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CONTENTS

Editor's Comment on an Article in Crux .......................... 308
The Olympiad Corner: 90 ............................................. 308
Problems: 1291-1300 ................................................. 319
Solutions: 1109, 1166-1172 ....................................... 322
Index to Volume 13, 1987 ............................................ 339

- 307 -
EDITOR'S COMMENT ON AN ARTICLE IN CRUX

The paper "A problem on lattice points", by B. Leeb and C. Stahlke, which appeared in the April 1987 issue [1987: 104-106], contained the result that among any 19 points in \( \mathbb{Z}^3 \) there are always three with integer centroid. I have since been informed that this result was already proved by H. Harborth in 1973 (Ein Extremalproblem für Gitterpunkte, J. Reine Angew. Math. 262/263, 356-360) and had also appeared as problem 6298 in the Amer. Math. Monthly (solution on pp.279-280 of the April 1982 issue). A more general problem, with references, occurs as problem #93 in W. Moser's Research Problems in Discrete Geometry.

I thank those readers, especially W. Moser, who brought these facts to my attention, and apologize for not including them in the published paper. Nevertheless, I hope that the paper's intended purpose, to illustrate several combinatorial techniques for solving problems, was served.

* * *

THE OLYMPIAD CORNER: 90

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this column with some further problems forwarded by Bruce Shawyer that were posed but not used for the 1987 I.M.O. I apologize for any errors in translation or resulting from my editing of the problems.

Belgium 1. Twenty-eight random draws are made from the set
\[ \{1,2,3,4,5,6,7,8,9,A,B,C,D,J,K,L,U,X,Y,Z\} \]
containing 20 elements. What is the probability that the sequence
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occurs in that order in the chosen sequence?

Bulgaria 1. A perpendicular dropped from the centre of the circumcircle to the bisector of the angle C of the triangle ABC divides it in the ratio \( \lambda \). Determine the length of the third side of the triangle if \( AC = b \) and \( BC = a \).

[Editor's note: I hope that the circumcircle is the correct circle.]
Bulgaria 2. Let $P_1, P_2, \ldots, P_n$ be $n$ points in the plane. What is the smallest number of segments $P_iP_j$ to join and how should they be selected so that whenever 4 points are chosen a triangle is formed by three of them (with edges amongst the selected segments)?

Bulgaria 3. Find all whole number solutions of the equation

$$\lceil n\sqrt{2} \rceil = 2 + m\sqrt{2}.$$  

(Of course $\lceil x \rceil$ means the integer part of $x$.)

Bulgaria 4. Let $P_1, \ldots, P_{2n+3}$ be $2n+3$ points selected in the plane so that no four lie on a circle (or on a straight line). Let $k$ be the number of circles (and straight lines) containing three of the points and which partition the remaining $2n$ points equally, with $n$ on each side. Show that

$$k > \frac{1}{\pi} \binom{2n+3}{2}$$

where the binomial coefficient $\binom{2n+3}{2} = \frac{(2n+3)(2n+2)}{2}$.

Finland 1. Let $A$ be an infinite set of integers such that every $a \in A$ is the product of at most 1987 prime numbers. Prove that there is an infinite set $B \subset A$ and a number $p$ such that the greatest common divisor of any two numbers in $B$ is $p$.

France 1. For each whole number $k > 0$ let $a_k^1 \ldots a_k^k$ ($a_k^k \neq 0$) denote the decimal representation of $(1987)^k$. (Thus $n_0 = 0$ and $a_{n_0}^0 = 1$ since $(1987)^0 = 1$. Also, $n_1 = 3$, $a_1^1 a_1^2 a_1^3 a_1^4 = 1987$, etc.) Form the infinite decimal

$$x = 0.1 \ 1987 \ a_2^1 \ldots a_2^2 \ldots a_k^1 \ldots a_k^k \ldots$$

Show that $x$ is irrational.

Poland 1. Let $ABC$ be a fixed non-equilateral triangle with the vertices listed anticlockwise. Find the locus of the centroids of those equilateral triangles $A'B'C'$ (the vertices listed anticlockwise) for which the points $A$, $B'$, $C'$, (respectively $A'$, $B$, $C$ and $A'$, $B'$, $C$) are collinear.

U.S.S.R. 1. The positive quantities $\alpha$, $\beta$ and $\gamma$ are such that $\alpha + \beta + \gamma < \pi$. Prove that a triangle can be formed from segments of length $\sin \alpha$, $\sin \beta$, and $\sin \gamma$ such that the area of the triangle does not exceed $(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)/8$.

U.S.S.R. 2. For each natural number $k \geq 2$ the sequence $a_n(k)$ is generated according to the rule
\[ a_0 = k, \quad a_n = \tau(a_{n-1}) \quad n = 1, 2, 3, \ldots, \]

where \( \tau(a) \) is the number of positive integral divisors of \( a \). Find all \( k \) for which the sequence \( a_n(k) \) does not contain squares of whole numbers.

**U.S.S.R. 3.** Find the largest value of the expression
\[
(a + b)^4 + (a + c)^4 + (a + d)^4 + (b + c)^4 + (b + d)^4 + (c + d)^4
\]
where \( a, b, c \) and \( d \) are real numbers satisfying
\[
a^2 + b^2 + c^2 + d^2 \leq 1.
\]

\[ \star \quad \star \quad \star \]

We now turn to solutions to problems posed in past issues of the Corner.

**14.** [1985: 38] Proposed by Mongolia.

Show that there exist distinct natural numbers \( n_1, n_2, \ldots, n_k \) such that
\[
\pi^{-1984} < 25 - \left[ \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} \right] < \pi^{-1960}.
\]

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let \( a \) and \( b \) be given real numbers, \( 0 < a < b \). We choose an integer \( n \) such that \( n > 1/a \) and \( n > 1/(b - a) \). Since \( 1/n \leq a \) and the harmonic series diverges, there is a largest non-negative integer \( x \) such that
\[
\sum_{i=0}^{x} \frac{1}{n + i} \leq a.
\]

Using \( n > 1/(b - a) \), it follows that
\[
\frac{x+1}{n + 1} \leq a + \frac{1}{n + x + 1} < a + \frac{1}{n} \leq b.
\]

In our problem, we set \( a = 25 - \pi^{-1960} \), \( b = 25 - \pi^{-1984} \) and choose \( n \) appropriately; then, with \( x \) as defined above, the \( x + 2 \) distinct natural numbers \( n, n + 1, \ldots, n + x, n + x + 1 \) solve the problem.


Let \( a \) and \( b \) be integers. Is it possible to find integers \( p \) and \( q \) such that the integers \( p + na \) and \( q + nb \) are relatively prime for any integer \( n \)?

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If \( a = 0 \) we set \( p = q = 1 \) and then \( \gcd(p + na, q + nb) = 1 \), for all \( n \). Similarly, we set \( p = q = 1 \) if \( b = 0 \). So we may assume \( ab \neq 0 \).
Let 

\[ r = \frac{\text{lcm}(a,b)}{a}, \quad s = \frac{\text{lcm}(a,b)}{b}. \]

Then \( \gcd(r,s) = 1 \) so there exist integers \( x, y \) such that \( rx - sy = 1 \). Set \( p = x, q = y \). For any integer \( n \), define \( d_n = \gcd(p + na, q + nb) \). Then 

\[ d_n | r(p + na) - s(q + nb), \]

so 

\[ d_n | (rp - sq) + n(ra - sb). \]

Also 

\[ d_n | (rx - sy) + n(\text{lcm}(a,b) - \text{lcm}(a,b)). \]

Hence 

\[ d_n | 1. \]

Thus \( \gcd(p + na, q + nb) = 1 \) for all integers \( n \).

22. [1985: 38] Proposed by the U.S.A.

Determine all pairs \((a, b)\) of positive real numbers with \( a \neq 1 \) such that 

\[ \log_a b < \log_{a+1} (b + 1). \]

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Define 

\[ f(x) \equiv \frac{\ln x}{\ln(x + 1)} \quad \text{for} \quad x > 0. \]

Then 

\[ f'(x) = \frac{\ln(x + 1) - \frac{\ln x}{x + 1}}{(\ln(x + 1))^2} = \frac{(x + 1) \ln(x + 1) - x \ln x}{x(x + 1)(\ln(x + 1))^2}. \]

Since, for \( x > 0 \),

\( (x + 1)\ln(x + 1) > x \ln(x + 1) > x \ln x, \)

\( f'(x) > 0 \) for \( x > 0 \). Thus \( f \) is increasing for \( x > 0 \). Thus, if \( x, y > 0 \), \( f(x) > f(y) \) iff \( x > y \).

The inequality \( \log_a b < \log_{a+1} (b + 1) \) is equivalent to 

\[ \frac{\ln b}{\ln a} < \frac{\ln(b + 1)}{\ln(a + 1)}. \tag{\text{(*)}} \]

If \( a > 1 \), we multiply (\text{(*)}) by the positive quantity \( \frac{\ln a}{\ln(b + 1)} \) to obtain the equivalent inequality 

\[ f(b) < f(a), \quad \text{or} \quad b > a. \]

If \( 0 < a < 1 \), we again multiply (\text{(*)}) by \( \frac{\ln a}{\ln(b + 1)} \), but now obtain \( f(b) > f(a) \), which is equivalent to \( b > a \). Therefore, the set of all admissible pairs is 

\[ \{(a,b): a > b > 0, a > 1\} \cup \{(a,b): a < b, 0 < a < 1\}. \]

The proper divisors of the natural number \( n \) are arranged in increasing order, \( x_1 < x_2 < \ldots < x_k \). Find all numbers \( n \) such that

\[ x_5^2 + x_6^2 - 1 = n. \]


Find all positive integers \( n \) such that

\[ n = d_1^2 + d_2^2 - 1, \]

where \( 1 = d_1 < d_2 < \ldots < d_k = n \) are all positive divisors of the number \( n \).

Solution by John Moruay, Dallas, Texas, U.S.A. and R.E.W.

The two problems have the same solution since for the former \( x_1, x_2, \ldots, x_k \) lists only the proper divisors of \( n \). We present the solution in the notation of 24 [1985: 39].

From \( x_5^2 + x_6^2 - 1 = n \) and \( x_5 | n, x_6 | n \) we conclude that \( x_5 \) and \( x_6 \) are coprime and \( n = tx_5x_6 \). Transforming the equation we have

\[ (x_5 - 1)(x_5 + 1) = n - x_5^2 = x_6(tx_5 - x_6) \]

from which we observe that either \( x_6 = x_5 + 1 \) or \( x_6 \) has prime divisors \( p, q \) (possibly equal) dividing \( x_5 - 1 \) and \( x_5 + 1 \), respectively. The argument breaks down into two main cases.

Case I. \( x_6 = ab \) with \( 1 < a, b < x \) and \( \gcd(a,b) = 1 \);

Case II. \( x_6 = p^a \) for some prime \( p \).

We show first that Case I is not possible. Then in Case II we shall find two solutions. In both, \( x_6 = x_5 + 1 \), but we have no easy direct justification of this fact. In each case we consider several subcases.

Case I. Let \( x_6 = ab \) with \( 1 < a, b < x_6 \) and \( \gcd(a,b) = 1 \). Note that \( a, b < x_5 \).

Subcase I.1. \( x_5 \) is not a prime power. Write \( x_5 = cd \) with \( 1 < c, d \) and \( \gcd(c,d) = 1 \). Now \( a, b, c, d \) are pairwise mutually coprime and \( \{x_1, x_2, x_3, x_4\} = \{a, b, c, d\} \). Now we cannot have \( c \geq a, b \) and \( d \geq a, b \) (for otherwise \( x_5 \geq x_6 \)). Without loss, let \( c < a \). Then \( cb < x_6 \) is a divisor of \( n \) which doesn't divide \( x_5 \). However, it is not among \( a, b, c, d \), showing this subcase cannot arise.

Subcase I.2. \( x_5 = q^a, \) \( q \) a prime. If both \( a \) and \( b \) are prime we must have \( a = 3 \). Then \( q < a \) or \( q < b \). Without loss \( q < a \). Then \( qb \) is a divisor of \( n \) which must occur among \( \{x_1, x_2, x_3, x_4\} \), which is impossible as \( \{a, b, q, q^2\} \) exhaust the list.
Thus we may assume that $a$ is not prime, and write $a = a_1 a_2$ with $1 < a_1 \leq a_2$. (Note we allow $a_1 = a_2$.) Then $a_1 b < x_6$ so, as above, $a_1, a_2b$ are distinct divisors of $n$ which make up $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ in some order. Then $a = 1$ and $b, a_1, a_2$ are prime and $a_1 = a_2$.

Now $x_6 = q$ is prime and $q > a_1 b$ so $q$ is odd. Also $x_6$ divides $(x_6 - 1)(x_6 + 1)$ so $x_6 = k x_6 + 1$ or $x_6 = k x_6 - 1$. Now

$$a_1^2 b, a_1 b < x_6 = q < x_6 = a_2^2 b = kq + 1$$

implies that $kq \leq a_1 q, bq$ whence $k \leq a, b < q$. From this it is easy to see that $k = 1$ and $x_6 = q + 1$ is the only possibility for $kq + 1 = x_6$ to divide $q^2 - 1$.

First suppose $(kq - 1) \ell = q^2 - 1$. Then $(k\ell - q)q = \ell - 1$ whence $k\ell = q$, $\ell = 1$ and $k = q$, which contradicts $k < q$; or $q|\ell - 1$, which is impossible unless $k = 1$ (else $kq - 1 \geq q$, $\ell > q$ which gives $(kq - 1)\ell > q^2$). However $k = 1$ is ruled out since $x_6 = x_6 - 1$ violates the order of $x_6$ and $x_6$. Next suppose $(kq + 1)\ell = q^2 - 1$. This gives $(k\ell - q)q = -(\ell + 1)$ which yields $k\ell = q$ and $\ell = -1$, $k = q$, an impossibility, or $q|\ell + 1$ whence $\ell = q - 1$, $b = 1$.

Now $x_6 = q + 1$, $q$ odd means that $2|x_6$. But $x_6$ odd and $x_6$ even entails from $(x_6^2 - 1) + x_6^2 = n$ that $4 < x_6$, so $a_1 = 2$ and $q = 4b - 1$. Now, however,

$$(4b - 1)^2 + (4b)^2 - 1 = n$$

gives $8|n$. Since $b$ is an odd prime we have $8 \leq 4b$, contradicting that $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is $a_1$, $a_1^2$, $b$, $a_1 b$, $a_1^2 b - 1$, $a_1^2 b$ in some order.

This finally shows that Case I cannot arise.

We now turn to

Case II. $x_6 = p^\alpha$, $p$ a prime.

Now $p^\alpha|(x_6 - 1)(x_6 + 1)$ implies $p^\alpha = x_6 + 1$ (since $p^\alpha \geq x_6 + 1$) unless $p = 2$.

Subcase II.1. $p = 2$.

We now consider the possible values of $\alpha$. Clearly $\alpha \leq 5$.

Subcase II.1(i). $\alpha = 5$. Then $\{x_1, x_2, x_3, x_4\} = 2, 4, 8, 16$ and $x_6$ is a prime, $16 < x_6 < 32$. Now $x_6$ divides $x_6^5 - 1 = 31 \cdot 33$. This gives $x_6 = 31$. Indeed $1984 = 31 \cdot 64$ is a solution.

Subcase II.1(ii). $\alpha \leq 4$. Then $x_6$ is not prime. The only composite numbers less than 16 not divisible by 2 are 9 and 15. This implies $\alpha = 4$.

However $x_6 = 9$ gives 7 divisors $2, 3, 4, 6, 8, 9, 16$ and $x_6 = 15$ gives 9 divisors, $2, 3, 4, 6, 8, 10, 12, 15, 16$.

The remaining possibility is

Subcase II.2. $x_6 + 1 = x_6 = p^\alpha$, where $p$ is an odd prime. Then

$$n = p^{2\alpha} - 2p^\alpha + p^2 = 2p^\alpha(p^\alpha - 1).$$

Thus 4 divides $n$. Therefore $\alpha \leq 3$. 
Subcase II.2(i). \( \alpha = 1 \). Then \( x_6 = p \) is prime. Write \( x_6 = p - 1 = 2^\beta c \) where \( c \) is odd. Now if \( c > 1 \) we either have \( c = c_1c_2 \) with \( c_1 \leq c_2 \), leaving no place for \( 2c_1 \) among \( 2, 4, c_1, c \) or we find \( \beta = 3 \) and no place for \( 2c \) among \( 2, 4, 8, c \), again impossible. Thus \( c = 1 \) and \( \beta = 5 \). But \( 2^6 + 1 = 33 \) which is not prime.

Subcase II.2(ii). \( \alpha = 2 \). Now \( p > 5 \) implies \( 2p \) and \( 4p \) are less than \( p^2 \), giving 5 divisors \( 2, 4, p, 2p, 4p \) less than \( x_1 = p^2 - 1 \). Thus \( p = 3 \). Now 144 is seen to be a solution with divisors \( 2, 3, 4, 6, 8, 9, \ldots \).

Subcase II.2(iii). \( \alpha = 3 \). Clearly \( 2p \) is not among \( 2, 4, p, p^2 \) and this case is impossible.

In summary, the only two values for \( n \) are 144 and 1984.

[Editor's note: The above argument is based on a solution submitted by Mr. Morvay, that was flawed by an oversight. I hope that I have not furnished an unduly complicated version.]


Prove that, for every natural number \( n \), the binomial coefficient \( \binom{2n}{n} \) divides the least common multiple of the numbers \( 1, 2, 3, \ldots, 2n \).

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let \( n \) be a given natural number. Let \( P_i, 1 \leq i \leq k \) be all the primes less than or equal to \( 2n \), and for each \( i \), let \( a_i \) be the largest integer \( s \) such that \( P_i^s \leq 2n \). Then

\[
\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \prod_{i=1}^{k} \left( \sum_{j=1}^{a_i} \left( \left\lceil \frac{2n}{P_i^j} \right\rceil - 2\left\lfloor \frac{n}{P_i^j} \right\rfloor \right) \right)
\]

where \( \lceil x \rceil \) is the greatest integer in \( x \).

Also

\[
\text{lcm}(1, 2, \ldots, 2n) = \prod_{i=1}^{k} P_i^{a_i}.
\]

Thus it suffices to prove

\[
\sum_{j=1}^{a_i} \left( \left\lceil \frac{2n}{P_i^j} \right\rceil - 2\left\lfloor \frac{n}{P_i^j} \right\rfloor \right) \leq a_i.
\]

Now let \( x \) be an arbitrary non-negative real number; then \( x = \lfloor x \rfloor + \{x\} \) with \( 0 \leq \{x\} < 1 \). Hence

\[
\left\lceil 2\lfloor x \rfloor \right\rceil - 2\lfloor x \rfloor = \left\lfloor 2\lfloor x \rfloor + 2\{x\} \right\rfloor - 2\lfloor x \rfloor
\]

is 0 or 1. Therefore

\[
\sum_{j=1}^{a_i} \left( \left\lceil \frac{2n}{P_i^j} \right\rceil - 2\left\lfloor \frac{n}{P_i^j} \right\rfloor \right) \leq \sum_{j=1}^{a_i} 1 = a_i,
\]

as required.

Given are a circle $\Gamma$ and a line $\ell$ tangent to it at $B$. From a point $A$ on $\Gamma$, a line $AP\ell$ is constructed, with $P \in \ell$. If the point $M$ is symmetric to $P$ with respect to $AB$, determine the locus of $M$ as $A$ ranges on $\Gamma$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let $O$ be the center of the circle. From isosceles triangle $\triangle AOB$, we have

$\angle BAO = \angle ABO$

$= \angle PAB$ (since $OB \parallel AP$)

$= \angle BAM$ (by construction of $M$).

Hence $A, O, M$ are collinear, and since $\angle APB = \angle AMB = 90^\circ$, also $\angle OMB = 90^\circ$.

The locus of points $M$ with $\angle OMB = 90^\circ$ is the circle with $OB$ as diameter, and so the desired locus lies on this circle.

Conversely, if $M$ lies on this circle, $\angle OMB = 90^\circ$. Let $A'$ be either point of intersection of the line through $OM$ with $\Gamma$, and let $P'$ be such that $A'P' \ell$. Then $\angle BMA' = \angle BMO = 90^\circ$ and

$\angle BAN = \angle BA'N$ (by construction of $A'$)

$= \angle BA'O$ (from isosceles $\triangle AA'O$)

$= \angle P'A'B$ ($OB \parallel A'P'$).

Also, $A'B = A'B$ so $\angle A'B \equiv \angle A'BA'B$ and $N$ is the reflection of $P'$ in $A'B$.

Thus the locus is the entire circle on $OB$ as diameter.


Let $m$ and $n$ be nonzero integers. Prove that $4mn - m - n$ can be a square infinitely often, but that this is never a square if either $m$ or $n$ is positive.

Solution by Daniel Ropp, Washington, St. Louis, MO, U.S.A.

For any integer $k$, let $m = 2k - 5k, n = -1$. Then

$4mn - m - n = 4(5k^2 - 2k) + (5k^2 - 2k) + 1 = (5k - 1)^2$

and so $4mn - m - n$ can be square infinitely often.

Suppose now that $mn \neq 0, n > 0$ and $4mn - m - n = k^2$, a square. Then

$k^2 = m(4n - 1) - n \equiv -n \mod(4n - 1)$

and

$(2k)^2 \equiv -4n \equiv -1 \mod(4n - 1)$.

Now $4n - 1$ is odd and positive and so it cannot have prime factors only of the form $4k + 1$. Thus, there is an integer $s$ and a prime $p$, with $p = 4s - 1$ and
If \( p \mid 4n - 1 \). Then
\[
(2k)^2 \equiv -1 \mod p,
\]
and so \((-1/p) = 1\), where \((a/p)\) is the Legendre symbol. But then
\[
1 = (-1/p) \equiv (-1)^{(p-1)/2} \mod p \equiv (-1)^{2s-1} \mod p.
\]
Thus, \( 2 \equiv 0 \mod p \). This is a contradiction. Hence \( n \), and similarly \( m \), cannot be positive.


You start with \( a \) white balls and \( b \) black balls in a container and proceed as follows:

**Step 1.** You draw one ball at random from the container (each ball being equally likely). If the ball is white, then stop.

**Step 2.** If the drawn ball is black, then add two black balls to the balls remaining in the container and repeat Step 1.

Let \( s \) denote the number of draws until stop. For the cases \( a = b = 1 \) and \( a = b = 2 \) only, determine
\[
a_n = \Pr(s = n), \quad b_n = \Pr(s < n), \quad \lim_{n \to \infty} b_n,
\]
and the expectation \( E(s) = \sum_{n \geq 1} n a_n \).

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

The procedure stops after the \( n \)th draw if the \( n \)th draw is a white ball while each preceding draw is a black ball. By the instructions of Step 2, for \( 1 \leq k \leq n - 1 \), there are \( a \) white balls and \( b + k \) black balls in the container after the \( k \)th draw. Hence
\[
a_n = \Pr(a = n) = \frac{a}{a + b + n - 1} \sum_{j=1}^{n-1} \frac{b + j - 1}{a + b + j - 1}
\]
\[
= \frac{a \left\{ \frac{a + b - 1}{b - 1} \right\}}{(a + 1) \left\{ \frac{a + b + n - 1}{a + 1} \right\}}
\]
where, as usual, the binomial coefficient
\[
\binom{m}{k} = \frac{m!}{k!(m-k)!}.
\]

(i) For \( a = b = 1 \),
\[
a_n = \frac{1}{n(n + 1)}
\]
\[
b_n = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k + 1} \right) = 1 - \frac{1}{n + 1}
\]
and
\[
\lim_{n \to \infty} b_n = 1.
\]
Also \(E(s) = \sum_{n=1}^{\infty} \frac{1}{n + 1} = \infty.\)

(ii) For \(a = b = 2,\)
\[
a_n = \frac{12}{(n + 1)(n + 2)(n + 3)},
\]
\[
b_n = \sum_{k=1}^{n} \frac{12}{(k + 1)(k + 2)(k + 3)}.
\]
\[
= \left[3 - \frac{6}{n + 2}\right] - \left[2 - \frac{6}{n + 3}\right] + \frac{6}{(n + 2)(n + 3)},
\]
\[
\lim_{n \to \infty} b_n = 1, \text{ of course, and}
\]
\[
E(s) = \sum_{n=1}^{\infty} n \frac{12}{(n + 2)(n + 3)} = -3 + 6 = 3.
\]


Let \(P\) be a convex \(n\)-gon with equal interior angles, and let \(\ell_1, \ell_2, \ldots, \ell_n\) be the lengths of its consecutive sides. Prove that a necessary and sufficient condition for \(P\) to be regular is that
\[
\frac{\ell_1}{\ell_2} + \frac{\ell_2}{\ell_3} + \ldots + \frac{\ell_n}{\ell_1} = n.
\]

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If \(P\) is regular, \(\ell_i = \ell_j, 1 \leq i, j \leq n\) so
\[
\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} = \sum_{i=1}^{n} 1 = n \quad \text{(where } \ell_{n+1} = \ell_1).\]

Conversely, suppose \(\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} = n.\) By the AM-GM inequality
\[
\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} \geq n \left( \frac{\prod_{i=1}^{n} \ell_i}{\ell_{i+1}} \right)^{1/n} = n
\]
with equality just in case \(\frac{\ell_i}{\ell_{i+1}} = \frac{\ell_j}{\ell_{j+1}}\) for \(1 \leq i, j \leq n.\) Since equality does hold, by assumption,
\[
\left( \frac{\ell_i}{\ell_{i+1}} \right)^n = \prod_{j=1}^{n} \frac{\ell_i}{\ell_{i+1}} = 1 \quad \text{or } \ell_i = \ell_{i+1}, 1 \leq i \leq n.
\]
Hence all the \(\ell_i\) are equal, and since the interior angles of \(P\) are equal, \(P\) is regular.
75. [1985: 105] Proposed by the U.S.A.

Inside triangle $ABC$, a circle of radius 1 is externally tangent to the incircle and tangent to sides $AB$ and $AC$. A circle of radius 4 is externally tangent to the incircle and tangent to sides $BA$ and $BC$. A circle of radius 9 is externally tangent to the incircle and tangent to sides $CA$ and $CB$. Determine the inradius of the triangle.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let $O$ be the incenter; $O_A$ the center of the circle tangent to $AB$, $AC$, and the incircle, $P$ and $Q$ the feet of the altitudes from $O_A$ and $O$, respectively, to $AC$, and $R$ the foot of the perpendicular from $O_A$ to $OQ$.

Since $O_A$ and $O$ are each equidistant from $AB$ and $AC$, $O_A$, $O$, and $A$ are collinear. Let $s$ denote the semiperimeter of $\triangle ABC$. Then it is known (and easy to prove) that

(i) $AQ = s - a$

and

(ii) $r^2 s = (s - a)(s - b)(s - c)$

where $r$ is the inradius of $\triangle ABC$. Let $r_A$, $r_B$, and $r_C$ denote the radii of the circles tangent to the incircle and to the two sides of $\triangle ABC$ passing through $A$, $B$, $C$, respectively. Then

$$O_A P = r_A, \quad OR = OQ - RQ = r - r_A, \quad OO_A = r + r_A,$$

and

$$O_A R = [(OQ_A)^2 - (OR)^2]^\frac{1}{2} = 2\sqrt{r_A}.$$

By similar triangles $\triangle APO_A$ and $\triangle AOQ$,

$$\frac{r_A}{r} = \frac{AP}{AQ} = \frac{AQ - O_A R}{AQ} = 1 - \frac{2\sqrt{r_A}}{s - a}.$$

(Note $OR > 0$ just in case $r > r_A$.) Therefore
Similarly
\[ s - a = \frac{2\sqrt{r_A}}{1 - r_A}, \]
\[
\begin{aligned}
\text{Similarly}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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1291. Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Evaluate
\[
\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} \, dx.
\]

1292. Proposed by Jack Garfunkel, Flushing, N.Y.

It has been shown (see Crux 1083 [1987: 96]) that if A, B, C are the angles of a triangle,
\[
\frac{2}{\sqrt{3}} \sum \sin A \leq \sum \cos \left( \frac{B - C}{2} \right) \leq \frac{2}{\sqrt{3}} \sum \cos \frac{A}{2},
\]
where the sums are cyclic. Prove that
\[
\sum \cos \left( \frac{B - C}{2} \right) \leq \frac{1}{\sqrt{3}} \left( \sum \sin A + \sum \cos \frac{A}{2} \right),
\]
which if true would imply the right hand inequality above.


Solve the following "twin" problems (in both problems, O is the center of the circle and OAAB).

(a) In Figure (a), AB = BC and \( \angle ABC = 60^\circ \). Prove CD = OA\(\sqrt{3}\).

(b) In Figure (b), OA = BC and \( \angle ABC = 30^\circ \). Prove CD = AB\(\sqrt{3}\).


Find a necessary and sufficient condition on a convex quadrangle \( AB\!CD \) in order that there exist a point \( P \) (in the same plane as \( AB\!CD \)) such that the areas of the triangles \( P\!AB, P\!BC, P\!CD, P\!DA \) are equal.

Let $A_1A_2A_3$ be a triangle with $I_1$, $I_2$, $I_3$ the excenters and $B_1$, $B_2$, $B_3$ the feet of the altitudes. Show that the lines $I_1B_1$, $I_2B_2$, $I_3B_3$ concur at a point collinear with the incenter and circumcenter of the triangle.

1296. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let $r_1$, $r_2$, $r_3$ be the distances from an interior point of a triangle to its sides $a_1$, $a_2$, $a_3$, respectively, and let $R$ be the circumradius of the triangle. Prove that

$$a_1r_1^n + a_2r_2^n + a_3r_3^n \leq (2R)^n a_1a_2a_3$$

for all $n \geq 1$, and determine when equality holds.

1297. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

(To the memory of Léo.)

(a) Let $C > 1$ be a real number. The sequence $z_1,z_2,...$ of real numbers satisfies $1 < z_n$ and $z_1 + ... + z_n < Cz_{n+1}$ for $n \geq 1$. Prove the existence of a constant $a > 1$ such that $z_n > a^n$, $n \geq 1$.

(b) Let conversely $z_1 < z_2 < ...$ be a strictly increasing sequence of positive real numbers satisfying $z_n \geq a^n$, $n \geq 1$, where $a > 1$ is a constant. Does there necessarily exist a constant $C$ such that $z_1 + ... + z_n < Cz_{n+1}$ for all $n \geq 1$?

1298. Proposed by Len Bos, University of Calgary, Calgary, Alberta.

Let $A = (a_{ij})$ be an $n \times n$ matrix of positive integers such that $|\det A| = 1$, and suppose that $z_1,z_2,...,z_n$ are complex numbers such that

$$\frac{a_{11}}{z_1} \frac{a_{12}}{z_2} ... \frac{a_{1n}}{z_n} = 1$$

for each $i = 1,2,...,n$. Show that $z_i = 1$ for each $i$.

1299* Proposed by Carl Friedrich Sutter, Viking, Alberta.

Three real numbers $a_1$, $a_2$, $a_3$ are chosen at random from the interval $[0,1]$ such that $\Sigma a_i = 1$. They are then rounded off to the nearest one-digit decimal to form $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$. What is the probability that $\Sigma \bar{a}_i = 1$?


ABC is a triangle, not right angled, with circumcentre $O$ and orthocentre $H$. The line $OH$ intersects $CA$ in $K$ and $CB$ in $L$, and $OK = HL$. Calculate angle $C$.

* * *
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


ABC is a triangle with orthocentre H. A rectangular hyperbola with centre H intersects line BC in A₁ and A₂, line CA in B₁ and B₂, and line AB in C₁ and C₂. Prove that the points P, Q, R, the midpoints of A₁A₂, B₁B₂, C₁C₂, respectively, are collinear.

[Editor's comment. The most interesting solution submitted to this problem, but also the most mysterious, is that of Jordi Dou. Unable to understand it, I asked my eminent colleague Richard K. Guy for help. This he kindly gave, and what follows is Dou's proof with elaborations (including references) by RKG. Can anyone find a less exotic, but still elegant, proof?]

Solution by Jordi Dou, Barcelona, Spain. (Translated and annotated by Richard K. Guy, University of Calgary.)

The locus of the midpoints of a system of parallel chords of a conic is a diameter of the conic. This is true in particular for the pair of asymptotes, so the midpoint of a chord of a hyperbola is also the midpoint of the segment intercepted on the chord by its asymptotes (see Theorem 28(iii), page 60 of [1]). So we may replace the rectangular hyperbola by a pair of perpendicular lines through H, the orthocentre of the triangle ABC.

If such a pair cuts the side BC of the triangle in A₁, A₂, and if P is the midpoint of A₁A₂, then H{A₁A₂; P∞} is a harmonic pencil, and since ∠A₁HA₂ = π/2, HA₁, HA₂ are the bisectors of the angles formed by HP and the parallel through H to BC (Theorem 35(2), page 68 of [2]). Alternatively, consider the circle on A₁A₂ as diameter. Its centre is P, and since ∠A₁HA₂ = π/2, H lies on the circle; ΔA₁PH is isosceles and PH makes twice the angle with BC that HA₁ does. In fact, as the pair of perpendicular lines rotates about H, the ray HP, and similarly the rays HQ, HR, rotate through twice the angle. In particular ∠QHR remains constant.

Consider the perpendicular lines when one is through A and the other is parallel to BC. Then P is at infinity in that direction, and the midpoints Q,
R are on the perpendicular bisector of AH. By reflexion in QR, we see that the constant angle QHR is equal to the angle A of the triangle. Similarly ∠RHP = B, ∠PHQ = C. Notice that the six angles formed at H by the altitudes of ΔABC are A, B, C, A, B, C. The six angles formed by HP, HQ and HR are the same, but in opposite cyclic order.

HQ, HR generate homographic (in fact, congruent) pencils, which intersect CA, AB in homographic ranges. The join, QR, of corresponding points envelops a conic (Theorem 44, Note, page 69 of [3]). Moreover, the constancy of the angle QHR tells us that H is a focus. Similarly RP, PQ envelop the same conic, and PQR is a straight line.

When R is at A, since ∠RHQ = A and ∠QHP = C, P will be at C, and AC is a position of the line. Thus the conic touches the sides of the triangle ABC.

We've seen that the perpendicular bisector of AH touches the conic, and similarly so do the perpendicular bisectors of BH and CH. These three lines form a triangle XYZ congruent to ΔABC and obtained from it by rotation through π about N, the common nine-point centre.

By symmetry, N is the centre of the conic and the second focus is O, the circumcentre of ΔABC (and orthocentre of the rotated triangle). The Euler line of ΔABC is the major axis of the conic, which is an ellipse, a hyperbola, or a degenerate pair of points (O and H), according as ΔABC is acute-angled, obtuse-angled, or right-angled. In the last case, H coincides with the right angle, and O is the midpoint of the hypotenuse.

References:

Also solved by the proposer; and partially by J.T. GROENMAN, Arnhem, The Netherlands*

* * *
Let $A$ and $B$ be positive integers such that the arithmetic progression $\{An + B: n = 0,1,2,\ldots\}$ contains at least one square. If $M^2$ ($M > 0$) is the smallest such square, prove that $M < A + \sqrt{B}$.


Suppose

$$An + B = m^2 \quad (1)$$

where $n \geq 0$, $m > 0$, and suppose that $m > A + \sqrt{B}$. We try to find a smaller solution. If we succeed then for the smallest solution we must have $m < A + \sqrt{B}$.

How do we get this smaller solution? If it is

$$A(n - \lambda) + B = q^2 \quad (2)$$

where $0 < \lambda \leq n$, then from (1) and (2)

$$A\lambda = (m + q)(m - q).$$

We try $A = m - q$, $\lambda = m + q$. Then $\lambda = 2m - A$, and

$$A(n - \lambda) + B = A(n - 2m + A) + B$$

$$= An + B - 2mA + A^2$$

$$= m^2 - 2mA + A^2 = (m - A)^2,$$

so we put $q = m - A$. Since $m > A + \sqrt{B}$, $0 < q < A$. Thus we need only show $n - \lambda \geq 0$. But this is equivalent to

$$n - 2m + A \geq 0,$$

$$An - 2mA + A^2 \geq 0,$$

$$m^2 - B - 2mA + A^2 \geq 0,$$

$$(m - A)^2 \geq B,$$

and finally

$$m - A \geq \sqrt{B},$$

which is true.

II. Solution by Walther Janous, Ursulinengymnastium, Innsbruck, Austria.

From the assumptions follows at once the existence of some integer $x$ such that $x^2 \equiv B \pmod{A}$. As the half-open interval $[\sqrt{B}, \sqrt{B} + A)$ contains $A$ consecutive natural numbers, one of them, say $y$, has to satisfy $y \equiv x \pmod{A}$. Then $y^2 \equiv B \pmod{A}$ and $\sqrt{B} \leq y < A + \sqrt{B}$. Thus $y^2 = B + An$ for some $n \geq 0$, and, as $M \leq y$, we are done.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; KENNETH M. WILKE, Topeka, Kansas; and the proposer.
(Dedicated to Léo Sauvé.)

Determine the greatest real number \( r \) such that for every acute triangle \( ABC \) of area 1 there exists a point whose pedal triangle with respect to \( ABC \) is right-angled and of area \( r \).

Solution by the proposer.

We shall show that the required value of \( r \) is \( 3/2 + \sqrt{3} \).

Consider a point \( U \) such that

(i) \( AU = BU \),

(ii) \( \angle AUB = 90^\circ - C \),

and

(iii) \( U \) and \( C \) are on different sides of \( AB \).

We denote the circumcircle of \( \Delta AUB \) by \( K_c \). Then for \( X \) on \( K_c \),

\[
\angle AXB = \begin{cases} 
90^\circ - C & \text{for } X \text{ on the same side of } AB \text{ as } U, \\
90^\circ + C & \text{otherwise.} 
\end{cases} 
\]  

(1)

Now let \( P \) be a point such that its pedal triangle \( P_bP_cP_a \) is right-angled with \( \angle P_bP_cP_a = 90^\circ \) (this includes the degenerate cases \( P = A \) and \( P = B \)). We shall show that the locus of \( P \) is \( K_c \).

Suppose \( P \) is inside \( \Delta ABC \). Then the quadrilaterals \( AP_bPP_a \) and \( BP_aPP_c \) are cyclic, therefore \( \angle P_bAP = \angle P_bP_cP, \angle P_aBP = \angle P_aP_cP \). Hence

\[
\angle APB = 180^\circ - (\angle PAB + \angle PBA) \\
= 180^\circ - (A - \angle P_bAP + B - \angle P_aBP) \\
= (180^\circ - A - B) + \angle P_bP_cP + \angle P_aP_cP \\
= C + \angle P_bP_cP_a. 
\]

Consequently, by (1), \( \angle P_bP_cP_a = 90^\circ \) if and only if

\[ \angle APB = C + 90^\circ = 180^\circ - \angle AUB, \]

i.e., if and only if \( P \in K_c \). The other case, when \( P \) lies outside \( \Delta ABC \), is quite similar.

Analogously, the locus of the point \( P \) such that \( \angle P_cP_bP_a = 90^\circ \), respectively \( \angle P_aP_bP_c = 90^\circ \), is the circle \( K_a \), respectively \( K_b \), defined similarly. Combining these results, we conclude that \( \Delta P_aP_bP_c \) is right-angled if and only if \( P \in K_a \cup K_b \cup K_c = K \).

Fix an arbitrary acute triangle \( ABC \) of area 1 and consider the function
\( F(P) = \text{area of } \triangle P_a P_b P_c \)

defined for \( P \) on \( K \). It is continuous, non-negative (with \( F(A) = 0 \), and, since \( K \) is compact and connected, its range is an interval \([0, s]\) where \( s \) depends on the triangle \( ABC \). Clearly \( r = \min s \), where the minimum is taken over all acute triangles \( ABC \) of area 1.

Let \( P \in K_c \) and let \( \theta = \angle PAB \). Then by (1),

\[
\frac{BP}{\sin \theta} = \frac{AB}{\sin(90^\circ - C)} = \frac{AP}{\sin(90^\circ + C - \theta)}
\]

if \( P \) is outside \( \triangle ABC \), and

\[
\frac{BP}{\sin \theta} = \frac{AB}{\sin(90^\circ + C)} = \frac{AP}{\sin(90^\circ - C - \theta)}
\]

if \( P \) is inside \( \triangle ABC \), so that

\[
BP = \frac{AB \cdot \sin \theta}{\cos C}
\]

and

\[
AP = \begin{cases} 
  \frac{AB \cdot \cos(C - \theta)}{\cos C} & \text{if } P \text{ is outside } \triangle ABC, \\
  \frac{AB \cdot \cos(C + \theta)}{\cos C} & \text{if } P \text{ is inside } \triangle ABC.
\end{cases}
\]

Remembering that \( AP_c P_b \) is cyclic and \( \angle AP_c P = 90^\circ \), we obtain

\[
P_c P_b = AP \cdot \sin A = \begin{cases} 
  \frac{AB \cdot \sin A \cos(C - \theta)}{\cos C} & \text{if } P \text{ is outside } \triangle ABC, \\
  \frac{AB \cdot \sin A \cos(C + \theta)}{\cos C} & \text{if } P \text{ is inside } \triangle ABC,
\end{cases}
\]

and similarly

\[
P_c P_a = BP \cdot \sin B = \frac{\sin B \sin \theta}{\cos C}.
\]

Hence

\[
F(P) = \frac{P_c P_a \cdot P_c P_b}{2} = \frac{AB^2 \cdot \sin A \sin B \sin \theta \cos \varphi}{2 \cos^2 C}
\]

where

\[
\varphi = \begin{cases} 
  C - \theta & \text{if } P \text{ is outside } \triangle ABC, \\
  C + \theta & \text{if } P \text{ is inside } \triangle ABC.
\end{cases}
\]

Since the area of \( \triangle ABC \) is 1,

\[
1 = \frac{AB \cdot AC \cdot \sin A}{2} = \frac{AB \cdot \sin A, AB \cdot \sin B}{\sin C},
\]

so

\[
F(P) = \frac{\sin C \sin \theta \cos \varphi}{\cos^2 C}
\]
\[
\sin C = \frac{(\sin(\theta + \varphi) + \sin(\theta - \varphi))}{2 \cos^2 \theta}
\]

\[
= \frac{\sin C}{2 \cos^2 \theta} \cdot g(\theta)
\]

where
\[
g(\theta) = \begin{cases} 
\sin C + \sin(2\theta - C) & \text{if } P \text{ is outside } \Delta ABC, \\
\sin(2\theta + C) - \sin C & \text{if } P \text{ is inside } \Delta ABC.
\end{cases}
\]

Thus the maximum value of \(g(\theta)\) is \(1 + \sin C\), attained when \(P\) is outside \(\Delta ABC\) and \(2\theta = C + 90^\circ\). Hence the maximum value of \(F(P)\) for \(P \in K_c\) is
\[
\frac{\sin C(1 + \sin C)}{2 \cos^2 C} = \frac{\sin C}{2(1 - \sin C)} = \frac{1}{2} \left( \frac{1}{1 - \sin C} - 1 \right). \tag{2}
\]

If we assume \(A \leq B \leq C\), then
\[
\max \left\{ \frac{1}{1 - \sin A}, \frac{1}{1 - \sin B}, \frac{1}{1 - \sin C} \right\} = \frac{1}{1 - \sin C}
\]
so the maximum value of \(F(P)\) over all \(P \in K\) is again (2). It remains to note that \(60^\circ \leq C < 90^\circ\), and therefore by (2)
\[
s \geq \frac{\sin 60^\circ(1 + \sin 60^\circ)}{2 \cos^2 60^\circ} = \frac{3}{2} + \sqrt{3},
\]
equality holding for \(\Delta ABC\) equilateral. Therefore
\[
r = \min s = \frac{3}{2} + \sqrt{3},
\]
attempted when \(\Delta ABC\) is equilateral and \(P\) is outside \(\Delta ABC\) such that \(\theta = 75^\circ\).

There were no other solutions received for this problem, and only one response. It seems to be a calculation of that acute triangle of area 1 for which the pedal triangle of the orthocentre is right-angled and has minimal area. Not the question, but, hey, the editor has an open mind\(^1\). (The answer given, for anyone who wishes to check it, was a minimum area of \((\sqrt{2} - 1)/2\).)

\* \* \* \* \* \* \* 

1168. [1986: 179] Proposed by Herta T. Freitag, Roanoke, Virginia. (Dedicated to Léo Sauvé.)

Let \(S = \sum_{n=1}^{k} F_{(2i-1)n}\) where \(n\) is odd and \(F_n\) denotes a Fibonacci number. Determine a Lucas number \(L_a\) such that \(L_aS\) is a Fibonacci number.

\(^1\)The empty set being open.
Solution by Kenneth M. Wilke, Topeka, Kansas.

We shall show that \( L_n \) satisfies the conditions of the problem and that
\[ L_n S = F_{2kn} \]
where \( S, k, \) and \( n \) are as defined in the problem.

Let \( m \) be an odd integer, \( i \) a positive integer, and \( L_j \) and \( F_j \) denote the
jth Lucas and Fibonacci numbers respectively. Then we claim that
\[ F_{2(i-1)m} + L_m F_{2(i-1)m} = F_{2im}. \]  \hspace{1cm} (1)

Letting
\[ \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \]
it is well known that
\[ F_j = \frac{\alpha^j - \beta^j}{\sqrt{5}}, \quad L_j = \alpha^j + \beta^j. \]

Thus
\[
F_{2(i-1)m} + L_m F_{2(i-1)m} = \frac{\alpha^{2m(i-1)} - \beta^{2m(i-1)}}{\sqrt{5}} + \frac{(\alpha^m + \beta^m)(\alpha^{2(i-1)m} - \beta^{2(i-1)m})}{\sqrt{5}} \\
= \frac{\alpha^{2im} - \beta^{2im}}{\sqrt{5}} + \frac{(\alpha^{2m(i-1)} - \beta^{2m(i-1)})(\alpha^m\beta^m + 1)}{\sqrt{5}} \\
= \frac{\alpha^{2im} - \beta^{2im}}{\sqrt{5}} = F_{2im}
\]
because \( m \) is odd and \( \alpha\beta = -1. \)

Now the formula
\[ L_n S = L_n \sum_{i=1}^{k} F_{2(i-1)n} = F_{2kn} \]
can be proved by induction on \( k. \) For \( k = 1 \) we have
\[ L_n S = L_n F_n = F_{2n} \]
directly or from (1). For the induction step, (1) provides the necessary
argument to show that once we assume the desired result holds for \( k - 1, \) it
must hold for \( k \) also.

Also solved by BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin;
DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

* * *

1169. [1986: 179] Proposed by Andy Liu, University of Alberta, Edmonton,
Alberta; and Steve Newman, University of Michigan, Ann Arbor,
Michigan. [To Léo Sauvé who, like J.R.R. Tolkien, created a
fantastic world.]

(i) The Fellowship of the Ring. Fellows of a society wear rings
formed of 8 beads, with two of each of 4 colours, such that no two adjacent
beads of the same colour. No two members wear indistinguishable rings. What is the maximum number of fellows of this society?

(ii) The Two Towers. On two of three pegs are two towers, each of 8 discs of increasing size from top to bottom. The towers are identical except that their bottom discs are of different colours. The task is to disrupt and reform the towers so that the two largest discs trade places. This is to be accomplished by moving one disc at a time from peg to peg, never placing a disc on top of a smaller one. Each peg is long enough to accommodate all 16 discs. What is the minimum number of moves required?

(iii) The Return of the King. The King is wandering around his kingdom, which is an ordinary 8 by 8 chessboard. When he is at the north-east corner, he receives an urgent summons to return to his summer palace at the south-west corner. He travels from cell to cell but only due south, west, or south-west. Along how many different paths can the return be accomplished?

Solution by the proposers.

(i) We consider more generally rings of two beads of each of \( n \) colours, but for now we allow beads of the same colour to be adjacent, and assume that rotations and reflections of a pattern are considered distinct.

The number of distinct rings is then \( (2n)!/2^n \). There are \( (2n)! \) permutations of the beads, but we must divide by \( 2^n \) to account for the fact that beads of the same colour are indistinguishable.

Let \( A \) denote the set of these \( (2n)!/2^n \) rings. For \( 1 \leq i \leq n \), let \( A_i \) denote the subset of rings where the two beads of the \( i \)th colour are adjacent (with possibly bead pairs of other colours adjacent as well). The number of rings with no adjacent beads of the same colour is then

\[
f(n) = |A| - |A_1 \cup A_2 \cup \ldots \cup A_n|.
\]

We claim that for all choices of \( k \) colours \( i(1), i(2), \ldots, i(k) \),

\[
|A_{i(1)} \cap A_{i(2)} \cap \ldots \cap A_{i(k)}| = \frac{(2n - k - 1)! (2n)}{2^{n-k}}.
\]

Place the first of two adjacent beads of colour \( i(1) \) in any of the \( 2n \) places. For each of the colours \( i(2), \ldots, i(k) \), merge the two beads of that colour into a single one, since they have to be adjacent. The \( k - 1 \) merged beads and the \( 2(n - k) \) other beads can now be placed in \( (2n - k - 1)! \) ways. Finally, we divide by \( 2^{n-k} \) to account for the indistinguishable beads.

Since there are \( \binom{n}{k} \) subsets of \( k \) colours, we have

\[
f(n) = \frac{n}{2^{n-1}} \sum_{k=0}^{n} (-1)^k (2n - k - 1)! \binom{n}{k} 2^k
\]
by inclusion-exclusion. Direct computation yields \( f(4) = 744 \).

Note that each rotationally distinct pattern has been counted \( 2n \) times in \( f(n) \), except for the \((n - 1)!\) patterns in which the two beads of each colour are diametrically opposite to each other. Each of these \((n - 1)!\) patterns is counted \( n \) times. Therefore, the number of rotationally distinct patterns is
\[
g(n) = \frac{f(n) + (n - 1)!n}{2n} = \frac{f(n) + n!}{2n}.
\]

Now each rotationally and reflectionally distinct pattern has been counted twice in \( g(n) \), except for the \( n!/2 \) rotationally distinct patterns which remain unchanged after reflection. Each of these \( n!/2 \) patterns is counted once. Therefore the number of distinguishable rings is given by
\[
g(n) + \frac{n!/2}{2} = \frac{f(n) + (n + 1)!}{4n}.
\]

In particular, with \( n = 4 \), the society has at most \((744 + 120)/16 = 54\) members.

(ii) We consider more generally towers of height \( n \). Let \( f(n) \) denote the minimum number of moves required. In the following diagram, we record the task for \( n = 1 \), so that \( f(1) = 3 \):

For higher values of \( n \), the bottom (largest) discs still eventually have to be moved in this way. Moreover, during these moves, all smaller discs must lie on top of the stationary large disc. This divides the task into four stages.

We suppose that initially the two towers occupy the left and right pegs in the above diagrams. To go from the original configuration to one in which the large disc on the left peg can be moved as in (a), we have to merge the two towers (minus the bottom discs) into a single tower above the large disc on the right peg. To go from the final move of a large disc, as in (c), to the final configuration, we simply reverse this process. To allow the intermediate moves (b) or (c) of a large disc, in each case we have first to transfer a doubled tower from one peg to another.
Let $g(n)$ denote the minimum number of moves required to merge two towers of $n$ discs as described above, and we use $g_1(n)$ if the merged tower stands on the peg not occupied by either tower before the merger. Let $h(n)$ denote the minimum number of moves required to transfer a doubled tower of $2n$ discs from one peg to another. Note that $g(1) = 1$ and $g_1(1) = h(1) = 2$.

Putting the four stages together, we have

$$f(n) = 2g(n-1) + 2h(n-1) + 3.$$  \hspace{1cm} (1)

Similar analysis yields

$$g(n) = g_1(n-1) + h(n-1) + 1,$$ \hspace{1cm} (2)

$$g_1(n) = g(n-1) + 2h(n-1) + 2,$$ \hspace{1cm} (3)

and

$$h(n) = 2h(n-1) + 2.$$ \hspace{1cm} (4)

From (4) we get

$$h(n) = 2^{n+1} - 2.$$ \hspace{1cm} (5)

(This is of course just the familiar Tower of Hanoi problem, but with each disc replaced by two discs of the same size, thus doubling the number of moves usually required.)

Eliminating $g_1(n)$ from (2) and (3), and using (5), we obtain

$$g(n) = g(n-2) + 2h(n-2) + h(n-1) + 3$$
$$= g(n-2) + 2^{n+1} - 3,$$ \hspace{1cm} (6)

and from (2) we also have

$$g(2) = g_1(1) + h(1) + 1 = 2 + 2 + 1 = 5.$$

Inspection of (6) leads us to conjecture that

$$g(n) = A \cdot 2^n + Bn + C$$

for some constants $A$, $B$, $C$, so that

$$A \cdot 2^n + Bn + C = A \cdot 2^{n-2} + B(n-2) + C + 2^{n+1} - 3.$$  

Equating the coefficients, we have $A = 8/3$ and $B = -3/2$. For odd $n$, we have $g(1) = 1$ so that $C = -17/6$. For even $n$, we have $g(2) = 5$ so that $C = -8/3$. Putting everything into (1), we obtain

$$f(n) = \begin{cases} 
\frac{7}{3} \cdot 2^{n+1} - 3n - \frac{10}{3} & n \text{ odd}, \\
\frac{7}{3} \cdot 2^{n+1} - 3n - \frac{11}{3} & n \text{ even}.
\end{cases}$$

In particular, $f(8) = 1167$.

(iii) We consider more generally an $n$ by $n$ board. A path for the King is equivalent to a sequence of $S$'s, $W$'s, and $D$'s, standing respectively for southward, westward, and southwestward moves. Such a sequence must contain
D's, \( n - i - 1 \) S's, and \( n - i - 1 \) W's for some \( 0 \leq i \leq n - 1 \).

The number of such sequences for a particular \( i \) is

\[
\binom{2n - i - 2}{i} \binom{2n - 2i - 2}{n - i - 1}
\]

(we choose \( i \) of the \( 2n - i - 2 \) places for the D's and \( n - i - 1 \) of the remaining \( 2n - 2i - 2 \) places for the S's, with the rest going to the W's).

The total number of such sequences is then

\[
\sum_{i=0}^{n-1} \binom{2n - i - 2}{i} \binom{2n - 2i - 2}{n - i - 1}
\]

For \( n = 8 \), the value is 48639.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

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1170. [1986: 179] Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana. (Dedicated to Léo Sauvé.)

In the plane of triangle \( ABC \), let \( P \) and \( Q \) be points having trilinears \( \alpha_1: \beta_1: \gamma_1 \) and \( \alpha_2: \beta_2: \gamma_2 \), respectively, where at least one of the products \( \alpha_1 \alpha_2, \beta_1 \beta_2, \gamma_1 \gamma_2 \) is nonzero. Give a Euclidean construction for the point \( P \ast Q \) having trilinears \( \alpha_1 \alpha_2: \beta_1 \beta_2: \gamma_1 \gamma_2 \). (A point has trilinears \( \alpha: \beta: \gamma \) if its signed distances to sides \( BC \), \( CA \), \( AB \) are respectively proportional to the numbers \( \alpha, \beta, \gamma \).)


Since we can construct three line segments of lengths proportional to \( \alpha_1: \beta_1: \gamma_1 \) (drop perpendiculars from \( P \) to the sides of \( \triangle ABC \)), and similarly for \( \alpha_2: \beta_2: \gamma_2 \), we can by a familiar construction form three line segments of (signed) lengths \( x, y, z \) proportional to \( \alpha_1 \alpha_2, \beta_1 \beta_2, \gamma_1 \gamma_2 \).

How to find the point \( P \ast Q \)? Draw the line \( l \parallel BC \) at signed distance \( x \) from \( BC \). Analogously construct lines \( m \parallel AC \) and \( n \parallel AB \) at distances \( y \) and \( z \) from \( AC \) and \( AB \) respectively. The lines \( l, m, n \) form a triangle \( \triangle A'B'C' \sim \triangle ABC \). The lines \( AA', BB', \) and \( CC' \) are then concurrent in the point \( P \ast Q \).

II. Remarks by the proposer.

The product \( \ast \) is commutative and associative. Thus \((G, \ast)\) is a group, where \( G \) is the set of all points not on the lines \( BC, CA, AB \). The incenter
(1:1:1) serves as the identity of \((G, \ast)\), and for \(P = (\alpha: \beta: \gamma) \in G\), \(P^{-1} = (\alpha^{-1}: \beta^{-1}: \gamma^{-1})\). \(P^{-1}\) is the isogonal conjugate of \(P\), constructed as follows: reflect line \(AP\) about the internal bisector of angle \(A\), and analogously for \(BP\) and \(CP\). The reflected lines concur at \(P^{-1}\).

Also solved by the proposer.

\[
\ast \ast \ast
\]

1171.\* [1986: 204] Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia. (Dedicated to Léo Sauvé.)

(i) Determine all real numbers \(\lambda\) so that, whenever \(a, b, c\) are the lengths of three segments which can form a triangle, the same is true for

\[(b + c)^\lambda, (c + a)^\lambda, (a + b)^\lambda.
\]

(For \(\lambda = -1\) we have Crux 14 [1975: 28].)

(ii) Determine all pairs of real numbers \(\lambda, \mu\) so that, whenever \(a, b, c\) are the lengths of three segments which can form a triangle, the same is true for

\[(b + c + \mu a)^\lambda, (c + a + \mu b)^\lambda, (a + b + \mu c)^\lambda.
\]

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We solve the general problem, i.e. part (ii).

First of all we note that always \(\mu \geq -1\). Indeed, for \(\mu < -1\) we could in view of the triangle inequality - find a triangle with sides \(a, b, c\) such that \(a + b + \mu c = 0\), say, so that the numbers

\[(b + c + \mu a)^\lambda, (c + a + \mu b)^\lambda, (a + b + \mu c)^\lambda
\]

(1)
could not be the sides of a triangle.

Next, let \(\mu = -1\) and \(\lambda \neq 0\). We consider triangles of sides \(a = 1, b = 2, c = 3 - t\), where \(t \to 0\). If \(\lambda < 0\), then a triangle with sides (1) must satisfy the inequality

\[(a + b - c)^\lambda < (b + c - a)^\lambda + (c + a - b)^\lambda,
\]

that is,

\[t^\lambda < (4 - t)^\lambda + (2 - t)^\lambda,
\]

which is impossible as \(t \to 0\). For \(\lambda > 0\), consider instead

\[(b + c - a)^\lambda < (a + b - c)^\lambda + (c + a - b)^\lambda,
\]
or

\[(4 - t)^\lambda < t^\lambda + (2 - t)^\lambda;
\]
as \(t \to 0\), this also yields a contradiction.

Moreover, if \(\mu = 1\) then all real \(\lambda\) are suitable, as (1) yields an equilateral triangle for all \(a, b, c\).
Thus we assume from now on that $\mu > -1$, $\mu \neq 1$, and put (without loss of generality) $a \leq b \leq c = 1$. We distinguish three cases.

Case I. $\lambda = 0$.

Then all $\mu > -1$ are suitable, since (1) yields equilateral triangles.

Case II. $\lambda > 0$.

We have to study the three possible triangle inequalities

\begin{align*}
(1 + a + \mu b)^\lambda &< (a + b + \mu)^\lambda + (1 + b + \mu a)^\lambda \quad (2) \\
(a + b + \mu)^\lambda &< (1 + a + \mu b)^\lambda + (1 + b + \mu a)^\lambda \quad (3) \\
(1 + b + \mu a)^\lambda &< (a + b + \mu)^\lambda + (1 + a + \mu b)^\lambda \quad (4)
\end{align*}

(2) is always satisfied, as

$$1 + a + \mu b \leq a + b + \mu$$ holds whenever $1 < \mu$ (since $b \leq 1$), and

$$1 + a + \mu b \leq 1 + b + \mu a$$ holds whenever $\mu < 1$ (since $a \leq b$).

For (3) and (4), we put $a + b = s$. Then $1 < s \leq 2$, since $a \leq b \leq c = 1$ and $a$, $b$, $c$ are the sides of a triangle. We first consider (3).

If $\lambda \geq 1$, then by the general mean-inequality applied to means $M_\lambda$ and $M_1$, we get

$$2^{1-1/\lambda} \cdot \frac{2 + s + \mu s}{s + \mu} =: f(s),$$

for $1 < s \leq 2$. Because

$$f'(s) = \frac{(s + \mu)(1 + \mu) - (2 + s + \mu s)}{(s + \mu)^2} = \frac{(\mu + 2)(\mu - 1)}{(s + \mu)^2},$$

we infer that $f$ decreases on $(1,2]$ if $-1 < \mu < 1$ and increases on $(1,2]$ if $\mu > 1$. Thus if $-1 < \mu < 1$ then

$$f(s) \geq f(2) = 2 > 2^{1-1/\lambda},$$

so (5) holds for all $\lambda \geq 1$. If $\mu > 1$ then

$$f(s) > f(1) = \frac{3 + \mu}{1 + \mu},$$

so (5) is true if

$$\frac{3 + \mu}{1 + \mu} \geq 2^{1-1/\lambda},$$

which leads to
If $0 < \lambda < 1$, then for (3) we consider a fixed $s$, $1 < s \leq 2$, and the function

$$g(a) = (1 + a + \mu s - \mu a)^\lambda + (1 + s - a + \mu a)^\lambda, \quad s - 1 \leq a \leq s/2,$$

which is just the right side of (3) with $b = s - a$. Then

$$g'(a) = \lambda (1 - \mu) [(1 + a + \mu s - \mu a)^{\lambda-1} - (1 + s - a + \mu a)^{\lambda-1}]$$

and

$$g''(a) = \lambda (\lambda - 1)(1 - \mu)^2 [(1 + a + \mu s - \mu a)^{\lambda-2} + (1 + s - a + \mu a)^{\lambda-2}].$$

Thus $g''(a) < 0$, i.e. $g$ is concave and attains its minimum value at an endpoint. Since

$$g(s - 1) = (s + \mu)^\lambda + (2 + \mu s - \mu)^\lambda \leq 2 \left[ \frac{2 + \mu s + s}{2} \right]^\lambda = \left[ 1 + \frac{s}{2} + \frac{\mu s}{2} \right]^\lambda + \left[ 1 + \frac{s}{2} + \frac{\mu s}{2} \right]^\lambda = g(s/2)$$

by the general mean-inequality applied to $M_\lambda$ and $M_1$, we have

$$g(a) \geq g(s - 1) > (s + \mu)^\lambda$$

for all $a$, $s - 1 \leq a \leq s/2$, which implies (3).

For inequality (4), if $\mu > 1$ we have

$$1 + b + \mu a \leq a + b + \mu \quad \text{and} \quad 1 + b + \mu a \leq 1 + a + \mu b$$

(since $a \leq b \leq 1$), either one of which implies (4) at once. Now let

$$-1 < \mu < 1 \quad \text{and consider the function}$$

$$h(a) = (1 + s - a + \mu a)^\lambda - (1 + a + \mu s - \mu a)^\lambda$$

for fixed $s$ and $s - 1 \leq a \leq s/2$. Then

$$h'(a) = \lambda (\mu - 1) [(1 + s - a + \mu a)^{\lambda-1} + (1 + a + \mu s - \mu a)^{\lambda-1}] < 0 \quad (6)$$

so $h$ decreases. Thus for (4) we need only consider

$$h(s - 1) < (a + b + \mu)^\lambda,$$

that is,

$$(2 + \mu s - \mu)^\lambda < (s + \mu)^\lambda + (s + \mu)^\lambda = 2(s + \mu)^\lambda,$$

or

$$2^{1/\lambda} > \frac{\mu s + 2 - \mu}{s + \mu} =: k(s), \quad 1 < s \leq 2. \quad (7)$$

Since

$$k'(s) = \frac{(s + \mu)\mu - (\mu s + 2 - \mu)}{(s + \mu)^2} = \frac{(\mu + 2)(\mu - 1)}{(s + \mu)^2} \quad (8)$$

and $-1 < \mu < 1$, $k$ decreases. Thus

$$k(s) < k(1) = \frac{2}{\mu + 1},$$

and (7), hence (4), holds if and only if
\[
\frac{2}{\mu + 1} \leq 2^{1/\lambda},
\]

implying
\[
\lambda \geq \frac{\log 2}{\log[2/(\mu + 1)]}.
\]

**Case III.** \(\lambda < 0\).

We put \(\lambda = -v\), where \(v > 0\), and rewrite inequalities (2), (3), (4) accordingly. Then (2) follows much as it did in Case II.

For (3) and (4), we again let \(a + b = s\), \(1 < s \leq 2\). By the general mean-inequality applied to \(M_v\) and \(M_{-1}\), we get
\[
\left[\frac{1}{1 + a + \mu b}\right]^b + \left[\frac{1}{1 + b + \mu a}\right]^b \geq 2\left[\frac{2}{2 + s + \mu s}\right]^b.
\]

For (3) we thus have to solve the inequality
\[
2\left[\frac{2}{2 + s + \mu s}\right]^b > \frac{1}{(s + \mu)^b},
\]
that is,
\[
2^{1+1/v} > \frac{2 + s + \mu s}{s + \mu} = f(s).
\]

It was already determined that \(f\) decreases on \((1,2]\) if \(-1 < \mu < 1\) and increases on \((1,2]\) if \(\mu > 1\). Thus if \(\mu > 1\) then
\[
f(s) < f(2) = 2,
\]
so (9) holds with no further limitation for \(v\). If \(-1 < \mu < 1\) then
\[
f(s) < f(1) = \frac{3 + \mu}{1 + \mu},
\]
so (9) is true if
\[
\frac{3 + \mu}{1 + \mu} \leq 2^{1+1/v},
\]
yielding
\[
v \leq \frac{\log 2}{\log[(\mu + 3)/(2\mu + 2)]}.
\]
i.e.
\[
\lambda \geq \frac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]}.
\]

For (4), if \(-1 < \mu < 1\) we have
\[
1 + b + \mu a \geq a + b + \mu \quad \text{and} \quad 1 + b + \mu a \geq 1 + a + \mu b
\]
(since \(a \leq b \leq 1\)), either one of which implies (4). Now let \(\mu > 1\), and consider again the function \(h(a)\) defined above for \(s - 1 \leq a \leq s/2\) where \(s\) is fixed. Then from (6), \(h'(a) < 0\) so \(h\) is decreasing on \([s - 1, s/2]\). Hence for (4) we need only consider
\[
h(s - 1) < (a + b + \mu)^\lambda,
\]
i.e.
\[
(2 + \mu s - \mu)^\lambda < (s + \mu)^\lambda + (s + \mu)^\lambda = 2(s + \mu)^\lambda
\]
or

\[ 2^{-1/v} < \frac{\mu s + 2 - \mu}{s + \mu} = k(s). \]  

(10)

From (8), since \( \mu > 1 \), \( k \) increases on \( (1,2] \). Thus

\[ k(s) > k(1) = \frac{2}{\mu + 1}, \]

and (10), hence (4), holds if and only if

\[ \frac{2}{\mu + 1} \geq 2^{-1/v} \]

yielding

\[ v \leq \frac{\log 2}{\log[(\mu + 1)/2]}, \]

i.e.

\[ \lambda \geq \frac{\log 2}{\log[2/(\mu + 1)]}. \]

Summarizing the results from above we get the following possibilities for \( \lambda \) and \( \mu \):

- if \( \mu = -1 \), \( \lambda = 0 \)
- if \( -1 < \mu < 1 \),
  \[ \frac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]} \leq \lambda \leq \frac{\log 2}{\log[2/(\mu + 1)]} \]
- if \( \mu = 1 \), \( \lambda \) is arbitrary
- if \( \mu > 1 \),
  \[ \frac{\log 2}{\log[2/(\mu + 1)]} \leq \lambda \leq \frac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]} \]

Concerning part (i), we get via \( \mu = 0 \) that

\[ -1.7 \approx \frac{\log 2}{\log(2/3)} \leq \lambda \leq 1. \]

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and
MURRAY S. KLAMKIN, University of Alberta.

Klamkin's proof was almost complete, with just one case left open.

Hess supplied no proofs for his answer, but did contribute a picture of the resulting set of ordered pairs \((\mu, \lambda)\):

Every point inside the "cross" works. The "boundary" contains degenerate cases.
Jonous and Klamkin both gave some further generalizations, none of which the editor had the energy to record.

Frankly, the editor wishes he had asked for part (i), and left it at that!

\[ 338 \]


Show that for any triangle \( ABC \), and for any real \( X > 1 \),
\[
\sum (a + b)\sec^{\lambda}(C/2) \geq 4(2/\sqrt{3})^{\lambda}s,
\]
where the sum is cyclic over \( \Delta ABC \) and \( s \) is the semiperimeter.

Solution by Murray S. Klamkin, University of Alberta.

By the power mean inequality,
\[
\frac{\sum (a + b)\sec^{\lambda}(C/2)}{4s} \geq \left[ \frac{\sum (a + b)\sec(C/2)}{\sum (a + b)} \right]^\lambda
\]
for any \( \lambda \geq 1 \). The desired result now follows from the inequality
\[
\sum (a + b)\sec(C/2) \geq \frac{2}{\sqrt{3}} \sum (a + b).
\]

Letting \( a = 2R \sin A \), etc., (1) becomes
\[
\sum (\sin A + \sin B)\sec(C/2) \geq \frac{4}{\sqrt{3}} \sum \sin A.
\]

Then using
\[
\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2},
\]
we get
\[
\sum \cos \frac{A - B}{2} \geq \frac{2}{\sqrt{3}} \sum \sin A.
\]
But this inequality is known (Crux 613 [1982: 55, 67, 138]).

Also solved by the proposer.

\* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* 

\* CRUX MATHEMATICA NUM \* 
\* wishes all of its readers a \* 
\* HAPPY NEW YEAR \* 
\* with many problems (suitable for Crux!) and solutions in 1988. \* 

\* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \*
INDEX TO VOLUME 13, 1987

ARTICLES AND NOTES

Editor’s Comment on an Article in Crux .............. 308
From the Editors and Publisher of Crux Mathematicorum .... 206
Léopold Sauvé (1921-1987). K.S. Williams .............. 240
Message from the Editor, A .................. 34
On a Note of Bottema and Groenman. Roland H. Eddy .......... 242
Problem on Lattice Points, A. B. Leeb and C. Stahlke .... 104
Reduced Subscription Rate for C.M.S. Members ............ 276
Relations Among Segments of Concurrent Cevians of a Simplex.
M.S. Klamkin .................................. 274

PROPOSALS AND SOLUTIONS

January: proposals 1201-1210; solutions 976, 1062-1068, 1070-1072
February: proposals 1211-1220; solutions 1069, 1073-1076, 1078-1080
March: proposals 1221-1230; solutions 1032, 1033, 1077, 1081-1086
April: proposals 1231-1240; solutions 1019, 1087-1095, 1097
May: proposals 1241-1250; solutions 478, 1039, 1096-1108, 1110
June: proposals 1251-1260; solutions 1111-1118, 1120-1128
September: proposals 1261-1270; solutions 1087, 1091, 1116, 1129-1143, 1149
October: proposals 1271-1280; solutions 1100, 1119, 1144-1147, 1150-1156
November: proposals 1281-1290; solutions 559, 1139, 1157-1165
December: proposals 1291-1300; solutions 1109, 1166-1172

PROPOSERS AND SOLVERS

The numbers refer to the pages in which the corresponding name appears with a problem proposal, a solution, or a comment.

Ahlburg, Hayo: 119
Arbel, Beno: 64, 258
Baethge, Sam: 195
Bankoff, Leon: 88
Bejlegaard, Niels: 53, 119
Bilchev, Svetoslav: 52, 296
Bondesen, Aage: 235, 264
Bos, Len: 130, 164, 290, 321
Bottema, O.: 57, 94, 299
Broline, Duane: 20, 166
Bulman-Fleming, Sydney: 216, 237
Chambers, G.A.: 15
Cooper, Curtis: 16
Coxeter, H.S.M.: 86
Cross, Donald: 219
Csirmaz, Laszlo: 166
Dilcher, Karl: 65
Dou, Jordi: 56, 87, 156, 168, 192, 232, 236, 257, 262, 268, 302, 322
Downes, Robert: 120, 217
Eddy, Roland H.: 163
Erdös, P.: 193
Festraets-Hamoir, C.: 184, 267
Fick, Gordon: 119
Fisher, J. Chris: 196
Freitag, Herta T.: 33, 53, 216, 327
Gardner, C.: 87
Garfunkel, Jack: 27, 87, 93, 96, 119, 132, 167, 184, 200, 225, 226, 264, 320
Gilbert, Peter: 199
Gmeiner, Wolfgang: 228
Grabiner, David: 190, 219
Guy, Richard K.: 54, 197, 266
Hess, Richard I.: 52, 67, 118, 179, 200, 261
Heydebrand, Ernst v.: 257
Ivady, Péter: 201, 217
Iwata, Shiko: 128
Izard, Roger: 163, 204, 268
Johnson, Allan Wm., Jr.: 32, 97, 183
Kierstead, Friend H., Jr.: 33, 159
Kimberling, Clark: 55, 128, 217, 231, 294, 298, 332
Larson, Loren C.: 93, 150, 226
Lau, Kee-Wai: 135, 265, 303
Li, Weixuan: 120
Liu, Andy: 20, 151, 328
Luthar, R.S.: 320
Lyness, Robert: 59
Margaliot, Zvi: 93
Maurer, Steve: 320
Messer, Peter: 133
Meyers, Leroy F.: 189, 190, 230, 290
Mitrofanovic, D.S.: 14, 24, 53, 67, 85, 150, 164, 193, 202, 333
Moore, Thomas E.: 15
Murty, Vedula N.: 96, 132, 201, 297
Moser, W.O.: 16
Meydenov, Milen N.: 14
Newman, Steve: 328
Parmenter, M.: 226
Pecaric, J.E.: 14, 53, 85, 150, 333
Pedoe, Dan: 127, 167, 262
Penning, P.: 257, 320
Prielipp, Bob: 200
Rabinowitz, Stanley: 15, 31, 64, 86, 120, 161, 179, 199, 215, 225, 257, 258, 289, 294, 320
Rassias, Themistocles M.: 216
Rennie, Basil: 67, 124
Sastry, K.R.S.: 290
Satyanarayana, Kesiraju: 152
Seifiya, Tosio: 218
Selby, M.A.: 125
Semenko, Lanny: 150, 238
Shafer, Robert E.: 150
Shan, Zun: 257
Shapiro, Daniel B.: 291
Singmaster, David: 86
Smeenk, D.J.: 15, 119, 126, 160, 180, 192, 199, 236, 258, 299, 321, 322, 332
Sokolowsky, Dan: 15, 53, 126, 152, 195, 204, 216, 224, 231, 232, 256, 304
Stoyanov, Jordan: 118, 180
Sturtevant, Helen: 189
Sutter, Carl Friedrich: 290, 321
Szekeres, Esther: 118
Szekeres, George: 85, 89, 118, 294
Tabov, Jordan B.: 23, 133, 149, 193, 291, 325
Trigg, Charles W.: 238, 291
Tsintsifas, George: 14, 16, 22, 60, 63, 86, 98, 100, 135, 149, 154, 156, 162, 179, 185, 216, 222, 227, 256, 259, 289, 297, 301
Upton, L.J.: 230
Utz, W.R.: 89
Veldkamp, G.R.: 297, 298
Velikova, Emilia: 52
Wang, Chung-lie: 27
Wilke, Kenneth M.: 328
Williams, Kenneth S.: 31, 324
Withheld, N.: 170
Witten, Ian: 180
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