

is equivalent to

\[ x^2 + 2x < 2, \]

which is true for \( 0 < x \leq 1/2 \).

Hence we have to show \( y \leq y_2 \), i.e.

\[ y\sqrt{3} \leq 1 + 2x + \sqrt{x^2 - 2x + 7}. \tag{2} \]

From item 5.4 of Bottema et al, Geometric Inequalities we take the inequality
and thus inequality (2) will follow if we prove

$$2\sqrt{3} + (9 - 4\sqrt{3})x \leq 1 + 2x + \sqrt{x^2 - 2x + 7},$$

i.e.

$$\ell(x) := 2\sqrt{3} - 1 + (7 - 4\sqrt{3})x \leq \sqrt{x^2 - 2x + 7} =: \tau(x).$$

Now \(\ell(x)\) increases, whereas \(\tau(x)\) decreases for \(0 < x < 1/2\). As furthermore

$$\ell(1/2) = 5/2 = \tau(1/2),$$

inequality (4) follows.

---


Solve the following "twin" problems (in both problems, \(O\) is the center of the circle and \(OA \perp AB\)).

(a) In Figure (a), \(AB = BC\) and \(\angle ABC = 60^\circ\). Prove \(CD = OA\sqrt{3}\).

(b) In Figure (b), \(OA = BC\) and \(\angle ABC = 30^\circ\). Prove \(CD = AB\sqrt{3}\).

Solution by Jordi Dou, Barcelona, Spain.

(a) Let \(E\) be on line \(AB\) such that \(\triangle EBD\) is equilateral. Then \(CD = EA\) and \(EO\) (extended) is the perpendicular bisector of \(BD\). Thus \(\angle OEA = 30^\circ\), so \(CD = EA = OA\sqrt{3}\).

(b) Let \(E\) be the other intersection of \(AB\) with the circle, and let \(F\) be on \(BD\) so that \(EF \perp BD\). Then \(EF = EB/2 = AB\). Since \(\angle EBD = 30^\circ\), we have \(\angle EOD = 60^\circ\) so that \(ED = EO = OB\). Therefore \(DF = OA = CB\), and thus \(CD = BF = EF\sqrt{3} = AB\sqrt{3}\).

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; S.C. CHAN, Singapore; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C.

Find a necessary and sufficient condition on a convex quadrangle $ABCD$ in order that there exist a point $P$ (in the same plane as $ABCD$) such that the areas of the triangles $PAB$, $PBC$, $PCD$, $PDA$ are equal.

Solution by Jordi Dou, Barcelona, Spain.

The simplest condition I have found is that the midpoint of one diagonal lies on the other. The point $P$ will then be the midpoint of the other diagonal. Convexity is not necessary.

If $M$ and $N$ are the midpoints of $AC$ and $BD$ respectively, the locus of points $P$ such that area $\triangle PAB = \triangle PBC$ is the line $BM$. This and analogous results show that if the areas of all four triangles $PAB$, $PBC$, $PCD$, $PDA$ are to be equal, $P$ must lie on lines $BM$ and $DM$, and also on lines $CN$ and $AN$. Thus either $M$ lies on $BD$ or $N$ lies on $AC$.

Also solved by J.T. Groenman, Arnhem, The Netherlands; George Tsintsifas, Thessaloniki, Greece; and the proposer. The conditions found appear to be more or less the same.


Let $A_1A_2A_3$ be a triangle with $I_1$, $I_2$, $I_3$ the excenters and $B_1$, $B_2$, $B_3$ the feet of the altitudes. Show that the lines $I_1B_1$, $I_2B_2$, $I_3B_3$ concur at a point collinear with the incenter and circumcenter of the triangle.
I. Solution by Shiko Iwata, Gifu, Japan.

\((I, r), (O, R), \text{ and } (O', R')\) are the centers and radii of the incircle and circumcircle of \(\triangle A_1A_2A_3\) and the circumcircle of \(\triangle I_1I_2I_3\), respectively. Then, since \(I\) is the orthocenter of \(\triangle I_1I_2I_3\) and the circumcircle \((O, R)\) of \(\triangle A_1A_2A_3\) is the nine—point circle of \(\triangle I_1I_2I_3\) ([1], page 197), \(O', O\), and \(I\) are collinear and \(R' = 2R\). Also,

\[\angle O'I_1A_3 = 90^\circ - \frac{1}{2}\angle I_1O'I_2 = 90^\circ - \angle I_3 = \angle A_2I_1I = \angle A_2A_3I = 90^\circ - \angle A_2A_3I_1,\]

so \(A_2A_3O'I_1\), i.e. \(A_1B_1||O'I_1\). Let \(P\) be the meeting point of \(I_1B_1\) and \(IO\), and let \(I'\) be on \(PI_1\) such that \(II'||O'I_1\). Then we have

\[PI:PO' = II'':O'I_1.\]  

(1)

On the other hand,

\[\frac{II'}{A_1B_1} = \frac{II_1}{A_1I_1} = \frac{r_1 - r}{r_1} = 1 - \frac{r}{r_1} = 1 - \frac{s - a_1}{s} = \frac{a_1}{s},\]

where \(s\) is the semiperimeter of \(\triangle A_1A_2A_3\), so that

\[II' = \frac{a_1A_1B_1}{s} = \frac{2\Delta}{s} = 2r,\]

where \(\Delta\) is the area of \(\triangle A_1A_2A_3\). Thus from (1)

\[PI:PO' = 2r.R' = r.R.\]

It follows that \(P\) is independent of \(I_1\), so lies on \(I_2B_2\) and \(I_3B_3\) too.

Reference:

II. Generalization by the proposer.

We prove more generally that if \(P\) is any point in the plane of \(\triangle A_1A_2A_3\), \(Q\) is its isogonal conjugate, and \(P_1\) is the intersection of \(A_1P\) and \(A_2A_3\) (with analogous definitions for \(P_2, P_3\)), then the lines \(P_1I_1, P_2I_2, P_3I_3\) are concurrent at a point collinear with \(Q\) and the incenter \(I\) of the triangle. The given result follows by letting \(P\) be the orthocenter so that \(Q\) is the circumcenter.

We use normal homogeneous triangular coordinates, with
\[ A_1 = (1,0,0), \quad A_2 = (0,1,0), \quad A_3 = (0,0,1), \]
\[ I = (1,1,1), \quad I_3 = (1,1,-1), \text{ etc.} \]

Let \( P \) have coordinates \((p_1,p_2,p_3)\). Then \( P_3 = (p_1,p_2,0) \), so \( P_3I_3 \) has the equation
\[ p_2x - p_1y + (p_2 - p_1)z = 0. \]

Analogous equations hold for \( P_1I_1 \) and \( P_2I_2 \). Thus to show these lines intersect we have to prove
\[
\begin{vmatrix}
  p_2 & -p_1 & p_2 - p_1 \\
  p_3 - p_2 & p_3 & -p_2 \\
 -p_3 & p_1 - p_3 & p_1
\end{vmatrix} = 0.
\]

This is easy as the sum of the three rows is zero. The coordinates of the intersection point \( S \) we get from the equations
\[
\begin{align*}
p_2x - p_1y + (p_2 - p_1)z &= 0 \\
(p_3 - p_2)x + p_3y - p_2z &= 0
\end{align*}
\]

from which comes
\[
S = (p_1p_2 + p_3p_1 - p_2p_3, \ p_1p_2 - p_3p_1 + p_2p_3, \ -p_1p_2 + p_3p_1 + p_2p_3). \]

Next we can put
\[
Q = (p_2p_3, p_3p_1, p_1p_2)
\]

so to show that \( S, Q, \text{ and } I \) are collinear we have to prove
\[
\begin{vmatrix}
  p_2p_3 & p_3p_1 & p_1p_2 \\
  p_1p_2 + p_3p_1 - p_2p_3 & p_1p_2 - p_3p_1 + p_2p_3 & -p_1p_2 + p_3p_1 + p_2p_3 \\
  1 & 1 & 1
\end{vmatrix} = 0,
\]
or
\[
\begin{vmatrix}
  p_2p_3 & p_3p_1 & p_1p_2 \\
  p_1p_2 + p_3p_1 + p_2p_3 & p_1p_2 + p_3p_1 + p_2p_3 & p_1p_2 + p_3p_1 + p_2p_3 \\
  1 & 1 & 1
\end{vmatrix} = 0,
\]

which is indeed true.

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; CLARK KIMBERLING, University of Evansville, Evansville, Indiana; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and JOSE YUSTY PITA, Madrid, Spain.

Kimberling notes that the proposal is a known result (John Casey, Analytic Geometry, 2nd ed., Hodges & Figgis, Dublin, 1893, p.85).

*  *  *


Let \( r_1, r_2, r_3 \) be the distances from an interior point of a triangle to its sides
$a_1, a_2, a_3,$ respectively, and let $R$ be the circumradius of the triangle. Prove that

$$a_1r_1^2 + a_2r_2^2 + a_3r_3^2 \leq (2R)^{n-2}a_1a_2a_3$$

for all $n \geq 1$, and determine when equality holds.

**Solution by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.**

Since $r_i < 2R$,

$$a_1r_1^2 + a_2r_2^2 + a_3r_3^2 \leq (a_1r_1 + a_2r_2 + a_3r_3)(2R)^{n-1} = 2 \cdot \text{Area}(A_1A_2A_3) \cdot (2R)^{n-1} = \frac{a_1a_2a_3}{2R} \cdot (2R)^{n-1} = a_1a_2a_3(2R)^{n-2}.$$ 

Equality holds when $n = 1$.

* * * 

**1297. [1987: 321]** Proposed by Walther Janous, Ursulengymnasium, Innsbruck, Austria. (To the memory of Léo.)

(a) Let $C > 1$ be a real number. The sequence $z_1, z_2, \ldots$ of real numbers satisfies $1 < z_n$ and $z_1 + \cdots + z_n < Cz_{n+1}$ for $n \geq 1$. Prove the existence of a constant $a > 1$ such that $z_n > a^n$, $n \geq 1$.

(b) Let conversely $z_1 < z_2 < \ldots$ be a strictly increasing sequence of positive real numbers satisfying $z_n > a^n$, $n \geq 1$, where $a > 1$ is a constant. Does there necessarily exist a constant $C$ such that $z_1 + \cdots + z_n < Cz_{n+1}$ for all $n \geq 1$?

**Solution by C. Wildhagen, Tilburg University, Tilburg, The Netherlands.**

(a) Let $\theta = C^{-1}$ so that $0 < \theta < 1$. Then

$$z_n > \theta(z_1 + z_2 + \cdots + z_{n-1})$$

for each $n \geq 2$. This gives

$$z_n > \theta(z_1 + z_2 + \cdots + z_{n-2} + \theta(z_1 + z_2 + \cdots + z_{n-2})) = \theta(1 + \theta)(z_1 + z_2 + \cdots + z_{n-2}),$$

and using an inductive argument

$$z_n > \theta(1 + \theta)^{k-1}(z_1 + z_2 + \cdots + z_{n-k})$$

for each $k$, $1 \leq k \leq n-1$. In particular when $k = n-1$,

$$z_n > \theta z_1(1 + \theta)^{n-2} = D_n \cdot \lambda^n$$

where

$$D_n = \frac{\theta z_1}{(1 + \theta)^2} \left[ \frac{1}{1 + \theta/2} \right]^n, \quad \lambda = 1 + \theta/2.$$ 

Take an $N \in \mathbb{N}$ such that $D_n > 1$ for each $n > N$ and hence $z_n > \lambda^n$ for $n > N$. Since $z_n > 1$ for each $n \geq 1$, there exists some $\mu > 1$ such that $z^n > \mu^n$ for $1 \leq n \leq N$. Letting
\( a = \min\{\lambda, \mu\} \), we have \( a > 1 \) and \( z_n > a^n \) for all \( n \geq 1 \).

(b) The answer is "no!" Take \( z_1 = 2 \) and, for each integer \( k \geq 2 \), let

\[
z_i = \left[ k + \frac{i}{k(k + 1)} \right]^{k(k + 1)}
\]

for each integer \( i \) satisfying \( k(k - 1) \leq i < k(k + 1) \).

Clearly \( z_{i+1} > z_i > 2^i \) for each \( i \). Let

\[
S_n = \sum_{i=1}^{n} z_i
\]

for each \( n \in \mathbb{N} \). For \( n = k(k + 1) \) we have

\[
S_{n-2} > \sum_{i=k(k-1)}^{n-2} z_i
\]

\[
= (2k-1)z_{k(k-1)}
\]

\[
= (2k-1)\left[ k + \frac{1}{k - 1} \right]^{k(k + 1)},
\]

while

\[
z_{n-1} < \left[ k + \frac{1}{k - 1} \right]^{k(k + 1)}.
\]

This implies

\[
\frac{S_{n-2}}{z_{n-1}} > (2k-1)\left( \frac{k + \frac{1}{k - 1}}{k + \frac{1}{k - 1}} \right)
\]

\[
= (2k-1)\left[ 1 - \frac{2}{(k - 1)(k + 1)\left[ k + \frac{1}{k - 1} \right]} \right]^{k(k + 1)}
\]

\[
\rightarrow \infty \quad (\text{as } k \rightarrow \infty).
\]

Hence the sequence \( (S_n/z_{n+1}) \) is unbounded.

Part (a) also solved by SEUNG-JIN BANG, Seoul, Korea; KEE-WAI LAU, Hong Kong; and the proposer.

* * *


Let \( A = (a_{ij}) \) be an \( n \times n \) matrix of positive integers such that \( |\det A| = 1 \), and suppose that \( z_1, z_2, \ldots, z_n \) are complex numbers such that

\[
z_1^{a_{11}} z_2^{a_{12}} \cdots z_n^{a_{1n}} = 1
\]

for each \( i = 1, 2, \ldots, n \). Show that \( z_i = 1 \) for each \( i \).
I. **Solution by Seung—Jin Bang, Seoul, Korea.**

Note that

\[ \begin{vmatrix} a_{i1} & z_1 \cdot a_{i2} & \cdots & z_n \cdot a_{in} \\ z_1 \cdot a_{i1} & a_{i2} & \cdots & \cdot a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ z_1 \cdot a_{i1} & z_2 \cdot a_{i2} & \cdots & a_{in} \end{vmatrix} = 1. \]

Taking logs, we have the system of linear equations

\[ a_{i1} \log |z_1| + \cdots + a_{in} \log |z_n| = 0 \quad (i = 1, \ldots, n) \]

with coefficient matrix \( A \). Since \( \det A \neq 0 \), we have

\[ \log |z_1| = \cdots = \log |z_n| = 0, \]

that is,

\[ z_1 = e^{i\theta_1}, \ldots, z_n = e^{i\theta_n} \] (1)

for some reals \( \theta_1, \ldots, \theta_n \). From the original equation

\[ z_1^{a_{i1}} z_2^{a_{i2}} \cdots z_n^{a_{in}} = 1 \]

we see that

\[ a_{i1} \theta_1 + \cdots + a_{in} \theta_n = 2k_i \pi, \quad i = 1, \ldots, n, \]

for integers \( k_i \). Since \( \det A = 1 \), by Cramer's rule we have \( \theta_i = 2\ell_i \pi \) for some integers \( \ell_i \), \( i = 1, \ldots, n \). Hence by (1) we know that \( z_1 = \cdots = z_n = 1 \).

II. **Solution by C. Wildhagen, Tilburg University, Tilburg, The Netherlands.**

We slightly extend the problem by allowing \( A \) to have arbitrary integer entries (provided still that \( \det A = 1 \)).

By applying a finite number of the following elementary row operations:

(i) multiplying a row by \(-1\);

(ii) interchanging two rows;

(iii) adding an integral multiple of one row to another row,

\( A \) can be transformed to an upper triangular integral matrix \( T = (t_{ij}) \) with \( \det T = 1 \) and \( t_{ii} = 1 \) for all \( i \). Moreover the property (of \( A \)) that

\[ z_1^{a_{i1}} z_2^{a_{i2}} \cdots z_n^{a_{in}} = 1 \quad (i = 1, 2, \ldots, n) \]

is preserved by each of (i), (ii), (iii), and therefore

\[ z_1^{t_{i1}} z_2^{t_{i2}} \cdots z_n^{t_{in}} = 1. \]

Now putting \( i = n \) we have \( t_{n1} = t_{n2} = \cdots = t_{n, n-1} = 0, t_{nn} = 1 \) so that \( z_n = 1 \); putting \( i = n-1 \) we have

\[ z_1 \cdots z_{n-2} \cdot z_{n-1}^{t_{n-1,n}} = 1 \]

so that \( z_{n-1} = 1 \); and so on. Continuing in this way we find \( z_i = 1 \) for all \( i \).

**Also solved by CHRIS GODSIL, University of Waterloo, and EDWARD T.H. WANG, Wilfrid Laurier University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; M.A. SELBY, University of Windsor; and the proposer. Two incorrect solutions were received.**

Except for Wildhagen's solution II, all solutions received were similar to solution I, and go through if the entries of \( A \) are arbitrary integers. This extension was only pointed out
by Wildhagen, however.

*  *  *


ABC is a triangle, not right angled, with circumcentre $O$ and orthocentre $H$. The line $OH$ intersects $CA$ in $K$ and $CB$ in $L$, and $OK = HL$. Calculate angle $C$.


Let $\alpha, \beta, \gamma$ be the angles and $a, b, c$ the sides of $\triangle ABC$, and let $R$ be the circumradius. Label the figure as shown, and also put $\theta = \angle CKO$. Since $\angle KCO = \angle HCL = 90^\circ - \beta$, we have

$$\frac{R}{\sin \theta} = \frac{CO}{\sin \theta} = \frac{KO}{\angle KCO} = \frac{HL}{\sin \angle HCL} = \frac{CH}{\sin \angle CLH} = \frac{2R \cos \gamma}{\sin \angle CLH}$$

and thus

$$\sin \angle CLH = 2 \sin \theta \cos \gamma = \sin(\theta + \gamma) + \sin(\theta - \gamma).$$

But $\theta + \gamma + \angle CLH = 180^\circ$, so

$$\sin \angle CLH = \sin(\theta + \gamma),$$

and thus

$$\sin(\theta - \gamma) = 0,$$

i.e. $\theta = \gamma$. Now

$$\tan \gamma = \tan \theta = \tan \angle JOH$$

$$= \frac{HN - OM}{AN - AM} = \frac{2R \cos \alpha \cos \gamma - R \cos \beta}{c \cos \alpha - b/2}$$

$$= \frac{3R \cos \alpha \cos \gamma - R \sin \alpha \sin \gamma}{2R \sin \gamma \cos \alpha - R \sin \beta}$$

$$= \frac{3 \sin \gamma \cos \alpha - \sin \alpha \cos \gamma}{\sin \gamma \cos \alpha - \sin \alpha \cos \gamma}$$

Thus $\tan^2 \gamma = 3$, so that $\gamma = 60^\circ$ or $120^\circ$.


We use triangular coordinates based on the distances to the sides of $\triangle ABC$.
\[ O(\cos A, \cos B, \cos C) \]

and

\[ H(\cos B \cos C, \cos C \cos A, \cos A \cos B), \]

so the straight line \( OH \) has equation

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\begin{bmatrix}
    \cos A \\
    \cos B \\
    \cos C
\end{bmatrix} = 0.
\]

Thus the intersection of \( OH \) with \( AC : y = 0 \) is

\[
K(\cos C(\cos^2 A - \cos^2 B), 0, \cos A(\cos^2 C - \cos^2 B)). \tag{1}
\]

The actual (signed) distance of \( O \) from \( BC \) is \( X_o = R \cos A \), of \( H \) from \( BC \) is \( X_h = 2R \cos B \cos C \), and of \( K \) from \( BC \) is (from (1))

\[
\frac{2R \sin A \sin B \sin C \cdot \cos C(\cos^2 A - \cos^2 B)}{\sin A \cdot \cos C(\cos^2 A - \cos^2 B) + \sin C \cdot \cos A(\cos^2 C - \cos^2 B)}, \tag{2}
\]

which, with quite some manipulation, equals

\[
X_k = \frac{2R \cdot \cos C(\cos^2 A - \cos^2 B)}{2 \cos A \cos C - \cos B}. \tag{3}
\]

The condition \( OK = HL \) implies

\[
X_k - X_o = X_h - X, \tag{4}
\]

which with (3) (and \( X = 0 \)) becomes

\[
\frac{2 \cos C(\cos^2 A - \cos^2 B)}{2 \cos A \cos C - \cos B} - \cos A = 2 \cos B \cos C,
\]

i.e.

\[
2 \cos C(\cos^2 A - \cos^2 B) = (2 \cos A \cos C - \cos B)(2 \cos B \cos C + \cos A).
\]

This leads to

\[
\cos A \cos B(4 \cos^2 C - 1) = 0,
\]

so \( \cos C = \pm 1/2 \), and thus \( C = 60^\circ \) or \( 120^\circ \). (Note that \( A = 90^\circ \) or \( B = 90^\circ \) are possible solutions, but are excluded in the problem.)

### III. Editor's comments.

After much manipulation indeed, the editor came up with the following argument to show (2) equals (3). We need to show

\[
\sin A \sin B \sin C(2 \cos A \cos C - \cos B) = \sin A \cos C(\cos^2 A - \cos^2 B) + \sin C \cos A(\cos^2 C - \cos^2 B)
\]

which can be written

\[
\cos A \cos C[\sin A(\sin B \sin C - \cos A) + \sin C(\sin A \sin B - \cos C)] + \cos^2 B(\sin A \cos C + \sin C \cos A) = \sin A \sin B \sin C \cos B. \tag{5}
\]

\(^1\)See III, Editor's comments.
Using
\[ \cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C, \quad \text{etc.} \]
(5) becomes
\[
\cos A \cos C (\sin A \cos B \cos C + \sin C \cos A \cos B) \\
+ \cos^2 B (\sin A \cos C + \sin C \cos A) = \sin A \sin B \sin C \cos B
\]
or
\[
\cos B (\cos A \cos C + \cos B) (\sin A \cos C + \cos A \sin C) = \sin A \sin B \sin C \cos B.
\]
This last equation now follows from
\[
\cos B = \sin A \sin C - \cos A \cos C
\]
and
\[
\sin A \cos C + \cos A \sin C = \sin(A + C) = \sin B.
\]

None of the solutions received for this problem were completely satisfactory, in that they don't appear to work in all cases. Groenman's proof, for instance, depends on the given diagram, while Penning's, which seems to be the most general, uses (at (4)) that point \( K \) is farther from line \( BC \) than the circumcentre \( O \) is. In fact, if (as is reasonable) one allows the points \( K \) and \( L \) to lie on the extended lines \( CA \) and \( CB \) respectively, the conclusion of the problem may not hold! An interesting counterexample is the triangle \( ABC \) illustrated at the right, where \( D \) is the foot of the perpendicular from \( B \), and \( DC = 1, AD = 2, BD = 3 \). The reader can check that \( OK = HL \), while of course \( \angle C \neq 60^0 \).

Also solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS–HAMOIR, Brussels, Belgium; GEORGE TSINTSIFAS, Thessaloniki, Greece; JOSE YUSTY PITA, Madrid, Spain; and the proposer.


Suppose \( \alpha_k > 0 \) for \( k = 1,2,\ldots,n \) and \( \sum_{k=1}^{n} \tanh^2 \alpha_k = 1 \). Prove that
\[
\sum_{k=1}^{n} \frac{1}{\sinh \alpha_k} \geq n \sum_{k=1}^{n} \frac{\sinh \alpha_k}{\cosh^2 \alpha_k}.
\]
Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg.

We may without loss of generality assume that $0 < \alpha_1 \leq \cdots \leq \alpha_n$. Let

$$ x_k = \frac{1}{\sinh \alpha_k}, \quad w_k = \tanh^2 \alpha_k, \quad k = 1, 2, \ldots, n. $$

Then it is easily seen that

$$ x_1 \geq x_2 \geq \cdots \geq x_n $$

and

$$ w_1 \leq w_2 \leq \cdots \leq w_n, $$

i.e. the correlation coefficient between the $x$'s and $w$'s is $\leq 0$. Therefore by Chebyshev's inequality

$$ \sum_{k=1}^{n} x_k w_k \leq \frac{1}{n} \left( \sum_{k=1}^{n} x_k \right) \left( \sum_{k=1}^{n} w_k \right). $$

Since

$$ \sum_{k=1}^{n} w_k = \sum_{k=1}^{n} \tanh^2 \alpha_k = 1 $$

is given, and

$$ \frac{\tanh^2 \alpha_k}{\sinh \alpha_k} = \frac{\sinh \alpha_k}{\cosh^2 \alpha_k}, $$

the required inequality follows.

Also solved by SEUNG-JIN BANG, Seoul, Korea; C. FESTRAETS-HAMOIR, Brussels, Belgium; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

As in Crux 1288 [1988: 312], about half the solvers simply applied Chebyshev's inequality.

* * *


Let $ABC$ and $A_1B_1C_1$ be two triangles with sides $a, b, c$ and $a_1, b_1, c_1$ and inradii $r$ and $r_1$, and let $P$ be an interior point of $\Delta ABC$. Set $AP = x$, $BP = y$, $CP = z$. Prove that

$$ \frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq 4rr_1. $$

I. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

First proof. Let $F$ and $F_1$ be the areas of triangles $ABC$ and $A_1B_1C_1$,
respectively. The given inequality can then be written as
\[ a_1x^2 + b_1y^2 + c_1z^2 \geq 8Fr_1. \]  
(1)

In [1] (see also item 12.56 of [3]) is shown the inequality
\[ (a_1x + b_1y + c_1z)^2 \geq \frac{M}{2} + 8FF_1, \]  
(2)

where
\[ M = \sum a_1^2(b^2 + c^2 - a^2), \]
the sum being cyclic. Moreover, in [2] inequality (2) is extended to all points \( P \) of the space. Applying the Cauchy–Schwarz inequality to (2) we get
\[ \left( \sum a_1x \right) \leq \left( \sum a_1 \right) \left( \sum a_1^2 \right) = 2s_1 \sum a_1x^2. \]  
(3)

Furthermore, we take from item 10.8 of [3] the "Neuberg–Pedoe" inequality
\[ M \geq 16FF_1. \]  
(4)

Finally (2), (3) and (4) lead to the better estimation
\[ \sum a_1x^2 \geq \frac{M}{4s_1} + 4Fr_1 \geq 8Fr_1, \]
the inequality holding for all points of the space.

**Second proof.** The polar moment of inertia inequality [4] states: if \( u, v, w \geq 0 \) then
\[ (u + v + w)(ux^2 + vy^2 + wz^2) \geq a^2uv + b^2wu + c^2uv. \]  
(5)

Furthermore, from *Crux* 1181(a) [1988: 25] the inequality
\[ a^2uv + b^2wu + c^2uv \geq 4F \sqrt{uvw(u + v + w)} \]  
(6)

is known. (5) and (6) yield
\[ ux^2 + vy^2 + wz^2 \geq 4F \frac{uvw}{u + v + w}. \]  
(7)

Putting in (7) \( u = a_1, v = b_1, w = c_1 \) and noting \( a_1b_1c_1 = 4F_1R_1 \) we get
\[ a_1x^2 + b_1y^2 + c_1z^2 \geq 4F \frac{4F_1R_1}{2s_1} = 4F \sqrt{2F_1R_1}. \]  
(8)

As \( R_1 \geq 2r_1 \), we obtain from (8) the desired inequality (1). Note that (8) gives another interpolation of (1). Again (as can be seen from [4]) there are no restrictions on the position of the point \( P \).

**References:**


II. **Solution by Murray S. Klamkin, University of Alberta.**

*First proof.* The given result will follow by successive use of some known stronger inequalities. First we use the polar moment of inertia inequality [1]

\[(w_1 + w_2 + w_3)(w_1w_2^2 + w_2w_3^2 + w_3w_1^2) \geq w_2w_3a^2 + w_3w_1b^2 + w_1w_2c^2\]

where \(w_1, w_2, w_3\) are arbitrary real numbers. (A simple proof follows by expanding out

\[(w_1X + w_2Y + w_3Z)^2 \geq 0,\]

where \(X, Y, Z\) are vectors from \(P\) to \(A, B, C\), respectively.) Letting \((w_1, w_2, w_3) = (a_1, b_1, c_1)\), we get

\[
\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq \frac{b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2}{(a + b + c)(a_1 + b_1 + c_1)} = \frac{(b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2)}{4FF_1^2}.
\]

It thus suffices to show that

\[
b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2 \geq 16FF_1, \tag{9}
\]

or equivalently,

\[
\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{4F}{R_1}. \tag{10}
\]

To prove (10), we use the known stronger inequality ([2], eq. 41)

\[
\frac{a^2}{a_2^2} + \frac{b^2}{b_2^2} + \frac{c^2}{c_2^2} \geq \frac{8F\sqrt{F_2\sqrt{3}}}{a_2b_2c_2}, \tag{11}
\]

where \(a_2, b_2, c_2, F_2\) are the sides and area of a third triangle. In (11), let

\[(a_2, b_2, c_2) = (\sqrt{a_1}, \sqrt{b_1}, \sqrt{c_1})\]

to give

\[
\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{8F\sqrt{F_2\sqrt{3}}}{\sqrt{a_1}b_1c_1}.
\]

It now remains to show that

\[
\frac{2\sqrt{F_2\sqrt{3}}}{\sqrt{a_1}b_1c_1} \geq \frac{1}{R_1},
\]

or

\[
4R_1^2F_2\sqrt{3} \geq a_1b_1c_1 = 4F_1R_1,
\]

or

\[
R_1F_2\sqrt{3} \geq F_1. \tag{12}
\]

But this follows from the Finsler-Hadwiger inequality ([3], item 10.3)

\[
4F_2^2 \geq F_1\sqrt{3},
\]

and the known inequality ([3], item 4.14)

\[
3R_1^2\sqrt{3} \geq 4F_1,
\]

which is equivalent to the fact that the largest triangle (in area) that can be inscribed in a circle is the equilateral one.
An inequality similar to (9), and due to the proposer, appears as problem E3154 of the *Amer. Math. Monthly* (solution in Vol. 95 (1988), pp.659–660). Here one was to show that

\[ b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2 \geq 4F^2 \]

where the triangle \( A_1B_1C_1 \) is inscribed in \( \Delta ABC \).

References:


[Editor's note. Klamkin gave a second solution, somewhat like Janous' first proof above. References to lines in this proof have been added by the editor.]

**Second proof.** Let \( r \geq 1 \). We start out with the power mean inequality

\[ \frac{a_1x^r + b_1y^r + c_1z^r}{a_1 + b_1 + c_1} \geq \left( \frac{a_1x + b_1y + c_1z}{a_1 + b_1 + c_1} \right)^r. \]

Then using (2) and (4),

\[ \frac{a_1x^r + b_1y^r + c_1z^r}{a + b + c} \geq \frac{(a_1x + b_1y + c_1z)^r}{2s(2s_1)^{r-1}} \geq \frac{(4\sqrt{FF_1})^r}{ss_1^{r-1}2^r} \frac{s_1^{r-2}}{ss_1^{r-1}}. \]

The given inequality corresponds to the case \( r = 2 \).

Also solved by SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; COLIN SPRINGER, student, University of Waterloo; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.

Bilchev and Velikova also obtained the stronger inequality (8) in Janous' second solution.

* * *


If \( p, q, r \) are the real roots of

\[ x^3 - 6x^2 + 3x + 1 = 0, \]

determine the possible values of

\[ p^2q + q^2r + r^2p \]
and write them in a simple form.

*Solution by Sam Baethge, Science Academy, Austin, Texas.*

Let

\[ A = p^2q + q^2r + r^2p, \quad B = p^2r + q^2p + r^2q, \]

the only two possible values of expressions of the given type. We also have

\[
\begin{align*}
p + q + r &= 6, \\
pq + qr + rp &= 3, \\
pqr &= -1.
\end{align*}
\]

In the equations that follow, all summations are symmetric over \( p, q \) and \( r \).

(i) \[ 18 = (p + q + r)(pq + qr + rp) = \sum p^2q + 3pqr = \sum p^2q - 3 \]

or

\[ A + B = \sum p^2q = 21. \]

(ii) \[ 216 = (p + q + r)^3 = \sum p^3 + 3 \sum p^2q + 6pqr \]

or

\[ \sum p^3 = 216 - 3(21) - 6(-1) = 159. \]

(iii) \[ 27 = (pq + qr + rp)^3 = \sum p^3q^3 + 3 \sum p^2q^2r + 6p^2q^2r^2 \]

or

\[ \sum p^3q^3 = 27 - 6(1) - 3pqr \sum p^2q = 21 - 3(-1)(21) = 84. \]

(iv) \[ AB = \sum p^4qr + \sum p^3q^3 + 3p^2q^2r^2 = pqr \sum p^3 + 84 + 3 \]

\[ = (-1)(159) + 87 = -72. \]

Using (i) and (iv), \( A \) and \( B \) are the roots of

\[ y^2 - 21y - 72 = 0, \]

so the possible values are 24 and -3.

*Also solved by FRANCISCO BELLOT ROSADO, Emilio Ferrari High School, and MARIA ASCENSION LOPEZ CHAMORRO, Leopoldo Cano High School, Valladolid, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Plushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SIDNEY KRAVITZ, Dover, New Jersey; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; G.R. VELDKAMP, De Bilt, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer. There were two partial solutions submitted.*
Some solvers, including the proposer, answered the problem for an arbitrary cubic.


Let \( A_1A_2A_3 \) be an acute triangle with circumcenter \( O \). Let \( P_1, Q_1 (Q_1 \neq A_1) \) denote the intersection of \( A_1O \) with \( A_2A_3 \) and with the circumcircle, respectively, and define \( P_2, Q_2, P_3, Q_3 \) analogously. Prove that

(a) \( \frac{OP_1 \cdot OP_2 \cdot OP_3}{P_1Q_1 \cdot P_2Q_2 \cdot P_3Q_3} \geq 1; \)

(b) \( \frac{OP_1}{P_1Q_1} + \frac{OP_2}{P_2Q_2} + \frac{OP_3}{P_3Q_3} \geq 3; \)

(c) \( \frac{A_1P_1 \cdot A_2P_2 \cdot A_3P_3}{P_1Q_1 \cdot P_2Q_2 \cdot P_3Q_3} \geq 27. \)

Solution by Colin Springer, student, University of Waterloo.

We write \( |OP_1| \) for \( OP_1 \) etc. and also write \( |T| \) for the area of the triangle \( T \).

Let

\[
\begin{align*}
\alpha_1 &= \angle A_2OA_3, \\
\alpha_2 &= \angle A_3OA_1, \\
\alpha_3 &= \angle A_1OA_2,
\end{align*}
\]

and, without loss of generality,

\( |OA_1| = |OA_2| = |OA_3| = 1. \)

Then, since \( \Delta A_1A_2A_3 \) is acute,

\[
|OP_1| = \frac{|OP_1|}{|OA_1|} = \frac{|\Delta OA_1A_2|}{|\Delta OA_1A_3|} = \frac{\sin \alpha_1}{\sin \alpha_3 + \sin \alpha_2},
\]

with similar expressions for \( |OP_2| \) and \( |OP_3| \). Thus

\[
|P_1Q_1| = 1 - |OP_1| = \frac{-\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{\sin \alpha_2 + \sin \alpha_3}, \text{ etc.}
\]

Let

\[
\begin{align*}
x &= -\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3, \\
y &= \sin \alpha_1 - \sin \alpha_2 + \sin \alpha_3, \\
z &= \sin \alpha_1 + \sin \alpha_2 - \sin \alpha_3;
\end{align*}
\]

then
\[
\frac{\left| OP_1 \right| \cdot \left| OP_2 \right| \cdot \left| OP_3 \right|}{\left| P_1 Q_1 \right| \cdot \left| P_2 Q_2 \right| \cdot \left| P_3 Q_3 \right|}
= \frac{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{(-\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 - \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 + \sin \alpha_2 - \sin \alpha_3)}
\]
\[
= \frac{y + z}{8xyz}
\]
\[
\frac{\left| OP_1 \right|}{\left| P_1 Q_1 \right|} + \frac{\left| OP_2 \right|}{\left| P_2 Q_2 \right|} + \frac{\left| OP_3 \right|}{\left| P_3 Q_3 \right|} \geq 3
\]
by the A.M.-G.M. inequality. This is (a).

For (b),
\[
\frac{\left| OP_1 \right|}{\left| P_1 Q_1 \right|} + \frac{\left| OP_2 \right|}{\left| P_2 Q_2 \right|} + \frac{\left| OP_3 \right|}{\left| P_3 Q_3 \right|} \geq 3
\]
by the A.M.-G.M. inequality and part (a).

Finally, since
\[
\left| A_1 P_1 \right| = 1 + \left| OP_1 \right| = \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{\sin \alpha_2 + \sin \alpha_3},
\]
we have
\[
\frac{\left| A_1 P_1 \right| \cdot \left| A_2 P_2 \right| \cdot \left| A_3 P_3 \right|}{\left| P_1 Q_1 \right| \cdot \left| P_2 Q_2 \right| \cdot \left| P_3 Q_3 \right|}
\]
\[
= \frac{(\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)^3}{(-\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 - \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 + \sin \alpha_2 - \sin \alpha_3)}
\]
\[
\geq \frac{27 \sin \alpha_1 \cdot \sin \alpha_2 \cdot \sin \alpha_3}{(-\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 - \sin \alpha_2 + \sin \alpha_3)(\sin \alpha_1 + \sin \alpha_2 - \sin \alpha_3)}
\]
\[
\geq 27,
\]
again by the A.M.-G.M. inequality and part (a).

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEI, Flushing, N.Y.; JORG HARTERICH, Winnenden, Federal Republic of Germany; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; D.J. SMEENK, Zaltbommel, The Netherlands; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.

Janous' solution to part (c) reduced it to Crux 1199 [1988: 87].

* * *
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