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The new problems I propose this month, for which I solicit elegant solutions from all readers, were all proposed (but unused) by various countries at the 26th International Mathematical Olympiad held this year in Finland. They were taken from the booklet *XXVI I.M.O., Results and Problems*, edited by Matti Lehtinen, to whom I am grateful for sending me a copy.

1. Proposed by Bulgaria.
   If $a, b, n$ are integers, with $n > 0$, prove that
   \[
   \frac{b^{n-1}a(a+b)(a+2b)\ldots(a+(n-1)b)}{n!}
   \]
   is an integer.

2. Proposed by Brazil.
   A convex quadrilateral is inscribed in a circle of radius 1. Prove that the (positive) difference $d$ between its perimeter and the sum of the lengths of its diagonals satisfies $0 < d < 2$.

3. Proposed by Canada.
   Determine the maximum value of
   \[
   \sin^2 \theta_1 + \sin^2 \theta_2 + \ldots + \sin^2 \theta_n,
   \]
   where $\theta_1 + \theta_2 + \ldots + \theta_n = \pi$ and all $\theta_i \geq 0$.

4. Proposed by Canada.
   Prove that
   \[
   \frac{x_1^2}{x_1^2 + x_2^2} + \frac{x_2^2}{x_2^2 + x_3^2} + \ldots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_n^2} + \frac{x_n^2}{x_n^2 + x_1^2} \leq n-1,
   \]
   where all $x_i > 0$.

5. Proposed by Czechoslovakia.
   Let $T$ be the set of all lattice points (i.e., all points with integer coordinates) in 3-dimensional space. Two such points, $(x, y, z)$ and $(u, v, w)$, are called neighbors if and only if
   \[
   |x-u| + |y-v| + |z-w| = 1.
   \]
   Prove that there exists a subset $S$ of $T$ such that, for each $P \in T$, there is exactly one point of $S$ among $P$ and its neighbors.
6. Proposed by Czechoslovakia.
Let $A$ be a set of positive integers such that $|x - y| \geq xy/25$ for any two elements $x, y$ of $A$. Prove that $A$ contains at most nine elements. Also, give an example of such a nine-element set.

7. Proposed by East Germany.
Determine whether or not there exist 100 distinct lines in the plane having exactly 1985 distinct points of intersection.

8. Proposed by France.
Determine eight positive integers $n_1, n_2, \ldots, n_8$ with the following property:
For every integer $k$ such that $-1985 \leq k \leq 1985$, there are eight integers $a_1, a_2, \ldots, a_8$, each belonging to the set \{-1, 0, 1\}, such that
$$a_1n_1 + a_2n_2 + \ldots + a_8n_8 = k.$$

$F_1, F_2,$ and $F_3$ are three mutually intersecting ellipses, all in the same plane, with respective foci $F_2$ and $F_3$, $F_3$ and $F_1$, and $F_1$ and $F_2$, which are not all three collinear. Prove that the common chords of each pair of ellipses are concurrent.

A sequence of polynomials $\{P_m\}$, $m = 0, 1, 2, \ldots$, in $x, y,$ and $z$ is defined by $P_0(x, y, z) \equiv 1$ and, for $m > 0$, by
$$P_m(x, y, z) = (x + z)(y + z)P_{m-1}(x, y, z) - z^2P_{m-1}(x, y, z).$$
Prove that each $P_m(x, y, z)$ is symmetric in $x, y, z$.

11. Proposed by Iceland.
If
$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0,$$
where $x_{i+n} = x_i$ and all $x_i^2 = 1$, prove that $n$ is divisible by 4.

12. Proposed by Israel.
1985 points are given inside a unit cube. Show that it is always possible to choose 32 of the points in such a way that every closed polygon (possibly degenerate) with these points as vertices has perimeter less than $8\sqrt{3}$.

13. Proposed by Italy.
Two persons $X$ and $Y$ play a game with a die. Whenever the toss of the die results in a 1 or a 2, $X$ wins the toss; otherwise $Y$ wins the toss. The game is won
by the first player who wins two consecutive tosses. Determine the probabilities that each of the players wins the game within 5 tosses. Also, determine the probabilities of winning for each player if there is no restriction on the number of tosses.

11. Proposed by Mongolia.

From each of the vertices of a regular n-gon a point starts to move with constant speed along the perimeter of the n-gon in the clockwise sense. Prove that if all the points get to a vertex A at the same time, then the points will never be together again at any other vertex. Can the points be together again at A?

15. Proposed by The Netherlands.

The solid $S$ is defined as the intersection of the six spheres whose diameters are the six edges of a given regular tetrahedron with unit edge length. Prove that no pair of points of $S$ is further apart than $1/\sqrt{6}$.

16. Proposed by Poland.

If $x_n = \sqrt[3]{2 + \sqrt{3} + \ldots + \sqrt{n}}$, prove that $x_{n+1} - x_n < 1/n!$ for $n = 2,3,4,\ldots$.

17. Proposed by Poland.

For which $k$ can the set $\{1,2,\ldots,k\}$ be partitioned into a prime number $p$ of subsets such that the sum of the elements in each subset is the same?

18. Proposed by Rumania.

For $k \geq 2$, let $n_1,n_2,\ldots,n_k$ be positive integers such that $n_2 | (2^{n_1} - 1)$, $n_3 | (2^{n_2} - 1)$, $\ldots$, $n_k | (2^{n_{k-1}} - 1)$, $n_1 | (2^{n_k} - 1)$.

Prove that $n_1 = n_2 = \ldots = n_k = 1$.


Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for $n = 1,2,3,\ldots$ (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.


Two equilateral triangles are inscribed in a circle with radius $r$. If $A$ is the area of the set consisting of all points interior to both triangles, prove that $2A \geq r^2\sqrt{3}$. 

If \( a, b, c \) are real numbers such that

\[
\frac{1}{bc-a^2} + \frac{1}{ca-b^2} + \frac{1}{ab-c^2} = 0,
\]

prove that

\[
\frac{a}{(bc-a^2)^2} + \frac{b}{(ca-b^2)^2} + \frac{c}{(ab-c^2)^2} = 0.
\]

22. Proposed by Spain.

Show how to construct a triangle ABC given the side AB, the length of segment OH and the fact that OH is parallel to AB, where O and H are the circumcenter and orthocenter, respectively, of the triangle.

23. Proposed by Spain.

Find all triples \((x, y, z)\) of positive integers such that

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}.
\]

24. Proposed by the Soviet Union.

Decompose the number \(5^{1985} - 1\) into a product of three integers each of which exceeds \(5^{100}\).

25. Proposed by the Soviet Union.

34 countries participated in a Jury session of the I.M.O., each country being represented by the leader and the deputy leader of the team. Before the meeting, some of the participants greeted each other by shaking hands, but no team leader shook hands with his or her deputy. After the meeting, the leader of the Illyrian team asked every other participant (including her own deputy) the number of people they had shaken hands with, and all the answers she got were different. How many people did the Illyrian team greet?

26. Proposed by Turkey.

Determine the smallest positive integer \(n\) such that

(i) \(n\) has exactly 144 distinct positive divisors, and

(ii) there are ten consecutive integers among the positive divisors of \(n\).

27. Proposed by the U.S.A.

Determine the range of \(w(x+y)(y+z)(z+w)\), where \(x, y, z, w\) are real numbers such that

\[x + y + z + w = x^7 + y^7 + z^7 + w^7 = 0.\]

28. Proposed by the U.S.A.

We are given \(n\) elements \(a_1, a_2, \ldots, a_n\) which are organized into \(n\) pairs
$P_1, P_2, \ldots, P_n$ in such a way that two pairs $P_i$ and $P_j$ share exactly one element when \{a_i, a_j\} is one of the pairs. Prove that every element is in exactly two of the pairs.

29, Proposed by Vietnam.

Prove that for every point $M$ on the surface of a regular tetrahedron there exists a point $M'$ such that there are at least three different curves joining $M$ to $M'$ on the surface with the smallest possible length among all the curves joining $M$ to $M'$ on the surface.

30, Proposed by Vietnam.

A circle with radius $R$ and center $O$, and a line $l$, are given in the plane. The distance of $O$ to $l$ is $d$, with $d > R$. Points $M$ and $N$ are chosen on $l$ such that the circle with diameter $MN$ is externally tangent to the given circle. Show that there exists a point $A$ in the plane such that all the segments $MN$ are seen in a constant angle from $A$.

* I now present solutions to some problems proposed in earlier columns.


Triangle $ABC$ is rotated about the centre of its circumscribed circle by an angle less than $180^\circ$ to form triangle $A_1B_1C_1$. If $BC \cap B_1C_1 = A_2$, $CA \cap C_1A_1 = B_2$, and $AB \cap A_1B_1 = C_2$, prove that triangles $ABC$ and $A_2B_2C_2$ are similar.

Solution by M.S.K.

Let the angle of rotation be $2\theta$, and let $A', B', C'$ and $A'', B'', C''$ be the feet of the perpendiculars from the center $O$ on $BC, CA, AB$ and $B_1C_1, C_1A_1, A_1B_1$, respectively (see figure). It follows easily that $A'B'C' \sim ABC$. It now suffices to show that the configuration $O-A'B'C'$ is similar to $O-A_2B_2C_2$. The first of the relations

$$OA_2 = \frac{OA'}{\cos \theta}, \quad OB_2 = \frac{OB'}{\cos \theta}, \quad OC_2 = \frac{OC'}{\cos \theta}$$

is clear from the figure, and the other two follow similarly. Since also

$$\angle A'O_2A = \angle B'O_2B = \angle C'O_2C = \theta,$$

it follows that

$$\angle B'O_2C = \angle B_2OC_2, \quad \angle C'O_2A = \angle C_2OA_2, \quad \angle A'O_2B = \angle A_2OB_2.$$

Thus $O-A'B'C' \sim O-A_2B_2C_2$, and then $A_2B_2C_2 \sim ABC$. *

Two players play the following game on a triangle ABC of unit area. The first player picks a point X on side BC, then the second player picks a point Y on CA, and finally the first player picks a point Z on AB. The first player wants triangle XYZ to have the largest possible area, while the second player wants it to have the smallest possible area. What is the largest area that the first player can be sure of getting?

Solution by M.S.K.

A point X having been chosen on side BC, let XP be parallel to AB, as shown in Figure 1. If Y is chosen on CP, then Z will be taken at A since this will maximize the altitude from Z to XY. It then follows that Y should be at P since this will minimize the altitude toXA. Similarly, if Y is chosen on PA, then Z will be taken at B and again Y should be P. We now wish to choose X so that triangle XPB, or equivalently triangle XPA, has a maximum area.

Although the calculation for point X is simple, it will be even simpler to take ABC as an isosceles right triangle with unit legs, as shown in Figure 2. This is permissible since we can transform the original triangle into the right triangle by an affine transformation (which preserves ratios of areas). Now the area of triangle XPB in Figure 2 is $\frac{1}{2}\lambda(1-\lambda)$, and this is maximized when $\lambda = 1/2$. Therefore the first player can always get an area of at least $1/4$.


In a plane is given a finite set of polygons, every two of which have a common point. Show that there exists a line which intersects all the polygons.

Solution by Andy Liu, University of Alberta.

Denote the polygons by $P_1, P_2, \ldots, P_n$. Choose an arbitrary coordinate system $x, y$ in the plane and orthogonally project each polygon onto the $x$-axis. Let the projection of $P_i$ be the interval $[a_i, b_i]$, $i = 1, 2, \ldots, n$. Since the polygons have pairwise nonempty intersections, so do the intervals. Let $a = \max_{1 \leq i \leq n} a_i$. Clearly $a$ belongs to all the intervals and the line $x = a$ is a common transversal of all the polygons.

*
Prove that, for positive $a, b, c$, we have

$$a^3 + b^3 + c^3 + 3abc \geq bc(b+c) + ca(c+a) + ab(a+b).$$

Solution by M.S.K.
The proposed inequality is equivalent to

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0. 
\tag{1}$$

This is the special case $t = 1$ of Schur's inequality

$$a^t(a-b)(a-c) + b^t(b-c)(b-a) + c^t(c-a)(c-b) \geq 0, 
\tag{1}$$

which is valid for all real numbers $t$ when $a, b, c > 0$, with equality just when $a = b = c$. □

We give a proof of (1) for completeness. Since the left member is symmetric in $a, b, c$, we may assume without loss of generality that $a \geq b \geq c > 0$ when $t \geq 0$ and $0 < a \leq b \leq c$ when $t < 0$. In each case $a^t(a-b)(a-c) \geq b^t(a-b)(b-c)$, or

$$a^t(a-b)(a-c) + b^t(b-c)(b-a) \geq 0, 
\tag{2}$$

and

$$c^t(c-a)(c-b) \geq 0. 
\tag{3}$$

Now (1) follows from (2) and (3). Note that when $t$ is an even integer, $a, b, c$ can be negative.

Find an integer $k \geq 1$ such that the expression

$$f(k, x) = \sin kx \cdot \sin^2 x + \cos kx \cdot \cos^2 x - \cos^2 kx$$

does not depend on $x$.

II. Comment by M.S.K.
Jordan B. Tabov notes that the above is only the first part of the Rumanian Olympiad problem. The missing second part asks for all integers $k \geq 1$ with the stated property. It has already been shown that $k = 3$ is the smallest such integer. We now show that $k = 3$ is the only such integer.

Suppose the integer $k \geq 3$ has the desired property. Since $f(k, 0) = 0$, we must also have

$$f(k, \frac{\pi}{2}) = \sin \frac{k\pi}{2} - (-1)^k = 0.$$

Therefore $k = 4n-1$ for some positive integer $n$. For $k = 4n-1$ and $x = \pi/(4n-1)$, we get
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\[f(k, \alpha) = -\cos^{4n-1} \frac{\pi}{4n-1} - \cos^{4n-1} \frac{2\pi}{4n-1} = 0;\]

hence

\[\cos \frac{\pi}{4n-1} + \cos \frac{2\pi}{4n-1} = 0, \quad \left(2 \cos \frac{\pi}{4n-1} - 1\right)\left(\cos \frac{\pi}{4n-1} + 1\right) = 0,\]

and finally \(\cos \{\pi/(4n-1)\} = 1/2\). Therefore \(n = 1\), and \(k = 3\) is the only such integer.

\(\star\)

\[3, \text{[1984: 215; 1985: 117]} \text{ From the 1984 British Mathematical Olympiad.} \]

(i) Prove that, for all positive integers \(m\),

\[(2 - \frac{1}{m})(2 - \frac{3}{m})(2 - \frac{5}{m})\ldots(2 - \frac{2m-1}{m}) \leq m!.\]

(ii) Prove that if \(a, b, c, d, e\) are positive real numbers, then

\[(\frac{a}{b})^4 + (\frac{b}{c})^4 + (\frac{c}{d})^4 + (\frac{d}{e})^4 + (\frac{e}{a})^4 \geq \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e}.\]

\text{II. (More elegant) solution of part (i) by Beno Arbel, Tel Aviv University, Israel.}

We prove a much stronger result, according to which the \(m!\) in the proposed inequality can be replaced by 1. By the A.M.-G.M. inequality,

\[
\prod_{k=1}^{m} \left(2 - \frac{2k-1}{m}\right) \leq \left\{\frac{1}{m} \sum_{k=1}^{m} \left(2 - \frac{2k-1}{m}\right)\right\}^m = 1. \quad \square
\]

On the other hand, if \(m \geq 4\), then

\[
\prod_{k=1}^{m} \left(3 - \frac{2k-1}{m}\right) \leq \left\{\frac{1}{m} \sum_{k=1}^{m} \left(3 - \frac{2k-1}{m}\right)\right\}^m = 2^m < m!.
\]

\(\star\)

\[20, \text{[1985: 51]} \text{ From the 1984 All-Union Olympiad (proposed by I.K. Zhuk and I.V. Voronovich).} \]

The numbers +1 or -1 are written in each of the nine unit squares of a 3x3 sheet of squared paper. Then the number in each unit square is replaced by the product of the numbers of its neighbours (two unit squares are neighbours if they have a common side). Prove that after a finite number of such operations each unit square will contain a +1.

\text{Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.}

After at most four operations, each unit square will contain a +1. To prove
this requires only simple calculations. Suppose the nine squares initially contain \(a, b, c, \ldots, i\), as in the figure. For a corner square such as the one containing \(a\), four operations give

\[a \rightarrow bd \rightarrow cg \rightarrow bdfh + 1.\]

For a boundary square such as the one containing \(b\), they give

\[b \rightarrow ace \rightarrow bh \rightarrow acei + 1.\]

And for the central square the results are

\[e \rightarrow bdfh + 1 + 1 + 1.\]


Three circles \(C_1, C_2, C_3\) of radii \(r_1, r_2, r_3\) in the plane are exterior to one another and \(r_1 > r_2, r_1 > r_3\). The two tangents to \(C_3\) are drawn from the intersection point of the exterior tangents to \(C_1\) and \(C_2\), and the two tangents to \(C_2\) from the intersection point of the exterior tangents to \(C_1\) and \(C_3\). Prove that these two pairs of tangents (to \(C_3\) and \(C_2\)) form a quadrilateral into which a circle can be inscribed. Find its radius.

Solution by Aage Bordingen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

[Readers are urged to make their own generous-size figure to follow this solution.]

Let \(O_i\) be the center of \(C_i, i = 1, 2, 3\); \(A\) the intersection of the exterior tangents to \(C_1\) and \(C_2\); \(B\) the intersection of the exterior tangents to \(C_1\) and \(C_3\); \(O\) the intersection of \(AO_3\) and \(BO_2\); \(D\) the intersection of \(OO_1\) and \(AB\); and \(x = AD, y = DB, s = O_2O, t = O_3O, u = O_3O, v = OA\). Note that, from similar triangles,

\[
\frac{BO_3}{O_3O_1} = \frac{r_3}{r_1 - r_3}, \quad \frac{BO_1}{O_1O_3} = \frac{r_1}{r_1 - r_3}, \quad \frac{O_1O_2}{O_2A} = \frac{r_1 - r_2}{r_2}, \quad \frac{O_2O_1}{O_1A} = \frac{r_1 - r_2}{r_1}.
\]

Applying Ceva's theorem to triangle \(ABO_1\) and cevian point \(O\), we get

\[
\frac{x}{y} \cdot \frac{r_3}{r_1 - r_3} \cdot \frac{r_1 - r_2}{r_2} = 1. \quad (1)
\]

Then applying Menelaus's theorem to triangle \(ABO_2\) and transversal \(DOO_1\) gives

\[
\frac{x}{y} \cdot \frac{s}{r_1} \cdot \frac{r_1 - r_2}{r_1} = 1, \quad (2)
\]

and doing the same to triangle \(ABO_3\) and transversal \(DOO_1\) gives

\[
\frac{x}{y} \cdot \frac{u}{r_1 - r_3} \cdot \frac{v}{r_1} = 1. \quad (3)
\]
From (1) and (2) we get

\[ \frac{t}{s+t} r_2 = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 - r_2 r_3} \]

and from (1) and (3)

\[ \frac{v}{u+v} r_3 = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 - r_2 r_3} \]

Hence a circle with center 0 and radius

\[ \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 - r_2 r_3} \]

will touch the tangents from A to C₃ as well as the tangents from B to C₂.

\[ \star \]


A math teacher wrote the quadratic trinomial \( x^2 + 10x + 20 \) on the blackboard. Then each student in turn either increased by 1 or decreased by 1 either the constant term or the coefficient of \( x \). Finally the trinomial \( x^2 + 20x + 10 \) appeared. Will a quadratic trinomial with integer roots necessarily appear on the blackboard in the process?

Solution by Lones Smith, Nepean, Ontario.

We begin with \( x^2 + bx + c \) where \( b-c = -10 \) and end up with \( b-c = +10 \), and at each stage \( b-c \) changes by \( \pm 1 \). Clearly, at one stage \( b-c = 1 \). Since our quadratic trinomial is then

\[ x^2 + (c+1)x + c = (x+1)(x+c) \]

it will have integer roots in the process.

\[ \star \]

3. [1980: 274] From the 8th U.S.S.R.

The vertices of a convex 32-gon lie on the points of a square lattice whose squares have sides of unit length. Find the smallest perimeter such a figure can have.

Solution by Andy Liu, University of Alberta.

If we allow polygons that are not strictly convex, the minimum perimeter is trivially 32 and is realized in rectangles of dimensions

\[ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \]
For strictly convex polygons, there cannot be three parallel edges. The figure on the preceding page yields 16 edges with different slopes. Their combined length is
\[ 2(1 + \sqrt{2} + 2\sqrt{5} + 2\sqrt{10} + 2\sqrt{13}) \approx 40.844. \]

These are the shortest edges except for those of lengths 2, 3, 2\sqrt{2}. But using the latter edges will prevent the use of edges of lengths 1 and \sqrt{2}, which are shorter. Finally, our polygon is a symmetric one using each of the edges in the figure twice. Hence the smallest perimeter is twice (1), or approximately 81.688.

\[ \star \]


On a 13 by 13 square piece of graph paper the centres of 53 of the 169 squares are chosen. Show that there will always be 4 of these 53 points which are the vertices of a rectangle whose sides are parallel to those of the paper.

Solution by Andy Liu, University of Alberta.

For \( i = 1,2,\ldots,13 \), let \( a_i \) be the number of points chosen in the \( i \)th row. Then \( \sum a_i = 53 \). (Sums, here and later, are for \( i = 1 \) to 13.) To each pair of chosen points in the same row we attach the label \((j,k)\) if they are in the \( j \)th and \( k \)th columns. The total number of such pairs of numbers is
\[
\sum \frac{a_i^2}{2} = \frac{1}{2} \sum a_i^2 - \frac{1}{2} \sum a_i \geq \frac{1}{26}(\sum a_i)^2 - \frac{1}{2} \sum a_i > 78 = \binom{13}{2},
\]
where we have used the power mean inequality. It follows from the pigeonhole principle that two of these pairs of numbers are identical, and the two pairs of points corresponding to these two pairs of numbers are the vertices of a rectangle with sides parallel to the grid. \( \Box \)

Such a rectangle can be avoided if only 52 points are chosen. This will occur, for example, if the 52 points are chosen according to the \((13,13,4,4,1)\) design: in a 13x13 configuration, pick the points so that there are 4 in each row and 4 in each column, with only one column coincidence for each pair of rows and only one row coincidence for each pair of columns. See Problem 9 below where the \((7,7,3,3,1)\) design is illustrated. (This problem is solved in Ross Honsberger, Mathematical Gems III, M.A.A., 1985, pp. 2-4. See also the block designs in Marshall Hall, Combinatorial Theory, Blaisdell, 1967, Appendix 1, p. 291.)

\[ \star \]


Three ants crawl along the sides of a triangle ABC in such a way that the centroid of the triangle they form at any given moment remains fixed. Show that
this centroid coincides with the centroid of triangle ABC if one of the ants travels along the entire perimeter of triangle ABC.

Solution by Andy Liu, University of Alberta.

Let $T$ be the centroid of the three ants, and suppose $T$ does not coincide with the centroid of the triangle. Now consider the three cevians through $T$. For at least one vertex, say $A$, $AT$ is longer than $2/3$ of the cevian through $A$. Then the point $A'$ on $AT$ produced such that $AT = 2TA'$ lies outside the triangle (see figure). Since $T$ is the centroid of the three ants, when one of them (the one that moves all around ABC) is at $A$, the midpoint of the other two ants must coincide with $A'$. However, this is impossible if the ants are on the perimeter of the triangle. Therefore $T$ coincides with the centroid of triangle ABC.

$\star$


A certain number of 0's, 1's, and 2's are written on a blackboard. Two unequal digits are erased and the third digit is written in their place (e.g., 2 is written if 0 and 1 are erased). This operation is repeated until no two distinct digits remain on the blackboard. Show that if only one digit remains at the end of the game, then this digit is independent of the order in which the digits were erased.

Solution by Andy Liu, University of Alberta.

Let $x$, $y$, and $z$ be three variables denoting the numbers of 0's, 1's, and 2's, respectively. With each operation, the parities of $x$, $y$, and $z$ all change. If $x \equiv y \equiv z \pmod{2}$, then it is impossible to have only one digit remaining. Hence exactly two of $x, y, z$ have the same parity, say $x \neq y \equiv z \pmod{2}$. In this case, the sole remaining digit must be 0, independently of the order in which the digits were erased.

$\star$


In a convex hexagon $A_1A_2A_3A_4A_5A_6$, let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of diagonals $A_5A_2, A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1$, respectively. Show that if hexagon $B_1B_2B_3B_4B_5B_6$ is convex, then its area is $\frac{1}{2}$ the area of $A_1A_2A_3A_4A_5A_6$.

Solution by M.S.K.

Suppose the $B$-polygon is convex and let $O$ be one of its interior points. Then $O$ is also an interior point of the $A$-polygon. For $i = 1, 2, \ldots, 6$, we will let $\vec{A}_i^+$ and $\vec{B}_i^+$ denote the vectors $OA_i$ and $OB_i$, respectively. We then have
2\^i = \tilde{A}_{i-1} + \tilde{A}_{i+1}, \quad \text{(1)}

where \( \tilde{A}_0 = \tilde{A}_6 \). With the brackets denoting area, we have

\[ [A_1 A_2 \ldots A_6] = \frac{1}{2} [A_1 A_2 + A_2 A_3 + \ldots + A_6 A_1]. \]

Finally, using (1) and cancelling terms like \( A_1 A_6 + A_6 A_1 \), we get

\[ [B_1 B_2 \ldots B_6] = \frac{1}{2} [B_1 B_2 + B_2 B_3 + \ldots + B_6 B_1] = \frac{1}{8} [A_1 A_2 + A_2 A_3 + \ldots + A_6 A_1] = \frac{1}{4} [A_1 A_2 \ldots A_6]. \]

\[ \Box \]


Show that with the digits 1 and 2 one can form \( 2^{n+1} \) numbers, each having \( 2^n \) digits, and every two of which differ in at least \( 2^{n-1} \) places.

Solution by Andy Liu, University of Alberta.

For any positive integer \( k \), let \( A_k \) be a set of positive integers such that

(a) every digit of each number in \( A_k \) is either 1 or 2;
(b) \( |A_k| = 2^{k+1} \);
(c) each number in \( A_k \) has \( 2^k \) digits;
(d) each pair of numbers in \( A_k \) differs in exactly \( 2^{k-1} \) places or exactly \( 2^k \) places.

Trivially,

\[ A_1 = \{11, 12, 21, 22\} \]

is such a set. Let

\[ A_2' = \{1111, 1212, 2121, 2222\}, \quad A_2'' = \{1122, 1221, 2112, 2211\}, \]

and

\[ A_2 = A_2' \cup A_2''. \]

Thus \( A_2' \) is obtained from \( A_1 \) by appending to each number of \( A_1 \) a copy of itself; and \( A_2'' \) is obtained from \( A_1 \) by appending to each number of \( A_1 \) a copy of itself with the 1's and 2's interchanged. In general, let

\[ A_n = A_n' \cup A_n''. \]

be obtained from \( A_{n-1} \) in an analogous manner.

Our proof is by induction. Suppose \( A_{n-1} \) has all four of the properties (a)-(d). Then, trivially, \( A_n \) has the three properties (a)-(c). Now consider two numbers in \( A_n \).
If they are both from $A^n_1$ or both from $A^n_2$, then they differ in the $i$th place, where $1 \leq i \leq 2^{n-1}$, and only if they differ in the $(2^{n-1}+i)$th place. If one number is from $A^n_1$ while the other is from $A^n_2$, then they differ in the $i$th place, $1 \leq i \leq 2^{n-1}$, if and only if they are identical in the $(2^{n-1}+i)$th place. It follows that $A^n_2$ has property (d) as well, and the induction is complete.

* *


On a 7x7 square piece of graph paper, the centres of $k$ of the 49 squares are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest $k$ for which this is possible?

Solution by Andy Liu, University of Alberta.

We first show that if 22 points are chosen, then four of them will be the vertices of a rectangle with sides parallel to the grid. The argument is the same as that used in Problem 4 (see page 314 in this issue). The key inequality is replaced by

$$\sum_{i=1}^{7} \frac{a_i}{2} = \frac{1}{2} \sum_{i=1}^{7} a_i^2 - \frac{1}{2} \sum_{i=1}^{7} a_i \geq \frac{1}{14} (\sum_{i=1}^{7} a_i)^2 - \frac{1}{2} \sum_{i=1}^{7} a_i > 21 = \binom{7}{2},$$

where all sums are for $i = 1$ to 7.

A rectangle can be avoided if only 21 points are chosen. This will occur, for example, if the 21 points are chosen according to the (7,7,3,3,1) design: in a 7x7 configuration, pick the points so that there are 3 in each row and 3 in each column, with only one column coincidence for each pair of rows and only one row coincidence for each pair of columns. The design is illustrated in the figure. (This problem is solved in Ross Honsberger, Mathematical Gems III, M.A.A., 1965, pp. 4-6. See also the Marshall Hall reference at the end of Problem 4.)

* *


A large cube measuring $k$ units on each edge is to be formed of smaller unit cubes, each coloured either black or white. Can this be done so that for any unit cube exactly two of its neighbours have the same colour as the unit cube itself? (Two cubes are called neighbours if they share a common face.)

Solution by Andy Liu, University of Alberta.

The arrangement is possible if and only if $k$ is even. For $k$ even, partition the top layer into 1x2x2 blocks. All cubes in the same block receive the same colour and adjacent blocks are given different colours. For succeeding layers, each
cube receives the colour different from that of the cube directly above. It now follows easily that each cube has exactly two neighbours, both on the same layer, with the same colour as itself.

We now use a parity argument to show that no such colouring is possible if \( k \) is odd. Our proof is indirect. Assume that such a colouring exists. Starting from any unit cube, it is possible to trace a closed path of cubes of the same colour. Since each cube has exactly two neighbours of the same colour, each of these paths consists of an even number of cubes, and furthermore the sets of cubes of different paths are disjoint. Therefore the total number of unit cubes is even, and we have a contradiction.

\[ \star \]


A horizontal strip is given in the plane, bounded by straight lines, and \( n \) lines are drawn intersecting this strip. Every two of these lines intersect inside the strip and no three of them are concurrent. Consider all paths starting on the lower edge of the strip, passing along segments of the given lines, and ending on the upper edge of the strip, which have the following property: travelling along such a path, we are always going upward, and when we come to the point of intersection of two of the lines we must change over to the other line to continue following the path. Show that, among these paths,

(a) at least \( \frac{3}{2}n \) of them have no point in common;
(b) there is some path consisting of at least \( n \) segments;
(c) there is some path passing along at most \( \frac{3}{2}n + 1 \) of the lines;
(d) there is some path which passes along each of the \( n \) lines.

Solution by Andy Liu, University of Alberta.

Let the lines be \( l_1, l_2, \ldots, l_n \), intersecting the lower edge of the strip at the respective points \( A_1, A_2, \ldots, A_n \) and the upper edge at \( B_1, B_2, \ldots, B_n \). We may assume that \( A_1, A_2, \ldots, A_n \) lie in that order from left to right. Since every pair of lines intersects within the strip, the points \( B_1, B_2, \ldots, B_n \) lie in that order from left to right. Let \( C_{i,j}^i = C_{j}^i \) be the point of intersection of \( l_i \) and \( l_j \). Then \( l_i \) is divided into \( n \) segments by

\[ C_1^i, C_2^i, \ldots, C_{i-1}^i, C_{i+1}^i, \ldots, C_n^i, \]

though not necessarily in that order. If \( i \) is odd, colour the segments along \( l_i \) alternately red and blue, with the bottom segment red. If \( i \) is even, colour the segments along \( l_i \) alternately blue and red, with the bottom segment blue. For each \( A_k \) between \( A_i \) and \( A_j \), \( l_k \) intersects one of \( A_i C_j^i \) and \( A_j C_i^i \). For each \( A_k \) not between \( A_i \) and \( A_j \), \( l_k \) intersects either both or neither of \( A_i C_j^i \) and \( A_j C_i^i \). It follows that
(\textsuperscript{*}) at any intersection, the two segments to the left are of one colour while the two segments to the right are of the other colour.

Denote by $p_i$ the path which starts from $A_i$. By (\textsuperscript{*}), the entire path is of the same colour. Moreover, $p_i$ is red if $i$ is odd and blue if $i$ is even. By (\textsuperscript{*}) again, paths of the same colour cannot intersect. Hence at least $\frac{3}{2}n$ of them have no point in common, proving (a).

Note that there are altogether $n^2$ segments, and each lies on some path. Since the number of paths is $n$, the average number of segments per path is $n$. Hence there is some path consisting of at least $n$ segments, proving (b).

We call $C_{i,j}$ exposed if it lies on $p_1 \cup p_n$ and protected otherwise. We claim that each $l_i$ contains at most two exposed intersection points. If $l_i$ contains at least three exposed points, then two of them, say $C_{i,j}$ below $C_{i,k}$, must both lie on $p_1$, say. There cannot be any $C_{i,j}$ between $C_{i,j}$ and $C_{i,k}$, as otherwise either $A_i C_{i,j}$ protects $C_{i,j}$ or $B_i C_{i,j}$ protects $C_{i,j}$. All points along $A_i C_{i,j}$ are protected by $A_j C_{i,j}$ and $A_k C_{i,j}$. Similarly, all points along $B_i C_{i,j}$ are protected by $B_j C_{i,j}$ and $B_k C_{i,j}$. Thus it is impossible for $l_i$ to contain a third exposed intersection point. It now follows that at most $n$ intersection points can lie on $p_1 \cup p_n$, and so the total number of segments along $p_1$ and $p_n$ is at most $n+2$. Thus either $p_1$ or $p_n$ has at most $\frac{3}{2}n+1$ segments along at most $\frac{3}{2}n+1$ of the lines, proving (c).

Finally, note that $p_k$ will end at $B_{n+1-k}$ by (\textsuperscript{*}). Let $k = \lfloor (n+1)/2 \rfloor$. Then $A_k$ and $B_{n+1-k}$ lie on opposite sides of $l_i$ for every $i \neq k$ or $n+1-k$. Hence $p_k$ crosses $l_i$, but it is impossible for $p_k$ not to pass along $l_i$ for at least one segment. It follows that $p_k$ passes along each of the $n$ lines, proving (d).


Given is a polynomial $P(x)$ whose coefficients are (i) natural numbers, (ii) integers. Denote by $a_n$ the sum of the digits in the decimal representation of $P(n)$. Show that there is some number which occurs infinitely often in the sequence $a_1, a_2, a_3, \ldots$.

Solution by M.S.K. and Andy Liu, University of Alberta.

(i) Let $k$ be a natural number such that $10^k$ exceeds all the coefficients of $P(x)$. Then, for $n = 10^k+i$, $i = 0, 1, 2, \ldots$, the sum $a_n$ of the digits of $P(n)$ is equal to the sum of the digits of the fixed coefficients of $P(x)$.

(ii) For a sufficiently large natural number $\alpha$, the coefficients of $P(x+\alpha)$ are all natural numbers. (Just let $\alpha$ be larger than the real part of each root of $P(x)$.) Now apply (i) to $P(x+\alpha)$.

Quadrilateral $ABCD$ is inscribed in a circle. It is rotated about the centre of the circle through an angle less than $180^\circ$ to form quadrilateral $A_1B_1C_1D_1$. Show that the points $AB \cap A_1B_1$, $BC \cap B_1C_1$, $CD \cap C_1D_1$, $DA \cap D_1A_1$ are the vertices of a parallelogram.

Solution by M.S.K.

Let $AB \cap A_1B_1 = A_2$, $BC \cap B_1C_1 = B_2$, $CD \cap C_1D_1 = C_2$, $DA \cap D_1A_1 = D_2$.

Also let the feet of the perpendiculars from center $O$ to $AB, BC, CD, DA$ and $A_1B_1, B_1C_1, C_1D_1, D_1A_1$ be $A', B', C', D'$ and $A'', B'', C'', D''$, respectively. It follows from Problem 1 (page 308 in this issue) that

$$O-A'B'C'D' \sim O-A_2B_2C_2D_2.$$ 

Finally, since it follows easily that $A'B'C'D'$ is a parallelogram, so is $A_2B_2C_2D_2$.


Twenty teams are participating in the competition for the championships both of Europe and the world in a certain sport. Among them, there are $k$ European teams (the results of their competitions for world champion count also towards the European championship). The tournament is conducted in round robin fashion. What is the largest value of $k$ for which it is possible that the team getting the (strictly) largest number of points towards the European championship also gets the (strictly) smallest number of points towards the world championship, if the sport involved is

(a) hockey (0 for a loss, 1 for a tie, 2 for a win);
(b) volleyball (0 for a loss, 1 for a win, no ties).

Solution by Andy Liu, University of Alberta.

(a) The number of points in the average score for the world championship is

$$\frac{(20)\cdot 2}{20} = 19.$$ 

Hence, to have the (strictly) smallest number of points, a team in the world championship can have at most 18 points. If there are at least 19 European teams, then the average score in the European championship is at least 18 points, so a team with at most 18 points cannot have the (strictly) largest number of points. Therefore $k < 19$. 

It is possible to have \( k = 18 \). Let \( A \) and \( B \) be two of the European teams. All games are ties except that \( A \) beats \( B \), \( B \) beats both non-European teams, and both of these beat \( A \). In the world championship, \( A \) has 18 points, \( B \) has 20 points, while every other team has 19 points. In the European championship, \( A \) has 18 points, \( B \) has 16 points, while every other team has 17 points.

(b) In the world championship, the average score is 9\( \frac{1}{2} \) points. Hence, to have the (strictly) smallest number of points, a team can have at most 8 points. If \( k \geq 16 \), then the average score in the European championship is at least 7\( \frac{3}{4} \) points, so a team with at most 8 points cannot have the (strictly) largest number of points. Hence \( k < 16 \).

It is possible to have \( k = 15 \). A possible table of standings is given above (in which a 1 means that the row team has beaten the column team, etc.). Team \( A \) has the (strictly) smallest number of points in the world championship but the (strictly) largest number of points in the European championship.

\[
\begin{array}{cccccccccccc}
\text{Teams} & \text{European} & \text{World} & \text{Scores} \\
A & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 & 11 \\
C & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 9 \\
D & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
E & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
F & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
G & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
H & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
I & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 9 \\
J & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 10 \\
K & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
L & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
M & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
N & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
O & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
V & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 7 & 10 \\
W & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 7 & 10 \\
X & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 10 \\
Y & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 7 & 10 \\
Z & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 7 & 10 \\
\end{array}
\]
Given real numbers
\[ a_1, a_2, \ldots, a_m \quad \text{and} \quad b_1, b_2, \ldots, b_n, \]
and positive numbers
\[ p_1, p_2, \ldots, p_m \quad \text{and} \quad q_1, q_2, \ldots, q_n, \]
we form an \( mxn \) array in which the entry in the \( i \)th row \((i = 1,2,\ldots,m)\) and \( j \)th column \((j = 1,2,\ldots,n)\) is
\[ \frac{a_i + b_j}{p_i + q_j}. \]
Show that in such an array there is some entry which is no less than any other in the same row and no greater than any other in the same column
(a) when \( m = 2 \) and \( n = 2 \),
(b) for arbitrary \( m \) and \( n \).

Solution by Andy Liu, University of Alberta.

For typographical convenience we let
\[ e(i,j) = \frac{a_i + b_j}{p_i + q_j}. \]

(a) We may assume without loss of generality that \( e(1,1) \) is no greater than the other three entries, so that, in particular,
\[ e(1,1) \leq e(1,2) \quad (1) \]
and
\[ e(1,1) \leq e(2,1). \quad (2) \]
We now assume that there is no entry \( e(i,j) \) with the desired property and find a contradiction. If \( e(1,2) \leq e(2,2) \), then \( e(1,2) \) has the desired property, so we must have
\[ e(2,2) < e(1,2). \quad (3) \]
If \( e(2,2) \geq e(2,1) \), then \( e(2,2) \) has the desired property, so we must have
\[ e(2,2) < e(2,1). \quad (4) \]
Since all the \( p_i \)'s and \( q_j \)'s are positive, we can clear the denominators in each of inequalities (1) to (4) and obtain four equivalent inequalities (the one corresponding to (1), for example, being
\[ a_1 p_1 + a_1 q_2 + b_1 p_1 + b_1 q_2 \leq a_1 p_1 + a_1 q_1 + b_2 p_1 + b_2 q_1. \]
Finally, adding these four new inequalities yields an inequality equivalent to $0 < 0$, and we have our contradiction.

(b) We pick the smallest entry in each column and then pick the greatest among these. We may assume without loss of generality that the resulting entry is $e(1,1)$. We now assume that there is no entry $e(i,j)$ with the desired property and find a contradiction. Since $e(1,1)$ does not have the desired property, we must have

$$e(1,1) < e(1,j)$$

for some $j > 1$. Now let $e(i,j)$ be the smallest entry in the $j$th column. Then $e(i,j) \leq e(1,1)$, so $i > 1$ and

$$e(i,j) < e(1,j).$$

Note that

$$e(1,1) \leq e(i,1),$$

from which it follows that

$$e(i,j) \leq e(i,1).$$

It is now easily verified that inequalities (5) to (8) yield the same contradiction as inequalities (1) to (4) in part (a).

* *

The following problems published in this column in Vol. 7 (1981) are still open, and readers are urged to send in elegant solutions.

Pages 15-16, 36th Moscow Olympiad (1973): Nos. 1, 8, 9, 10, 11, 13, 15.
Pages 46-47, Rumanian Olympiad (1978): 11th class, Nos. 1, 2, 3; 12th class, No. 2.
Pages 74-78, Rumanian Selection Tests (1978): First Selection Test, Nos. 2, 5, 6, 8, 9; Second Selection Test, Nos. 1, 2, 3, 4; Third Selection Test, Nos. 1, 2, 3, 4, 5, 6, 7, 8; Fourth Selection Test, Nos. 1, 2, 3, 4, 5, 6, 7, 8.
Pages 114-115, Bulgarian Competitions: Nos. B-1, B-4, B-6.

Editor's note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

*  *  *
NEW GEOMETRY PROGRAM FOR THE IBM PC

Named EUCLID, this new program for the IBM Personal Computer will do all possible Euclidean constructions and conics, except for extreme cases.

Keys F1 to F5 on the keyboard allow the user to plot points on the screen, join them with segments, extend segments or choose random points on them, plot circles, and compute points of intersection of lines and circles. These capabilities are sufficient to carry out virtually all straightedge-and-compass constructions.

Keys F6 and F7 bisect angles and construct perpendiculars, and Key F8 provides access to other time-savers: triangle analysis, including plotting of circumcircle, incircle, and excircles, if desired; instant polygons, regular or random, up to 15 vertices; the unique conic through any five distinct points; nine conic-plotting options for user-chosen foci, vertices, etc.; points of intersection of any two conics, or of a conic and a line.

All points are automatically labeled when input or computed, and Key F9 allows labels to be erased or changed. Key F10 lists in double-precision all labeled points.

The program EUCLID serves two purposes: students can practise (and teachers demonstrate) constructions; and students and teachers can enjoy making discoveries, e.g., Problems 1042 [1985: 146] and 1091* [on this page]. (For the latter problem, the EUCLID program has supported the conjecture on every choice of random triangles tried, always with 14 decimal-place agreement on the point of concurrence, so the theorem is very likely true.)

EUCLID runs on IBM Personal Computers with 256K and DOS 2.1 or later. To order the EUCLID disk and manual, send check for $37.95 or purchase order to University of Evansville Press, 1800 Lincoln Avenue, Evansville, Indiana 47714.

PROBLEMS -- PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1091*, Proposed by Clark Kimberling, University of Evansville, Indiana.

Let $A_1A_2A_3$ be a triangle and $\gamma_i$ the excircle opposite $A_i$, $i = 1,2,3$. Apollonius knew how to construct the circle $\Gamma$ internally tangent to the three excircles and encompassing them. Let $B_i$ be the point of contact of $\Gamma$ and $\gamma_i$, $i = 1,2,3$. Prove that the lines $A_1B_1$, $A_2B_2$, and $A_3B_3$ are concurrent.
1092, Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let \( n \) be a natural number. Show that, for all natural numbers \( m \), the sum

\[
S_m = \sum_{k=1}^{n} \left( \sec \frac{2km}{2n+1} \right)^m
\]

is an integer.

1093, Proposed by Jack Garfunkel, Flushing, N.Y.

Prove that

\[
\left( \sum \sin A \right)^3 \geq 8 \Pi \sin \frac{A}{2},
\]

where the sums and product are cyclic over the angles \( A, B, C \) of a triangle. When does equality occur?

1094, Proposed by Peter Messer, M.D., Mequon, Wisconsin.

A skew quadrilateral consists of two triangular surfaces \( ABC \) and \( ABD \) with common edge \( AB \). Prove that

\[
\text{dihedral angle with edge } AB = \arccos \left( \frac{\tan S}{\tan C} \right) + \arccos \left( \frac{\tan S}{\tan D} \right),
\]

where \( S \) is half the central angle of chord \( AB \) in the circumsphere \( ACBD \).

1095, Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let \( N_n = \{1, 2, \ldots, n\} \), where \( n \geq 4 \). A subset \( A \) of \( N_n \) with \(|A| \geq 2\) is called an RC-set (relatively composite) if \((a,b) > 1\) for all \( a, b \in A \). Let \( f(n) \) be the maximum cardinality of all RC-sets \( A \) in \( N_n \). Determine \( f(n) \) and find all RC-sets in \( N_n \) of cardinality \( f(n) \).

1096, Proposed by M.S. Klamkin, University of Alberta.

Determine the maximum and minimum values of

\[
S = \cos \frac{A}{4} \cos \frac{B}{4} \cos \frac{C}{4} + \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4},
\]

where \( A, B, C \) are the angles of a triangle. (No calculus, please!)

1097, Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \( AD, BE, CF \) be the angle bisectors and \( AM, BN, CP \) the medians of a triangle \( ABC \). Prove that

\[
AD \cdot AM + BE \cdot BN + CF \cdot CP = s^2,
\]

where \( s \) is the semiperimeter.
1098, Proposed by Jordi Dou, Barcelona, Spain.
Characterize all trapezoids for which the circumscribed ellipse of minimal area is a circle.

Find a solution to each of the Diophantine equations
\[ x^2 - 726y^2 = 3 \quad \text{and} \quad x^2 - 363y^2 = -2 \]
in which the integers \( x \) and \( y \) are both greater than 500.

\( \triangle ABC \) is a triangle with \( \angle C = 30^\circ \), circumcentre \( O \) and incentre \( I \). Points \( D \) and \( F \) are chosen on \( BC \) and \( AC \), respectively, such that \( BD = AE = AB \). Prove that \( DE = 0I \) and \( DE \perp 0I \).

\[ \star \quad \star \quad \star \]

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

Solve the piscine alphametic
\[ \text{FISH + FISH + FISH + ... + FISH = SHOAL}. \]
This SHOAL has 73 FISH.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Clearly, \( F = 1 \) and \( S = 7, 8, \) or \( 9 \). If \( S = 9 \), then \( I = 2, H = 4, \) and \( L = 2 \) (duplication). If \( S = 8 \), then \( I = 1 \) (duplication). Therefore \( S = 7, I = 0, H = 8, \) and we have the unique solution

\[ 1078 \times 73 = 78694. \]

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHE, Culver City, California; FRED A. MILLER, Elkins, West Virginia; GLEN E. MILLS, Maitland, Florida; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; JAMES WATSON, student, Erskine College, Due West, South Carolina; PETER WATSON-HURTHIG, Columbia College, Vancouver; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

\[ \star \quad \star \quad \star \]


(a) Prove that two equilateral triangles of unit side cannot be placed inside a unit square without overlapping.
(b) What is the maximum number of regular tetrahedra of unit edge that can be packed without overlapping inside a unit cube?

(c) Generalize to higher dimensions.

Partial solution by Jordan B. Tabov, Sofia, Bulgaria.

(a) The diameter (least upper bound of the set of distances between pairs of points) of a unit square is \( \sqrt{2} \), and it is easy to see that the diameter of the configuration consisting of two nonoverlapping equilateral triangles of unit side is at least \( \sqrt{3} \). Since \( \sqrt{3} > \sqrt{2} \), it follows that the statement in part (a) is true.

(b) At least two regular tetrahedra of unit edge can be packed without overlapping inside a unit cube. To see this, let \( ABCD_1B_1C_1D_1 \) be a unit cube (see figure), \( P \) and \( Q \) being the midpoints of edges \( BB_1 \) and \( DD_1 \), respectively, choose points \( E \) and \( E_1 \) on \( PQ \) such that \( PE = QF_1 = (\sqrt{2}-1)/2 \). It is easy to verify that tetrahedra \( AA_1EE_1 \) and \( CC_1FE_1 \) are regular of unit edge, and they are placed without overlapping inside the unit cube.

Also partially solved by GEORGE TSINT-SIFAS, Thessaloniki, Greece; and the proposer.

\[
\lim_{n \to \infty} \frac{P_n}{n} = 2^{n+1} \sin \left( \frac{\pi}{2^n} \right), \quad n = 1, 2, 3, \ldots
\]

Solution by Walther Janous, Ursulinenegymnasium, Innsbruck, Austria.

Via induction, it quickly follows that \( P_n = 2^{n+1} \sin \left( \frac{\pi}{2^n} \right), \quad n = 1, 2, 3, \ldots \).

Thus

\[
\lim_{n \to \infty} \frac{P_n}{n} = \frac{2\pi \lim_{n \to \infty} \frac{\sin \left( \frac{\pi}{2^n} \right)}{\frac{\pi}{2^n}}}{2\pi} = 2\pi.
\]

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; KARL DILCHER, Dalhousie University, Halifax; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; PETER WATSON-HURTHIG, Columbia College, Vancouver; and the proposer.
Several solvers showed that \( P_n \) is the perimeter of a regular \( 2^n \)-gon inscribed in a circle of unit radius, and the required limit follows at once.

\[ \text{Editor's comment.} \]

974. [1984: 262] Proposed by Jack Garfunkel, Flushing, N.Y.

Consider the following double inequality, where \( A, B, C \) are the angles of any triangle:

\[ \cos A \cos B \cos C \leq 8 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \leq \frac{1}{8}. \]

The inequality involving the first and third members and that involving the second and third members are both well known. Prove the inequality involving the first and second members.

Comment by M.S. Klamkin, University of Alberta.

The proposed inequality is equivalent to

\[(1 - \cos A)(1 - \cos B)(1 - \cos C) - \cos A \cos B \cos C \geq 0,\]

an inequality already established in Problem 836 [1984: 228].

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; ROLAND H. EDDY, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.


Parabolas \( y^2 = 2px \) and \( x^2 = 2qy \) are given. A triangle with vertices \( P_i(x_i,y_i), \ i = 1,2,3 \), is inscribed in \( y^2 = 2px \) and the lines containing two of its sides are tangent to \( x^2 = 2qy \).

(a) Prove that \( 2p^2q = y_1y_2y_3 \).

(b) Deduce from (a), or otherwise, that the line containing the third side of the triangle is also tangent to \( x^2 = 2qy \).

I. Solution by Gali Salvatore, Perkins, Québec.

(a) Since two of the lines \( P_2P_3, \ P_3P_1, \) and \( P_1P_2 \) are tangent to \( x^2 = 2qy \), we may assume that \( P_i = (0,0) \) for \( i = 1,2,3 \) and that the two tangents are both nonvertical (and so each has a slope). We represent the parabola \( y^2 = 2px \) parametrically by \( x = 2pt^2, \ y = 2pt \), so

\[ P_i = (2pt_i^2, 2pt_i), \ i = 1,2,3, \]

where \( t_1, t_2, t_3 \) are distinct nonzero real numbers. We note that the nonvertical line
\[ y = mx + b \text{ is tangent to } x^2 = 2qy \text{ if and only if} \]
\[ qm^2 + 2b = 0. \quad (1) \]

The equation of \( P_1P_2 \) is
\[ x - (t_1 + t_2)y + 2pt_1t_2 = 0. \]

Hence, using (1), \( P_1P_2 \) is tangent to \( x^2 = 2qy \) if and only if \( t_1 + t_2 \neq 0 \) and
\[ q + 4pt_1t_2(t_1 + t_2) = 0. \quad (2) \]

Similarly, \( P_1P_3 \) is tangent to \( x^2 = 2qy \) if and only if \( t_1 + t_3 \neq 0 \) and
\[ q + 4pt_1t_3(t_1 + t_3) = 0. \quad (3) \]

Subtracting (2) and (3) yields
\[ 4pt_1(t_2 - t_3)(t_1 + t_2 + t_3) = 0, \]

or, since \( 4pt_1(t_2 - t_3) \neq 0, \)
\[ t_1 + t_2 + t_3 = 0. \quad (4) \]

From (4) and (2) or (3), we get
\[ q = 4pt_1t_2t_3, \quad (5) \]

so
\[ 2p^2q = 2pt_1 \cdot 2pt_2 \cdot 2pt_3 = \nu_1 \nu_2 \nu_3. \]

Note that (4) and (5) must both hold if \( P_1P_2 \) and \( P_1P_3 \) are both tangent to \( x^2 = 2qy. \)

(b) We assume that \( P_1P_2 \) and \( P_1P_3 \) are both tangent to \( x^2 = 2qy, \) so (4) and (5) both hold. The line \( P_2P_3 \) is tangent to \( x^2 = 2qy \) if and only if \( t_2 + t_3 \neq 0, \) which is guaranteed by (4) since \( t_1 \neq 0, \) and
\[ q + 4pt_2t_3(t_2 + t_3) = 0, \]

which follows from (5) and (4).

II. Joint comment on part (b) by J. Chris Fisher, University of Regina; and Michal Szurek, (Visiting Professor from) University of Warsaw.

This is a special case of a result known as Poncelet’s Porism [7, Vol. 1, p. 316]:

Two conics \( C, C' \) and a point \( P_0 \in C \) exterior to \( C' \) are given. Define \( P_1 \) to be the point where one of the tangents from \( P_0 \) to \( C \) meets \( C, \) and \( P_j \) \((j \geq 2)\) to be the point (different from \( P_{j-2} \)) where the tangent to \( C' \) from \( P_{j-1} \) meets \( C. \) If \( P_n = P_0 \) for some \( n, \) then the polygon closes with \( n \) vertices from any starting point \( P_0 \) of \( C. \)

There are many proofs of this porism [1, Vol. 2, p. 55; Vol. 5, p. 65], [2], [3], [5, p. 55], [6], [7]. The pair of parabolas given in the problem satisfy the
hypothesis with \( n = 3 \). One way to see this is to consider the degenerate triangle with \( P_0 \) at the origin and \( P_1 = P_2 \) where the \( x \)-axis meets the line at infinity.

An analytic approach (due to Cayley [2], [3]) is outlined in [8, Articles 370 and 376]. The condition that \( n = 3 \) for the two conics

\[
C: \ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0
\]

and

\[
C': a'x^2 + b'y^2 + c'z^2 + 2f'yx + 2g'zx + 2h'xy = 0
\]

is that \( \theta = 4\Delta \theta' \), where

\[
\Delta = abc + 2fgh - af^2 - bg^2 - ch^2,
\]

\[
\theta = (bc - f^2) a' + (ca - g^2) b' + (ab - h^2) c' + 2(gh - af) f' + 2(hf - bg) g' + 2(fg - ch) h',
\]

and \( \theta' \) is obtained from \( \theta \) by interchanging the primed and unprimed letters. In the given problem \( \theta = \theta' = 0 \), so the condition is satisfied.

Readers might be interested in other special cases that are more or less elementary.

(i) If \( C \) is the circumcircle of a given triangle and \( C' \) is either the incircle or an excircle, then any other triangle inscribed in \( C \) that has two sides tangent to \( C' \) will also have its third side tangent. (This is an immediate consequence of Euler's formula \( d^2 = R^2 \pm 2Rr \).)

Note that the original problem with parabolas is projectively equivalent to the case of unit circles whose centres are separated by a distance of \( d = \sqrt{3} \). This may be seen as follows: Call the points of intersection of the two circles \( C \) and \( C' \), and let \( R \) and \( S \) be the points where the tangents to each circle at \( P \) meet the other circle. One shows that \( RS \) is a common tangent to the circles (as in the figure). Let \( \phi \) be the projectivity that takes \( P \) to the origin, \( R \) and \( S \) to the points at infinity of the \( x \)- and \( y \)-axes, and \( Q \) to the second point of intersection of the two parabolas. The proof that \( \phi \) takes the circles to the parabolas is completed by invoking the theorem that there is exactly one conic tangent to a pair of lines at specified points that passes through a point not on either line.

(ii) The case when \( C \) and \( C' \) are nonintersecting circles and \( n \geq 3 \). (An inversion that takes \( C \) and \( C' \) to concentric circles takes the polygon to a chain of \( n \) evenly spaced congruent circles passing through the common centre.)
(iii) If two triangles have six distinct vertices, all lying on a conic, there is a polarity for which both triangles are self-polar. Conversely, if two triangles, with no vertex of either on a side of the other, are self-polar for a given polarity, then their six vertices lie on a conic and their six sides touch another conic [4, Theorems 9.41 and 9.42].

Also solved by S.C. CHAN, Singapore; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

REFERENCES


\[
\begin{array}{|c|c|}
\hline
n & f(n) = p^2 \\
\hline
0 & 5^2 \\
17 & 29^2 \\
116 & 169^2 \\
593 & 985^2 \\
4056 & 5741^2 \\
23657 & 33461^2 \\
\hline
\end{array}
\]

Let \( f(n) = 2n^2 + 14n + 25 \). It is easy to verify that \( f(17) = 29^2 \). Find two more positive integers \( n \) such that \( f(n) \) is a perfect square.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

We will show that there are, in fact, infinitely many such integers. We seek nonnegative integers \( n \) such that

\[
f(n) = (n + 3)^2 + (n + 4)^2 = p^2
\]

for some positive integer \( p \). Thus the problem is equivalent to the well-known problem of finding Pythagorean triangles whose legs differ by unity. This problem is fully discussed in Beiler [1]. The first six solutions are given in the adjoining table. Additional solutions can be obtained from

\[
n_i = 6n_{i-1} - n_{i-2} + 14 \quad \text{and} \quad p_i = 6p_{i-1} - p_{i-2}.
\]
But why bother? Beiler has provided us with a table of the first 100 solutions, the last of which corresponds to an \( n \) with 77 digits.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; S.C. CHAN, Singapore; CURTIS COOPER, Central Missouri State University, Warrensburg; KARL DILCHER, Dalhousie University, Halifax; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California (two solutions); FRED A. MILLER, Elkins, West Virginia; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; DAVID R. STONE, Georgia Southern College, Statesboro; CHARLES W. TRIGG, San Diego, California; PETER WATSON-HURTHIG, Columbia College, Vancouver; and the proposer (two solutions). Two incorrect solutions were received.

REFERENCE


Proposed by Andy Liu, University of Alberta.

Determine the smallest positive integer \( m \) such that

\[ 529^n + m \cdot 132^n \]

is divisible by 262417 for all odd positive integers \( n \).

I. Solution by Curtis Cooper, Central Missouri State University, Warrensburg.

It is clear from

\[ 529 + 1984 \cdot 132 = 262417 \]

that no \( m < 1984 \) will do. Moreover, it follows from (1) that

\[ 529^n + 1984 \cdot 132^n \equiv 0 \pmod{262417} \]

holds for \( n = 1 \), and from \( 529^2 \equiv 132^2 \pmod{262417} \) and an easy induction that (2) holds for all odd positive integers \( n \). Thus the required number is \( m = 1984 \).

II. Solution by Peter Watson-Hurthig, Columbia College, Vancouver.

It follows from

\[ a^{2k+1} + mb^{2k+1} = a(a^{2k} - b^{2k}) + b^{2k}(a + mb) \]

that \( a^2 - b^2 \) divides \( a^{2k+1} + b^{2k+1} \) for any nonnegative integer \( k \) if and only if it divides \( b^{2k}(a + mb) \). With \( a = 529 \) and \( b = 132 \), we have

\[ a^2 - b^2 = (a + b)(a - b) = 661 \cdot 397 = 262417. \]

Thus 262417 divides \( 529^{2k+1} + m \cdot 132^{2k+1} \) for any nonnegative integer \( k \) if and only if it divides \( 132^{2k}(529 + 132m) \). Since 262417 is relatively prime to 132, the
necessary and sufficient condition is that \( \frac{262417}{132} \) divide \( 529 + 132m \). The smallest \( m \) for which this occurs is

\[ m = \frac{262417 - 529}{132} = 1984. \]

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; KARL DILCHER, Dalhousie University, Halifax; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; M.S. Klamkin, University of Alberta; KEE-WAI LAU, Hong Kong; J.A. McCallum, Medicine Hat, Alberta; STEWART METCHETT, Culver City, California; D.J. Smeenk, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; ALEKSANDAR ZOROVIC, Elizabethtown High School, Kentucky; and the proposer.

\[ \text{Q80, [1984: 263] Proposed by Leon Bankoff, Los Angeles, California.} \]

Show that

\[ \frac{\prod \sin A}{\sum \sin A} + \frac{\sum \sin^2 A}{2} = 1, \]

where the sums and product are cyclic over the angles \( A, B, C \) of a triangle.

Composite of the solutions of Blundon, Eddy, Garfunkel, Janous, Klamkin, and Marty (who are further identified below).

The proposed equality follows immediately from the following known results, where \( R, r, s \) have their usual meanings:

\[ \prod \sin A = \frac{rs}{2}, \quad \sum \sin A = \frac{R}{r}, \quad \sum \sin^2 A = 1 - \frac{r}{2R}. \]

Solutions were received from W.J. BLUNDON, Memorial University of Newfoundland; DUANE BROLINE, University of Evansville, Indiana; ROLAND H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; FRED A. MILLER, Elkins, West Virginia; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; PETER WATSON-HURTHIG, Columbia College, Vancouver; ALEKSANDAR ZOROVIC, Elizabethtown High School, Kentucky; and the proposer.

\[ \text{Q81, [1984: 291] Proposed by Allan Wm. Johnson Jr., Washington, D.C.} \]

Solve the doubly true multiplication

\[ 5 \cdot \text{ELEVEN} = \text{FIFTY5} \]

where FIVE is, of course, prime.

\[ \text{Solution by Edwin M. Klein, University of Wisconsin-Whitewater.} \]

Clearly \( F = 1 \) and \( \{T, I\} = \{0, 5\} \). Since \( L \) and \( V \) produce the same carry when
multiplied by 5, we must have

\[ \{L, V\} = \{2,3\}, \{6,7\}, \text{ or } \{8,9\}; \]

but the last possibility must be excluded because it leads to the duplication \( F = 9 \).

Now

\[
\begin{align*}
L = 2 & \implies I = 0, T = 5, F = 6, V = 3 \implies \text{FIVE} = 6031 = 37 \cdot 163; \\
L = 6 & \implies I = 0, T = 5, F = 8, V = 7 \implies \text{FIVE} = 8071 = 7 \cdot 1153; \\
L = 7 & \implies I = 5, T = 0, F = 8, V = 6 \implies \text{FIVE} = 8561 = 7 \cdot 1223.
\end{align*}
\]

Therefore \( L = 3, I = 5, T = 0, F = 6, V = 2 \), so \( \text{FIVE} = 6521 \), a prime, and then \( N = 7, Y = 8 \). The unique reconstruction is

\[ 5 \cdot 131217 = 656085. \]

Also solved by JAMES BOWE, Erskine College, Due West, South Carolina; KENNETH ALLAN BRADLEY, student, The Ohio State University; MARGARET A. BROWNING, student, The Ohio State University; J. ALAN BURKHOLDER, student, The Ohio State University; DAVID J. COLEMAN, student, The Ohio State University; RICHARD I. HESS, Rancho Palos Verdes, California; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; LORI KRAUS, student, The Ohio State University; KATHI MAUK, student, The Ohio State University; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Maitland, Florida; MARJORIE ANNE NELSON, student, The Ohio State University; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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