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1. **Dirichlet sets.**

Let $\Sigma$ be the set of all segments in space. Each segment is, of course, an infinite point set. We select one particular segment, $T$, which we will call the **target set**, and find the cardinality of $S \cap T$ when $S$ ranges over $\Sigma$. For any $S \in \Sigma$, $\text{Card } S \cap T$ is either 0 (when $S$ and $T$ are disjoint) or 1 (when they touch or cross) or $\infty$ (when they overlap). So if we set

$$\chi(\Sigma, T) = \{\text{Card } S \cap T : S \in \Sigma\},$$

then we have $\chi(\Sigma, T) = \{0, 1, \infty\}$.

There is an analogue of this in number theory. Let the target set $T$ be the set of all primes, and let $\Sigma$ be the set of all arithmetic progressions of positive integers. More precisely, let

$$\Sigma = \{\mathcal{S}(d,a) : d, a = 1, 2, 3, \ldots\}, \quad (1)$$

where

$$\mathcal{S}(d,a) = \{dn+a : n = 0, 1, 2, \ldots\}. \quad (2)$$

For any $S \in \Sigma$, Card $S \cap T$ is either 0 (when $(d,a) > 1$ and $a$ is composite) or 1 (when $(d,a) > 1$ and $a$ is prime) or $\infty$ (when $(d,a) = 1$). So here again we have $\chi(\Sigma, T) = \{0, 1, \infty\}$. Because the only nontrivial case, Card $S \cap T = \infty$, is just a restatement of Dirichlet's Theorem, we will say that the set $T$ of all primes is a **Dirichlet set** (with respect to $\Sigma$). More generally, we will say that an infinite set $T$ of positive integers is a **Dirichlet set** (with respect to $\Sigma$ as defined in (1) and (2)) if and only if

$$\chi(\Sigma, T) \in \{0, 1, \infty\}.$$

Of course, one can artificially create infinite sets $T$ of positive integers that are not Dirichlet sets. But the surprising thing is that, as we shall see, all the infinite sets of positive integers that arise naturally in elementary number theory, with one certain and one doubtful exception, are Dirichlet sets (with respect to $\Sigma$). We will call them **elementary Dirichlet sets**. The doubtful exception is, as we shall see later, the image of the Euler $\phi$-function. The certain exception is the set of factorials $T = \{1!, 2!, 3!, \ldots\}$. For though we can have Card $S \cap T = 0$ with the progression $S(4,3) = \{4n+3\}$, Card $S \cap T = 1$ for $S(4,1) = \{4n+1\}$, and Card $S \cap T = \infty$ for $S(d,d) = \{dn+d\}$, for then $S \cap T$ contains (at least) $d!$, $(d+1)!$, $(d+2)!$, $\ldots$, we can also have Card $S \cap T = 2$ with the progression $S(5,1) = \ldots$
Therefore the set of factorials is not a Dirichlet set.

Let the target set \( T \) be an arbitrary infinite set of positive integers. Instead of considering the set \( \Sigma \) of all arithmetic progressions as defined by (1) and (2), we consider only the subset \( \Sigma' \) of \( \Sigma \) defined by

\[
\Sigma' = \{ S'(d,a') : d = 1,2,3,\ldots ; a' = 1,2,\ldots,d \},
\]

where

\[
S'(d,a') = \{ dn+a' : n = 0,1,2,\ldots \}.
\]

Since \( S \subseteq S' \), it is clear that

\[
\text{Card } S'nT = 0 \implies \text{Card } SnT = 0,
\]

\[
\text{Card } S'nT = 1 \implies \text{Card } SnT = 0 \text{ or } 1,
\]

and

\[
\text{Card } S'nT = \infty \implies \text{Card } SnT = \infty,
\]

and so

\[
\chi(\Sigma',T) \in \{ 0,1,\infty \} \implies \chi(\Sigma,T) \in \{ 0,1,\infty \}.
\]

Therefore \( T \) is a Dirichlet set with respect to \( \Sigma' \) if and only if it is a Dirichlet set with respect to \( \Sigma \). What we have done, in effect, is to partition the elements of \( T \) into equivalence classes modulo \( d \), so that, for example, if \( S' = S'(d,a') \), then \( S'nT \) is the set of elements of \( T \) that are congruent to \( a' \) modulo \( d \).

When the target set \( T \) is arbitrary, one would expect that \( \text{Card } S'nT \) would equal \( \infty \) for most, if not all, of the \( S' \in \Sigma' \), and that, when not \( \infty \), it could take on any of a number of values. The astonishing result is that so many elementary target sets \( T \) (possibly all but the factorials, as mentioned earlier) have the Dirichlet property of the primes, viz., that \( \text{Card } S'nT \) must equal \( 0 \) or \( 1 \) if it is not \( \infty \).

2. Elementary Dirichlet sets.

Throughout this section, \( \Sigma \) is the set of all arithmetic progressions of positive integers, as defined in (1) and (2). As a first example, we take a polynomial \( P \) with integer coefficients that has infinitely many positive integer values, and take those positive integer values as our target set \( T \). (In particular, this \( T \) could be an element of \( \Sigma \).) If \( S = \{dn+a\} \) and \( P(s) = dt+a \) for some \( s \) and \( t \), then, for all \( k \), \( P(s+kd) \equiv P(s) \pmod{d} \). Therefore \( \text{Card } SnT = \infty \), and \( \chi(\Sigma,T) \in \{ 0,\infty \} \). So the positive integer values of all such polynomials \( P \) form elementary Dirichlet sets.

Now let the target set \( T \) be the set of all powers \( b^m \), \( m = 0,1,2,\ldots \), for a fixed integer \( b > 1 \). We show that, if \( \text{Card } SnT \geq 2 \) for \( S = \{dn+a\} \), then \( \text{Card } SnT = \infty \).

Suppose, then, that \( b^\ell = dr+a \) and \( b^m = ds+a \), where \( \ell < m \). We use induction on \( u \) to
show that $e^t = dt + a$, where $e = um - (u - 1)l$. This is certainly true for $u = 0$ and $u = 1$. For the induction step, we show that $b^{e'} = dt' + a$, where $e' = (u + 1)m - ul$.

This is seen from

$$b^{e'} = b^{e-2}(b^m - b^l) + b^e = b^{e-2}d(s-t)+dt+a = dt'+a.$$ 

Therefore $x(z, T) \subseteq \{0, 1, \infty\}$ and $T$ is an elementary Dirichlet set. In particular, if $b = 2$, then $\text{Card } SnT = 0$ if $S = \{4n+3\}$ and $\text{Card } SnT = 1$ if $S = \{4n+2\}$.

The target set $T = \{2^{m+3}m: m = 0, 1, 2, \ldots\}$ does not have the Dirichlet property, since $\text{Card } S^nT = 2$ for $S = \{8n+5\}$. But we do not consider that this set $T$ is one that arises "naturally" in elementary number theory. Note, however, that this example shows that the Dirichlet property is not unrestrictedly "additive". But there is a restricted kind of additivity in the sense that, for any positive integer $k$, $kT = \{ky. \in T\}$ is a Dirichlet set whenever $T$ is a Dirichlet set. The proof of this is straightforward. As a consequence of this and the Dirichlet character of sets of powers, all geometric progressions of positive integers are elementary Dirichlet sets.

The set of all binomial coefficients is the set of all positive integers, so it trivially forms an elementary Dirichlet set. Moreover, when they are written out in the form of Pascal's rectangle, in which the element in the $i$th row and $j$th column is

$$a_{i,j} = \frac{(i+j-2)!}{(i-1)!(j-1)!}, \quad i, j = 1, 2, 3, \ldots,$$

each row after the first is the set of values of a polynomial in $j$ and thus constitutes an elementary Dirichlet set.

We now take as our target set $T$ an arbitrary Lucas sequence,

$$T = \{b_0, b_1, b_2, \ldots\},$$

where $b_0$ and $b_1$ are arbitrary positive integers and $b_{i+2} = b_i + b_{i+1}$, $i = 0, 1, 2, \ldots$ . (When $b_0 = b_1 = 1$, then $T$ is the Fibonacci sequence.) For some positive integer $d$, we reduce each $b_i$ to $b_i^{'} \pmod d$, so that $b_i^{'} \equiv b_i \pmod d$ and $1 \leq b_i^{'} \leq d$, and then set

$$T' = \{b_0^{'}, b_1^{'}, b_2^{'}, \ldots\}.$$ 

Since the numbers $b_i^{'}$ can take on at most $d$ values, there are at most $d^2$ distinct consecutive pairs of values $(b_i^{'}, b_{i+1}^{'})$, so

$$(b_i^{'}, b_{i+1}^{'})(b_{i+k}^{', b_{i+k+1}^{'}}, \quad (3)$$

for some $i$ and $k$. It now follows from the recurrence relation of the Lucas sequence
that (3) holds for that same \( k \) and for all \( i \). Therefore, for any \( b_i \in T \), \( b_{i+sk} \equiv b_i \) (mod \( d \)) for \( s = 0,1,2,... \). Since the Lucas sequence is monotone increasing, every equivalence class of \( T \) has cardinality 0 or \( \infty \), and \( T \) is an elementary Dirichlet set.

The positive integers that occur as sides of primitive Pythagorean triangles are those of one of the forms

\[
\left| r^2 - s^2 \right|, \quad 2rs, \quad r^2 + s^2,
\]

where the integers \( r,s > 0, (r,s) = 1 \), and \( r \neq s \) (mod 2). No such number is of the form \( 4n+2 \), \( 2rs \) generates all multiples of \( 4 \), and \( |r^2 - s^2| \) generates all odd integers greater than \( 1 \). So the only interesting case is to take for our target set \( T \) the numbers of the form \( r^2 + s^2 \) (the hypotenuses). Let \( S = \{dn+a\} \), and suppose \( r^2 + s^2 = \hat{d}n+a \). If \( t = s+rh \), where \( h \) is any positive integer, then clearly \( r^2 + t^2 = \hat{d}m+a \) for some \( m \). Now \((r,t) = 1 \), for otherwise \((r,s) > 1 \). If \( r \) is even, then \( s \) is odd and \( t \) is odd for all \( h \). If \( r \) is odd, \( s \) is even and \( t \) is even (at least) for all even \( h \). In any case \( r \neq s \) (mod 2) for infinitely many \( h \). So \( \text{Card} \ S \cap T = \infty \), \( \chi(S,T) \leq \{0, \infty\} \) and \( T \) is an elementary Dirichlet set.

The image of the number of positive divisors function \( \tau \) is the set of all positive integers, and this is trivially an elementary Dirichlet set.

Now let the target set \( T \) be the image of the sum of positive divisors function \( \sigma \), and let \( S = \{dn+a\} \). Suppose \( \sigma(r) = \hat{d}s+a \) for some \( r \) and \( s \). Then, by Dirichlet's Theorem, there are infinitely many primes \( p \) of the form \( \hat{d}m+d-1 \) that do not divide \( r \). Computing \( \sigma(p^2-r) \) for all those primes yields infinitely many members of \( S \). Therefore \( \chi(S,T) \leq \{0, \infty\} \) and \( T \) is an elementary Dirichlet set.

We conjecture that the image of the Euler \( \phi \)-function is also an elementary Dirichlet set. Several well-known mathematicians have looked at the problem, and one of them gave orally what he claimed was a correct proof, but that proof was never written down. See Problem 964 on page 215 of this issue.

108 North Second Street, Unit C, Alhambra, California 91801.

* *

MATHEMATICAL CLERIHEWS

Maria Gaetana Agnesi
Bewitched by a shape that was crazy,
Computed all night
To get her curve right.

Pythagoras
Did stagger us—
Gave us that gem,
His Theorem.

ALAN WAYNE
Holiday, Florida

* *

*
With as few as 23 (but not 22) people chosen at random, it is more likely than not that there is a day of the (365-day) year which "covers" two birthdays (a birthday coincidence), that is, that there is a day on which two (or more) of the people celebrate the anniversaries of their birth. This is so well known that there are few people you could surprise with it. It is possible, however, that you would be surprised to know that with as few as 14 (but not 13) people, it is more likely than not that there are two consecutive days (here and throughout the article, December 31 and January 1 are consecutive days) which cover two birthdays. Still not surprised? Well, perhaps you will be when you learn that with as few as 11 (but not 10) people, it is more likely than not that there are three consecutive days which cover two birthdays.

More generally, for any nonnegative integer \( w \), let \( f(w) \) denote the smallest integer such that, for \( f(w) \) people chosen at random, it is more likely than not that some \( w+1 \) consecutive days will cover two of their birthdays. Table I lists the numbers \( f(w) \) for \( w = 0, 1, 2, \ldots, 9 \).

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Let \( P_w(k) \) denote the probability that, for \( k \) people chosen at random, no \( w+1 \) consecutive days cover two birthdays. Clearly, \( f(w) \) is the smallest integer \( k \) for which

\[
1 - P_w(k) > \frac{1}{2}.
\]

The entries in Table II exhibit some of the probabilities \( 1 - P_w(k) \) for \( 0 \leq w \leq 9 \) and \( 5 \leq k \leq 40 \), and the underlined entries yield the information in Table I.

The probabilities in Table II are larger than common sense would lead us to estimate. As a result of \( 1 - P_1(21) = .836 \), for example, if you should find yourself in a room with 21 people you would do well to wager five dollars against one dollar that there are two consecutive days of the year which cover two (or more) of their birthdays. You might even find some simple soul who will take on an even money bet!

We will show that

\[
P_w(k) = \frac{365}{365-kw} \cdot \binom{365-kw}{k} \cdot \frac{k!}{365^k},
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</table>

The entries in this table are the values of

$$1 - P_\omega(\kappa) = \text{the probability that, for } \kappa \text{ people chosen at random, some } x+1 \text{ consecutive days cover at least two of their birthdays.}$$
but first we establish several combinatorial counts of certain restricted choices.

Let \( x_1, x_2, \ldots, x_k \) be \( k \) integers from the set \( S_n = \{1, 2, \ldots, n\} \) named so that
\[
1 \leq x_1 < x_2 < x_3 < \ldots < x_k \leq n.
\] (2)

The number of such \( k \)-choices from \( S_n \) is \( \binom{n}{k} \). Conditions (2) can be written in the form
\[
x_i - x_{i-1} \geq 0, \quad x_i - x_{i-1} - 1 \geq 0 \quad (i = 2, 3, \ldots, k),
\]
and this suggests the following generalization: For a given integer \( w \geq 0 \), find
\[
[n|k]_w \equiv \text{the number of } k \text{-choices (2) satisfying}
\]
\[
x_i - x_{i-1} \geq 0, \quad x_i - x_{i-1} - 1 \geq w \quad (i = 2, 3, \ldots, k),
\]
These conditions mean that, in the linear display
\[
1, 2, 3, \ldots, n,
\] (3)
each of the integers \( x_1, \ldots, x_{k-1} \) is followed (to the right) by \( w \) non-chosen integers. Setting
\[
y_i = x_i - (i-1)w, \quad i = 1, 2, \ldots, k,
\]
yields a \( k \)-choice
\[
1 \leq y_1 < y_2 < \ldots < y_k \leq n - (k-1)w,
\]
and hence
\[
[n|k]_w = \binom{n-(k-1)w}{k}.
\]
Thus \( [n|k]_0 = \binom{n}{k} \), while \( [n|k]_1 = \binom{n-k+1}{k} \) is the number of \( k \)-choices from \( S_n \) with no two chosen integers adjacent in the display (3).

A "circular" problem arises when we display the integers in a circle (see figure) and ask for \( (n|k)_w \equiv \text{the number of } k \text{-choices (2) with the property that every chosen integer is followed (clockwise) in the figure by } w \text{ non-chosen integers, or, equivalently, the number of } k \text{-choices (2) such that}
\]
\[
x_i - x_{i-1} - 1 \geq w \quad (i = 2, 3, \ldots, k),
\]
We count these \( k \)-choices in two subsets: \( x_1 \geq w+1 \) and \( 1 \leq x_1 \leq w \). In the first case, \( x_1 + n - x_k - 1 \geq w \) is automatically satisfied, so (4) reduces to
\[
\omega+1 \leq x_1, \quad x_i - x_{i-1} - 1 \geq w \quad (i = 2, 3, \ldots, k)
\]
or, equivalently, with \( y_i = x_i - w \), to
1 \leq y_1, \quad y_i - y_{i-1} - 1 \geq \omega \quad (i = 2, 3, \ldots, \kappa), \quad y_{\kappa} \leq n - \omega,
and there are \([n-\omega|k]_\omega\) such \(k\)-choices. In the second case, \(x_1\) can be chosen in \(\omega\) ways; then the \(\omega\) integers following \(x_1\) and the \(\omega\) integers preceding \(x_1\) are excluded from the choice and it is easy to see that \(x_2, x_3, \ldots, x_{\kappa}\) can be chosen in \([n-2\omega-1|k-1]_\omega\) ways. Hence

\[
\binom{n}{k}_\omega = \binom{n-\omega}{k}_\omega + \omega \binom{n-2\omega-1}{k-1}_\omega
\]

\[
= \frac{n-\omega-(k-1)\omega}{k} + \omega \binom{n-2\omega-1-(k-2)\omega}{k-1}_\omega
\]

\[
= \frac{n}{n-k\omega} \binom{n-k\omega}{k}_\omega.
\]

Now we establish (1). First choose \(k\) days of the year, say days \(x_1, x_2, \ldots, x_k\) satisfying (4). This can be done in \((365|k)_\omega\) ways. The birthdays of \(k\) people can fall on such a choice of \(k\) days in \(k!\) ways. Hence there are \(k!(365|k)_\omega\) birthday distributions satisfying the condition that no \(\omega+1\) consecutive days cover two birthdays. Since there are 365\(^k\) birthday distributions, (1) follows.

In this Corner I give the problems set at the 1983 Dutch Olympiad, at the final round of the 1983 Swedish Mathematical Contest, and at the 1984 British Mathematical Olympiad, which I received through the courtesy of Jan van de Craats, Ake Samuelsson, and Robert Lyness, respectively. For all of these problems, I solicit elegant solutions from all readers. I will give next month the problems and the results of the 25th International Mathematical Olympiad, which took place this year in Prague, Czechoslovakia, together with some reflections on this competition over the last ten years.

1, A triangle ABC can be divided into two isosceles triangles by a line through A. Given that one of the angles of the triangle is 30°, determine all possible values of the other two angles.
2. Prove that if \( n \) is an odd positive integer, then the last two digits in base ten of \( 2^n(2^{n+1} - 1) \) are 28.

3. Let \( a, b, c, p \) be real numbers, with \( a, b, c \) not all equal, such that
\[
\frac{1}{b} + \frac{1}{c} = \frac{1}{a} = p.
\]
Determine all possible values of \( p \) and prove that \( abc + p = 0 \).

4. Within an equilateral triangle of side 15 are 111 points. Prove that it is always possible to cover at least 3 of these points by a round coin of diameter \( \sqrt{3} \), part of which may lie outside the triangle.

1983 SWEDISH MATHEMATICAL CONTEST (Final Round)

1. The positive integers are summed in groups in the following way:
\[
1, \ 2+3, \ 4+5+6, \ 7+8+9+10, \ \ldots
\]
Find the sum of the \( n \)th group.

2. Prove that, for all real numbers \( x \) and \( y \),
\[
\cos x^2 + \cos y^2 - \cos xy < 3.
\]

3. Prove that if there exist \( n \) positive integers \( x_1, x_2, \ldots, x_n \) satisfying the \( n \) equations
\[
\begin{align*}
2x_1 - x_2 &= 1 \\
-x_{k-1} + 2x_k - x_{k+1} &= 1, \quad k = 2, 3, \ldots, n-1 \\
-x_{n-1} + 2x_n &= 1
\end{align*}
\]
then \( n \) is even.

4. Two concentric circles have radii \( r \) and \( R \). A rectangle has two adjacent vertices on one of the circles. The two other vertices are on the other circle. Determine the length of the sides of the rectangle when its area is maximal.

5. A unit square is to be covered by three congruent circular disks.
   (a) Show that there are disks with radii less than half the diagonal of the square that provide a covering.
   (b) Determine the smallest possible radius.

6. Prove that \((x, y) = (1, 2)\) is the unique (real) solution of the system
\[
\begin{align*}
x(x + y)^2 &= 9 \\
x(y^3 - x^3) &= 7.
\end{align*}
\]

*
BRITISH MATHEMATICAL OLYMPIAD
March 13, 1984. Time: 3½ hours

Candidates are not expected to attempt all seven questions.

1. P, Q, R are arbitrary points on the sides BC, CA, AB, respectively, of triangle ABC. Prove that the triangle whose vertices are the centres of the circles AQR, BRP, CPQ is similar to triangle ABC.

2. For \(0 \leq r \leq n\), let \(a_n\) be the number of binomial coefficients \(\binom{n}{r}\) which leave remainder 1 on division by 3, and let \(b_n\) be the number which leave remainder 2. Prove that \(a_n > b_n\) for all positive integers \(n\).

3. (i) Prove that, for all positive integers \(m\),

\[
(2 - \frac{1}{m})(2 - \frac{3}{m})(2 - \frac{5}{m}) \ldots (2 - \frac{2m-1}{m}) \leq m!.
\]

(ii) Prove that if \(a, b, c, d, e\) are positive real numbers, then

\[
\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4 \geq \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e}.
\]

4. Let \(N\) be a positive integer. Determine, with proof, the number of solutions of the equation

\[
x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2
\]
lying in the interval \(1 \leq x \leq N\). (The square brackets denote the greatest integer function.)

5. A plane cuts a right circular cone with vertex \(V\) in an ellipse \(E\) and meets the axis of the cone at \(C\); \(A\) is an extremity of the major axis of \(E\). Prove that the area of the curved surface of the slant cone with \(V\) as vertex and \(E\) as base is

\[
\frac{VA}{AC} \cdot \text{(area of } E)\]

6. Let \(a\) and \(m\) be positive integers. Prove that, if there exists an integer \(x\) such that \(a^2x - a\) is divisible by \(m\), then there exists an integer \(y\) such that both \(a^2y - a\) and \(ay^2 - y\) are divisible by \(m\).

7. ABCD is a quadrilateral which has an inscribed circle. With the side \(AB\) is associated

\[
u_{AB} = p_1 \sin (\angle DAB) + p_2 \sin (\angle ABC),
\]
where \(p_1\) and \(p_2\) are the perpendiculrars from \(A\) and \(B\), respectively, to the opposite

(continued on p. 240)
Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be resubmitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

961. Proposed by Allan Wm. Johnson Jr., Washington, D.C.

If the digit(s) of \( n \) can be reused in \( EGG \) and \( NEST \), then in the decimal alphametic

\[ n \times EGG = NEST \]

the maximum \( EGG \) is 988 and there are solutions when \( n \) is in geometric progression \((n = 2, 4, 8)\) and when \( n \) is consecutive \((n = 2, 3, 4)\):

\[
egin{align*}
2 \times 988 &= 1976, \\
3 \times 988 &= 2964, \\
4 \times 988 &= 3952, \\
8 \times 988 &= 7904.
\end{align*}
\]

Find the minimum \( EGG \) for which solutions exist when \( n \) is in arithmetic progression.


Two gamblers play a game with 5 coins, taking alternate turns. At each turn the player calls either "Heads" or "Tails" and then throws the 5 coins. If fewer than 3 coins fall as he has called, he pays one dollar into the pool; if 3 or 4 coins fall as he has called, he does not pay anything into the pool; if all 5 coins fall as he has called, he receives the contents of the pool.

Determine the probabilities
(a) that the pool is paid out after exactly \( n \) turns;
(b) that after \( n \) turns since the pool was last paid out it contains \( r \) dollars;
(c) that when the pool is paid out it contains \( r \) dollars.


Find consecutive squares that can be split into two sets with equal sums.
Let \( T \) be the image of the Euler \( \phi \)-function, that is,

\[
T = \{ \phi(n) : n = 1, 2, 3, \ldots \}.
\]

Prove or disprove that \( T \) is a Dirichlet set, as defined in the proposer's article "Elementary Dirichlet Sets" [1984: 206-209, esp. p. 206 and last paragraph p. 209].

Let \( A_1A_2A_3 \) be a nondegenerate triangle with sides \( A_2A_3 = a_1, A_3A_1 = a_2, \)
\( A_1A_2 = a_3 \), and let \( PA_i = x_i \) \((i = 1, 2, 3)\), where \( P \) is any point in space. Prove that

\[
\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} \geq \sqrt{3},
\]

and determine when equality occurs.

A topless prism has square vertical faces hinged to the sides of a regular \( n \)-gonal base, edge \( e \). The top coinciding vertices of adjacent vertical faces are connected with cords of length \( e \). The vertical faces are permitted to fall outward until stopped by the cords. What angles do the faces then make with the surface upon which the box rests for various values of \( n \)?

(The case \( n = 4 \) is considered in Problem 3898, School Science and Mathematics. as (February 1983) 177-179.)

Let \( ABC \) be a triangle with sides \( b = CA \) and \( c = AB \) of fixed length, with \( b < c \), and variable angle \( A \). If \( S \) is the intersection of the symmedian of vertex \( A \) with the circumcircle of the triangle, construct the limit segment of \( AS \) as angle \( A \) tends to 180°.

For real numbers \( a, b, c \), let \( S_1 = a^n + b^n + c^n \). If \( S_1 \geq 0 \), prove that

\[
12S_5 + 33S_1S_2^2 + 3S_1^5 + 6S_2^2S_3 \geq 12S_1S_4 + 10S_2S_3 + 20S_1^2S_2.
\]

When does equality occur?

Find a 3-parameter solution of the Diophantine equation

\[
\frac{x}{x^2 + w^2} + \frac{y}{y^2 + w^2} + \frac{z}{z^2 + w^2} = \frac{2w^2}{\sqrt{(x^2 + w^2)(y^2 + w^2)(z^2 + w^2)}}.
\]
Let $a, b, c$ and $m_a, m_b, m_c$ denote the side lengths and median lengths of a triangle. Find the set of all real $t$ and, for each such $t$, the largest positive constant $\lambda_t$, such that

$$\frac{m_a m_b m_c}{abc} \geq \lambda_t \frac{m_t}{a} + \frac{m_t^2}{b} + \frac{m_t^3}{c}$$

holds for all triangles.

**SOLUTIONS**

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


Let $n$ be a fixed natural number. We are interested in finding an infinite sequence $(v_0, v_1, v_2, \ldots)$ of strictly increasing positive integers, and a finite sequence $(u_0, u_1, \ldots, u_n)$ of nonzero integers such that, for all integers $m \geq n$,

$$u_0 v_m^2 + u_1 v_{m-1}^2 + \ldots + u_n v_{m-n}^2 = u_0 v_m^2 + u_1 v_{m-1}^2 + \ldots + u_n v_{m-n}^2. \quad (1)$$

(a) Prove that (1) holds if

$$u_m = \text{coefficient of } x^m \text{ in } (1-x)^n$$

and

$$v_m = \text{coefficient of } x^m \text{ in } (1-x)^{-n-1}.$$

(b) Find other sequences $(u_m)$ and $(v_m)$ for which (1) holds.

III. Comment by the proposer.

The published proof of part (a), as well as my own, was based on an identity of Bruce (see equation (2) in [1984: 30]) for which Bruce did not give a formal proof. Later Klamkin showed that the Bruce identity was equivalent to the following identity of Le-Jen Shoo,

$$\sum_{r=0}^{m} \binom{n}{m+r}^2 \binom{n+2m-r}{2m} = \binom{n+m}{m}^2, \quad (2)$$

and he gave several references, not easily accessible to most readers, to proofs of (2). I give here a combinatorial proof of (2). The proof is similar to that in Section 4 of my recent article in this journal [1]. Lemmas 1 and 2 referred to below are in Section 2 of [1].
Consider the number, \( N \), of distinct committees of \( m \) men and \( 2n \) women that can be formed from \( m \) married couples, \( m \) single men, and \( n+m \) single women, with the proviso that no married couple is allowed on any committee. We evaluate \( N \) in two ways.

Any such committee must contain \( r \) married men for some \( r = 0,1,2,\ldots,m \). These can be chosen in \( \binom{m}{r} \) ways; the \( m-r \) single men can be chosen in \( \binom{m-r}{2m-r} \) ways; and the \( 2m \) women can be chosen, from the \( n+m \) single women and the \( m-r \) married women whose husbands have not been chosen, in \( \binom{n+2m-r}{2m} \) ways. Hence

\[
N = \sum_{r=0}^{m} \binom{m}{r} \binom{m-r}{2m-r} = \sum_{r=0}^{m} \binom{m}{r}^2 \binom{n+2m-r}{2m}, \quad (3)
\]

Any such committee must contain \( s \) married women for some \( s = 0,1,2,\ldots,m \). These can be chosen in \( \binom{m}{s} \) ways; the \( 2m-s \) single women can be chosen in \( \binom{n+m}{2m-s} \) ways; and the \( m \) men can be chosen, from the \( m \) single men and the \( m-s \) married men whose wives have not been chosen, in \( \binom{2m-s}{m} \) ways. Hence

\[
N = \sum_{s=0}^{m} \binom{m}{s} \binom{n+m}{2m-s} \binom{m-s}{m} = \sum_{s=0}^{m} \binom{m}{s} \binom{n}{m-s}, \quad \text{by Lemma 1},
\]

\[
= \binom{n+m}{m} \sum_{s=0}^{m} \binom{m}{s} \binom{n}{m-s}, \quad \text{by Lemma 2}. \quad (4)
\]

Now (2) follows from (3) and (4).

*Editor's comment.*

A more careful reading of the proposer's original solution to part (a) shows that he gave a valid proof of the Bruce identity for \( n = 3 \) and said that the proof could be easily generalized, as in fact it can. He has subsequently produced the above combinatorial proof of the equivalent Le-Jen Shoo identity, which must surely be one of the easiest on record.

We have recently received two more accessible references to the generalization of the Le-Jen Shoo identity which appears as equation (3) in [1984: 90]. M.S. Klamkin found one in Lovasz [2] and Vedula N. Murty found one in Riordan [3]. Both of these proofs are based on the Vandermonde Convolution. We have also come across a reference to another elementary proof of the Le-Jen Shoo identity by Kaucky [4], which we have not seen.
II. Solution by Harley Flanders, Florida Atlantic University, Boca Raton.
Unfortunately only the proposer's rather computational solution appeared. Another more conceptual solution should be of interest.

Consider the linear transformation

\[ \phi: X \rightarrow AX - XB \]

on the \( n^2 \)-dimensional space of \( n \times n \) matrices (over a field) into itself. Its matrix is

\[ A \otimes I - I \otimes B. \]

Since \( A \otimes I \) and \( I \otimes B \) commute, the characteristic roots of their difference are all the \( \lambda_i - \mu_i \),

where \( \lambda_1, \ldots, \lambda_n \) are the characteristic roots of \( A \) and \( \mu_1, \ldots, \mu_n \) are those of \( B \). Now \( \phi \) is singular if and only if it has at least one characteristic root 0. The proposer's result is now obvious.

Let \( S \) be a subset of an \( m \times n \) rectangular array of points, with \( m, n \geq 2 \). A circuit in \( S \) is a simple (i.e., nonself-intersecting) polygonal closed path whose vertices form a subset of \( S \) and whose edges are parallel to the sides of the array.

Prove that a circuit in \( S \) always exists for any subset \( S \) with \( |S| \geq m+n \), and show that this bound is best possible.
Editor's comment.

Nearly all solvers showed that the proposal is incorrect as stated. The proposer may have confused the graph-theoretic concept of a simple graph (one with no loops or multiple edges) with the topological concept of a simple curve (one which is nonself-intersecting). The proposer's solution, which is given below, has been edited to be consistent with the following reformulation of the problem. Unless otherwise indicated, the word "simple" is used throughout in its graph-theoretic sense.

The problem reformulated. Let $S$ be a subset of an $m \times n$ rectangular array of points, with $m, n \geq 2$. A rectangular circuit in $S$ is a simple circuit whose vertices form a subset of $S$ and whose edges are parallel to the sides of the array. Prove that a rectangular circuit in $S$ always exists for any subset $S$ with $|S| \geq m+n$, and show that this bound is best possible.

Solution by the proposer.

Let the rectangular array of points have $m$ rows and $n$ columns, and let $(i, j)$ denote the point in the $i$th row and $j$th column. Suppose $S$ is a subset of the array such that $|S| \geq m+n$. We associate with $S$ a bipartite graph $G$ whose vertices are

$$\{V_1, V_2, \ldots, V_m\} \cup \{V'_1, V'_2, \ldots, V'_n\}$$

and whose edges are determined by the rule: join $V_i$ and $V'_j$ if and only if $(i, j) \in S$. Thus $G$ is a simple graph with $m+n$ vertices and at least $m+n$ edges. It is a well-known graph-theoretic result that such a graph (i.e., one with at least as many edges as vertices) must have at least one simple circuit. And a simple circuit in $G$ corresponds to a rectangular circuit in $S$.

To show that the bound is best possible, let $S$ consist of all the points in the first row and first column of the array. Then $|S| = m+n-1$, and there is no rectangular circuit in $S$. □

![Figure 1](image1.png)  
![Figure 2](image2.png)
A special case, with \( m = 6 \) and \( n = 4 \), is illustrated in Figures 1 and 2. In Figure 1, the elements of \( S \) are represented by small circles and the remaining elements of the array by dots. Here \(|S| = 6+4 = 10\), there is only one rectangular circuit in \( S \) (shown by solid lines), and this circuit is not topologically simple. In Figure 2, solid lines are used to represent the simple circuit in the associated bipartite graph \( G \).

Also solved by PAUL R. BEESACK, Carleton University, Ottawa; CURTIS COOPER, Central Missouri State University at Warrensburg; G.P. HENDERSON, Campbellcroft, Ontario; WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LEROY F. MEYERS, The Ohio State University.

Editor's comment.

Henderson and Meyers both conjectured that there is a topologically simple rectangular circuit in \( S \) if \(|S| \geq m + n + \min\{m,n\} - 2\).


(a) What is the largest integer, a permutation of the nine nonzero digits, that is divisible by 99?

(b) What is the smallest such number divisible by 99?

(c) If the nine nonzero digits are arranged at random, what is the probability that the integer formed will be divisible by 99?

(d) Answer (a), (b), and (c) as applied to integers formed from the ten distinct digits (initial zeros excluded).

I. Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Since all pandigital numbers are divisible by 9, we need only investigate those that are divisible by 11.

An integer is divisible by 11 if and only if the sum of the odd-position digits differs from the sum of the even-position digits by a multiple of 11. Since the sum of the digits of a pandigital number is 45, this can happen only if one sum is 39 and the other is 6, or one sum is 28 and the other is 17. But the sum of five distinct digits cannot exceed 35, so we need consider only the 28-17 case.

It is easy to set down all the five-digit combinations that add up to 17. There are only 11, and they are shown in the first row of the table below, with the other five digits shown in the second row:

<table>
<thead>
<tr>
<th>01259</th>
<th>01268</th>
<th>01349</th>
<th>01358</th>
<th>01367</th>
<th>01457</th>
<th>02348</th>
<th>02357</th>
<th>02456</th>
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<th>12356</th>
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<tbody>
<tr>
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<td>87652</td>
<td>97642</td>
<td>98542</td>
<td>98632</td>
<td>98651</td>
<td>98641</td>
<td>98731</td>
<td>98650</td>
<td>98740</td>
</tr>
</tbody>
</table>

The nine-digit pandigital numbers of parts (a), (b), and (c) are simply the ten-digit pandigital numbers with a leading zero, while the ten-digit numbers of
part (d) are those without a leading zero. We will start with the latter set.

(d,a) The largest ten-digit pandigital number divisible by 99 must, if possible, start with 98765. This is possible only with the second column of the table, and the resulting number is

9876524130.

(d,b) The smallest must, if possible, start with 102. There are two possibilities (the last two columns in the table), but the smaller resulting number is

1024375869.

(a) Here the largest number must, if possible, start with (0)98765. Not surprisingly, the only possibility is the answer to (d,a) with the trailing zero moved to the head:

(0)987652413.

(b) The smallest number must, if possible, start with (0)1234. Again, there is only one choice (column 9 in the table), and the resulting number is

(0)123475869.

(c;d,c) There are \( \binom{10}{5} = 252 \) ways to select the odd-position digit set, and only 22 of them yield numbers that are divisible by 99. In each case, exactly \( 1/10 \) of the permutations have a leading zero, so the answer to both (c) and (d,c) is

\[
\frac{22}{252} = \frac{11}{126}.
\]

II. Comment by the proposer.

The total number \( f_n \) of permutations of the nine nonzero digits which are divisible by \( 11^n \), and the smallest \( S_n \) and largest \( L_n \) in each class, are given below. No permutation of the nine nonzero digits is divisible by \( 11^n \) for \( n > 5 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f_n )</th>
<th>( S_n )</th>
<th>( L_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31680</td>
<td>123475869</td>
<td>987652413</td>
</tr>
<tr>
<td>2</td>
<td>2912</td>
<td>123495867</td>
<td>987641325</td>
</tr>
<tr>
<td>3</td>
<td>273</td>
<td>124365978</td>
<td>987165432</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>126893547</td>
<td>936745821</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>436287159</td>
<td>842135679</td>
</tr>
</tbody>
</table>

\( S_5 \) and \( L_5 \) are both also divisible by 7, and \( S_5 \) is divisible as well by its initial dyad, 43. The first four digits of \( S_1 \) and \( S_2 \) are consecutive in increasing order of magnitude. The leading five, four, and three digits of \( L_1 \), \( L_2 \), and \( L_3 \),
respectively, are consecutive in decreasing order of magnitude. Indeed, only the intercalated 1 keeps \( L_3 \) from having all of its digits consecutive. Centrally imbedded in \( S_3 \) is the number of days in a normal year.

III. Comment by Jordi Dou, Barcelona, Spain.

Parts (a), (b), and (c) of this problem were proposed in Madrid in 1933 at an examination for professors in which I took part. We were also asked to find the sum, \( \Sigma \), of all the numbers concerned (permutations of nine nonzero digits which are divisible by 99). If \( \Sigma(A) \) denotes the sum of all those whose odd-position digits sum to 28, then

\[
\Sigma(A) = 9 \cdot 4! \cdot 5! \left(101010101 \cdot \frac{28}{5} + 101010101 \cdot \frac{17}{4}\right) = 157745441952;
\]

and if \( \Sigma(B) \) denotes the sum of all those whose odd-position digits sum to 17, then

\[
\Sigma(B) = 2 \cdot 4! \cdot 5! \left(101010101 \cdot \frac{17}{5} + 101010101 \cdot \frac{28}{4}\right) = 2385454541184.
\]

Thus

\[
\Sigma = \Sigma(A) + \Sigma(B) = 18159999983136.
\]

It was a pleasure for me to rework this problem (proposed here in 1983) on its fiftieth anniversary, but armed, this time, with a calculator and in the tranquillity of retirement.

Also solved by SAM BAETHGE, San Antonio, Texas; the COPS of Ottawa; JORDI DOU, Barcelona, Spain; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; Richard I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin-Whitewater; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; STEWART METCHETTE, Culver City, California; LEROY F. MEYERS, The Ohio State University; GLEN E. MILLS, Pensacola Junior College, Florida; DAVID R. STONE, Georgia Southern College, Statesboro; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer.


For fixed positive integer \( n \) and arbitrary \( \theta \), simplify the product

\[
\sin \theta \sin(\theta + \frac{2\pi}{n}) \sin(\theta + \frac{4\pi}{n}) \ldots \sin(\theta + \frac{2(n-1)\pi}{n}).
\]

I. Solution by Curtis Cooper, Central Missouri State University at Warrensburg.

Let \( P_n(\theta) \) denote the given expression. The following result is given without proof in [1, p. 121]:

\[
P_n(\theta) = 2^{-n} \{\cos \frac{n\pi}{2} - \cos(n(\theta + \frac{\pi}{2}))\}.
\]

To establish (1), we set \( \psi = \pi/2 \) and \( \phi = \theta + \pi/2 \) in the following identity, which is
given and proved in [1, p. 117]:

\[ \cos n\psi - \cos n\phi = 2^n \prod_{s=0}^{n-1} (\cos \psi - \cos (\phi + \frac{s\pi}{n})). \]

Thus

\[ \cos \frac{n\pi}{2} - \cos \left( \theta + \frac{\pi}{2} \right) = 2^{n-1} \prod_{s=0}^{n-1} \left( \cos \frac{\pi}{2} - \cos \left( \theta + \frac{s\pi}{n} \right) \right) \]

\[ = 2^{n-1} \prod_{s=0}^{n-1} \sin \left( \theta + \frac{s\pi}{n} \right) \]

\[ = 2^{n-1} P_n(\theta), \]

and (1) follows. According to the parity of \( n \), (1) can be simplified to

\[ P_n(\theta) = \begin{cases} (-1)^{(n-1)/2} 2^{1-n} \sin n\theta, & \text{if } n \text{ is odd}, \\ (-1)^{n/2} 2^{2-n} \sin 2n\theta, & \text{if } n \text{ is even}. \end{cases} \]

II. Generalization by A.P. Guinand, Trent University, Peterborough, Ontario.

More generally, we express in closed form the product

\[ P_{m,n}(\theta) = \prod_{s=0}^{n-1} \sin \left( \theta + \frac{m\pi s}{n} \right), \quad (1) \]

where \( m \) and \( n \) are positive integers. The special case \( m = 1 \) is well known [1, p. 119]:

\[ P_{1,n}(\theta) = \prod_{s=0}^{n-1} \sin \left( \theta + \frac{s\pi}{n} \right) = 2^{1-n} \sin n\theta, \quad (2) \]

and our problem corresponds to the case \( m = 2 \). We will deduce a formula for (1) from (2) in two stages. Let \( d = (m,n) \) be the g.c.d. of \( m \) and \( n \).

Stage 1. Suppose \( d = 1 \). As \( s \) runs through 0,1,2,...,\( n-1 \), the least positive residues of \( ms \) modulo \( n \) also run, in some order, through the same set [2]. If \( r \) is the least positive residue of \( ms \) modulo \( n \), then \( ms = qn + r \), where \( q = \lfloor ms/n \rfloor \). Hence

\[ \sin \left( \theta + \frac{ms\pi}{n} \right) = \sin \left( \theta + \frac{qn + r}{n} \right) = (-1)^q \sin \left( \theta + \frac{r}{n} \right) \]

and, on multiplication,

\[ P_{m,n}(\theta) = (-1)^d \prod_{s=0}^{n-1} \sin \left( \theta + \frac{r}{n} \right) = (-1)^d 2^{1-n} \sin n\theta, \]

where
(The term \( s = 0 \) is omitted since its contribution is zero.) It is a simple exercise (see [3]) to show that \( k = (m-1)(n-1)/2 \). Therefore, if \( d = (m,n) = 1 \), we have

\[
P_{m,n}(\theta) = (-1)^{(m-1)(n-1)/2} 2^{-1-n} \sin n\theta.
\]  

(3)

Stage 2. Assume more generally that \( d = (m,n) > 1 \). Then \( m = \mu d \) and \( n = \nu d \), where \( (\mu,\nu) = 1 \), and

\[
P_{m,n}(\theta) = \frac{\nu d - 1}{d - 1} \prod_{s=0}^{d-1} \sin \left( \theta + \frac{s\pi}{\nu} \right).
\]

Put \( s = \alpha \nu + \beta \), where \( \alpha = 0,1,2,...,d-1 \) and \( \beta = 0,1,2,...,\nu-1 \). Then

\[
P_{m,n}(\theta) = \frac{\nu d - 1}{d - 1} \prod_{\alpha=0}^{d-1} \prod_{\beta=0}^{\nu-1} \sin \left( \theta + \frac{\alpha \pi}{\nu} + \frac{\beta \pi}{\nu} \right)
\]

\[
= \frac{\nu d - 1}{d - 1} \prod_{\alpha=0}^{d-1} \left( \prod_{\beta=0}^{\nu-1} \sin \left( \theta + \frac{\alpha \pi}{\nu} + \frac{\beta \pi}{\nu} \right) \right).
\]

(4)

By (3), the inner product equals \((-1)^{(\mu-1)(\nu-1)/2} \nu \sin \theta\nu\), and in the repeated product (4) each such factor occurs \( d \) times. Hence, altogether,

\[
P_{m,n}(\theta) = (-1)^{k_2 d (1-\nu)} \sin^d \theta = (-1)^{k_2} \sin^d \left( \frac{n\theta}{d} \right),
\]

where

\[
k = \mu \nu (1+2+...+(d-1)) + \frac{(\mu-1)(\nu-1)d}{2}
\]

\[
= \frac{\mu \nu d (d-1)}{2} + \frac{(\mu-1)(\nu-1)d}{2}
\]

\[
= \frac{(\mu d-1)(\nu d-1)}{2} + \frac{d-1}{2}
\]

\[
= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}.
\]

Hence we have:

**Theorem.** If \( m \) and \( n \) are positive integers and \( d = (m,n) \), then, for all \( \theta \),

\[
P_{m,n}(\theta) = \prod_{s=0}^{n-1} \sin \left( \theta + \frac{s\pi}{n} \right) = (-1)^{(m-1)(n-1)+(d-1)/2} 2^{-d-n} \sin \left( \frac{n\theta}{d} \right).
\]
Also solved by PAUL R. BEESACK, Carleton University, Ottawa; S.C. CHAN, Singapore; RICHARD I. HESS, Rancho Palos Verdes, California; J.T. GROENMAN, Arnhem, The Netherlands; WALTER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer (two solutions). Comments were received from W.J. BLUNDON, Memorial University of Newfoundland; and M.S. KLAMKIN, University of Alberta.

REFERENCES


Let $ABC$ be a triangle with sides $a, b, c$, and let $R_m$ be the circumradius of the triangle formed by using as sides the medians of triangle $ABC$. Prove that

$$R_m \geq \frac{a^2 + b^2 + c^2}{2(a + b + c)}.$$

Solution by Leon Bankoff, Los Angeles, California.

Let $m_a, m_b, m_c$, and $R$ denote the median lengths and circumradius, respectively, of triangle $\Delta = ABC$, and let $m'_a, m'_b, m'_c$ be the median lengths of the triangle $\Delta'_m$ whose sides are $m'_a, m'_b, m'_c$. It is well known [1, p. 66] that

$$m'_a = \frac{3}{4}a, \quad m'_b = \frac{3}{4}b, \quad m'_c = \frac{3}{4}c$$

and [1, p. 70] that

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

The following inequality is known to hold for every triangle $\Delta$ (see Crux 733 [1983: 121, 149]):

$$2R(m_a + m_b + m_c) \geq a^2 + b^2 + c^2.$$  \hspace{1cm} (3)

Applying (3) to triangle $\Delta'_m$, we get

$$2R_m(m'_a + m'_b + m'_c) \geq m'_a^2 + m'_b^2 + m'_c^2,$$

and, using (1) and (2), the desired inequality follows from

$$2R_m \frac{3}{4}(a+b+c) \geq \frac{3}{4}(a^2+b^2+c^2).$$
Also solved by WALther JANous, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; Vedula N. MURty, Pennsylvania State University, Capitol Campus; Bob PRIELIPP, University of Wisconsin-Oshkosh; Kesiraju SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

For a related problem see Crux 970 in this issue.

REFERENCE


(a) If A, B, C are the angles of a triangle, prove that

\[(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C,\]

with equality if and only if the triangle is equilateral.

(b) Deduce from (a) Bottema's triangle inequality [1982: 296]:

\[(1 + \cos 2A)(1 + \cos 2B)(1 + \cos 2C) + \cos 2A \cos 2B \cos 2C \geq 0.\]

Solution by M.S. Klamkin, University of Alberta.

(a) Let I, H, r, R be the incenter, orthocenter, inradius, and circumradius, respectively, of triangle ABC. It is known [1, p. 200] that

\[IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C\]

\[= 4R^2 \{(1 - \cos A)(1 - \cos B)(1 - \cos C) - \cos A \cos B \cos C \}.\]

Since \(IH^2 \geq 0\), the desired inequality follows, with equality just when I and H coincide, that is, just when the triangle is equilateral. \[\square\]

This result is not new. Bager [2, p. 20] had already pointed out that \(IH^2 \geq 0\) was equivalent to inequality (a), and this fact was mentioned in connection with Crux 544 [1981: 151-152].

(b) We assume without loss of generality that \(A \geq B \geq C\). If \(A = \pi/2\), then Bottema's inequality reduces to

\[\frac{1}{2}(1 - \cos 2(B-C)) \geq 0,\]

and this clearly holds, with equality just when \(B = C = \pi/4\). If \(A < \pi/2\), then the angles of the orthic triangle are

\[\pi - 2A, \quad \pi - 2B, \quad \pi - 2C;\]

and Bottema's inequality results if we apply inequality (a) to the orthic triangle.
If \( A > \frac{\pi}{2} \), then the angles of the orthic triangle are
\[ 2A - \pi, \quad 2B, \quad 2C. \]

If we now apply inequality (a) to the orthic triangle, the result is
\[ (1 + \cos 2A)(1 - \cos 2B)(1 - \cos 2C) \geq -\cos 2A \cos 2B \cos 2C, \]
and Bottema's inequality will follow if
\[ (1 + \cos 2B)(1 + \cos 2C) \geq (1 - \cos 2B)(1 - \cos 2C), \]
or, equivalently, if
\[ \cos 2B + \cos 2C = 2 \cos (B+C) \cos (B-C) \geq 0. \]
This is clearly true, since both \( B+C \) and \( B-C \) are less than \( \pi/2 \).

Having shown that \( (a) \implies (b) \), we show that also \( (b) \implies (a) \). Let \( \triangle ABC \) be any
triangle. Then
\[ A' = \frac{\pi-A}{2}, \quad B' = \frac{\pi-B}{2}, \quad C' = \frac{\pi-C}{2} \]
are also the angles of a triangle, and \( (a) \) results if we apply \( (b) \) to triangle \( \triangle A'B'C' \). Note however that, even though inequalities \( (a) \) and \( (b) \) are equivalent, the corresponding equalities are not, for equality holds in \( (b) \) but not in \( (a) \) when \( \triangle ABC \) is an isosceles right triangle.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; CURTIS COOPER, Central Missouri State University at Warrensburg; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; JOHN OMAN and BOB PRIELIPP, University of Wisconsin-Oshkosh (joint solution); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (part (a) only); GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

Cooper conjectured (with an incomplete proof) the following generalization of Bottema's inequality:
\[ (1 + \cos 2^n A)(1 + \cos 2^n B)(1 + \cos 2^n C) + \cos 2^n A \cos 2^n B \cos 2^n C \geq 0, \quad n = 1, 2, 3, \ldots. \]
It would be interesting to have a proof or disproof of this conjecture.

REFERENCES

Let $G$ be a group with normal subgroups $H$ and $K$. Prove or disprove:

(a) if $H$ and $K$ are isomorphic, then $G/H$ and $G/K$ are also isomorphic;
(b) if $G/H$ and $G/K$ are isomorphic, then $H$ and $K$ are also isomorphic.

Solution by Curtis Cooper, Central Missouri State University at Warrensburg.

The substance of this solution comes from [1]. We will disprove both statements. Let

$$G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b : a^4 = b^2 = 1, ab = ba\}.$$ 

$G$ is abelian and isomorphic to the direct product $C_4 \times C_2$ of a cyclic group of order 4 with one of order 2. Since $G$ is abelian, any subgroup, $H$, of $G$ must be normal, so that the quotient group, $G/H$, can be constructed.

(a) Let $H = \{1, a^2\}$ and $K = \{1, b\}$. Then $H \cong K \cong C_2$, but $G/H \not\cong G/K$ since $G/H \cong C_2 \times C_2$ (Klein's group) and $G/K \cong C_4$.

(b) Let $H = \{1, a, a^2, a^3\}$ and $K = \{1, a^2, b, a^2b\}$. Then $G/H \cong G/K \cong C_2$ but $H \not\cong K$ since $H \cong C_4$ and $K \cong C_2 \times C_2$.

Also solved by DAVID R. STONE, Georgia Southern College, Statesboro; and the proposer.

REFERENCE

1. K.R. McLean, "When isomorphic groups are not the same", The Mathematical Gazette, 57 (1973) 207-208.

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For a given regular tetrahedron of edge length 2, there is a point $M$ such that the distance between $M$ and each of the four vertices is an integer. Prove that $M$ must coincide with one of the vertices of the tetrahedron.

Solution by the proposer.

Let $ABCD$ be the given regular tetrahedron, labeled so that $MA \geq MB \geq MC \geq MD$. By the triangle inequality, $MA-MD \leq AD = 2$. Therefore $MA-MD = 2, 1, \text{ or } 0$. We consider separately these three cases. We will see that the first case holds just when $M$ coincides with $D$, and that the remaining two cases have to be rejected as impossible.

Case 1: $MA-MD = 2$. Here $M$ must lie at or beyond $D$ on the ray $AD$ (see Figure 1). If $P$ is the midpoint of $AD$, then $MP = MA-1 = n$, where $n$ is a positive integer. By the Pythagorean theorem, $MB^2 = n^2 + 3$, so
(MB + n)(MB - n) = 3.

Since 3 is a prime, we must have MB+n = 3, MB-n = 1, so n = 1, MA = 2, and M coincides with D.

**Case 2:** MA-MD = 1. Here we consider three subcases.

**Case 2.1:** MA = k and MB = MC = MD = k-1, where k is a positive integer. Let G be the center of triangle BCD. The equalities MB = MC = MD imply that M lies on the ray AG (see Figure 2). Here triangle MAB has integral sides but the cosine of its angle A is the irrational number \(\sqrt{2/3}\). This is impossible, by the law of cosines.

**Case 2.2:** MA = MB = MC = k and MD = k-1, where k is a positive integer. This case is similar to the preceding one, with the same result: impossibility.

**Case 2.3:** MA = MB = k and MC = MD = k-1, where k is a positive integer. Let ST be the bimedian relative to edges AB and CD (whose length is \(\sqrt{2}\)), as shown in Figure 3. It follows from the assumed equalities that M lies on the ray ST. Triangles MAS and MCT are both right-angled; hence

\[ k^2 = MS^2 + 1 \quad \text{and} \quad (k-1)^2 = MT^2 + 1. \]

Thus

\[ k^2 - (k-1)^2 = MS^2 - MT^2 = (ST\pm MT)^2 - MT^2 = 2 \pm 2\sqrt{2k^2-2k}. \]

This is equivalent to

\[ 2k - 3 = \pm 2\sqrt{2k^2-2k}, \]

which implies

\[ (2k - 3)^2 = 8(k^2 - 2k). \]

This last equality is impossible, since the left side is odd and the right side even.
Case 3: MA-MD = 0. Here MA = MB = MC = MD. This implies that M is the circumcenter of the tetrahedron and that MA equals the circumradius R. This is impossible because R is not an integer.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and JACK LESAGE, Eastview Secondary School, Barrie, Ontario (incomplete solution).

Editor's comment.
The proposer noted that he had proposed this problem for the 25th Bulgarian Mathematical Olympiad in 1976.

839. [1983: 113] Proposed by W.J. Blundon, Memorial University of Newfoundland.
Prove that the Diophantine equation
\[ x^n + y^n = z^{n+1} \]
has infinitely many solutions \((x,y,z)\) for every natural number \(n\).

I. Solution by R.B. Killgrove, Alhambra, California.
One learns in logic courses that the position of quantifiers in propositions is crucial. With the quantifier at the end of the proposal, one "logical" interpretation is:

There are infinitely many positive integer triples \((x,y,z)\) such that
\[ x + y = z^2, \quad x^2 + y^2 = z^3, \quad x^3 + y^3 = z^4, \quad \ldots, \quad x^n + y^n = z^{n+1}, \quad \ldots \]

This statement is false. For suppose \((x,y,z)\) is a solution. Then
\[ (x + y)(x^3 + y^3) = (x^2 + y^2)^2. \]
This equation is equivalent to \((x - y)^2 = 0\), so \(x = y\). Now we have \(2x = z^2\) and \(2x^2 = z^3\), so \(x = z = 2\), and the only solution is \((2,2,2)\).

We reformulate the proposal to give the meaning undoubtedly intended by the proposer:

Let \(n\) be an arbitrary, but fixed, positive integer. Then there are infinitely many positive integer triples \((x,y,z)\) such that
\[ x^n + y^n = z^{n+1}. \tag{1} \]

This statement is true. Indeed, for any positive integer \(k\),
\[ (x, y, z) = (2k^{n+1}, 2k^{n+1}, 2k^n) \tag{2} \]
is a solution of (1). There are, as we shall see, many more solutions.
A solution \((x,y,z)\) of (1) is said to be a primitive solution if and only if
there is no integer \( d > 1 \) such that
\[
d^{n+1} \mid x, \quad d^{n+1} \mid y, \quad \text{and} \quad d^n \mid z.
\]
(Thus (2) is a primitive solution if and only if \( k = 1 \).) For each pair \((a, b)\) of relatively prime positive integers, we will find a primitive solution of (1) corresponding to \((a, b)\). All solutions of (1) can then be generated from these primitive solutions.

Let \((a, b)\) be a pair of relatively prime positive integers, and let
\[
c = a^n + b^n.
\]
If the canonical factorization of \( c \) is
\[
c = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \cdots (p_m)^{\alpha_m}
\]
and if, by the division algorithm,
\[
\alpha_i = \beta_i(n+1) + \gamma_i, \quad 0 \leq \gamma_i < n+1, \quad i = 1, 2, \ldots, m,
\]
then
\[
c = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \cdots (p_m)^{\gamma_m} \{ (p_1)^{\beta_1}(p_2)^{\beta_2} \cdots (p_m)^{\beta_m}\}^{n+1}. \tag{3}
\]

Now, for \( i = 1, 2, \ldots, n \), let
\[
\omega_i = \begin{cases} 
1, & \text{if no } \gamma_j = i, \\
\prod_{j=1}^{m} p_j^{\gamma_j}, & \text{otherwise},
\end{cases}
\]
and let
\[
\omega_{n+1} = \prod_{i=1}^{m} (p_i)^{\beta_i}.
\]
Then, from (3),
\[
c = \omega_1 \omega_2 \cdots \omega_n \omega_{n+1},
\]
and so
\[
a^n + b^n = \prod_{i=1}^{n+1} \omega_i. \tag{4}
\]
We now multiply both sides of (4) by \((\omega_1 \omega_2 \cdots \omega_n)^n\) and obtain (1) with
\[
\begin{align*}
\alpha &= a \prod_{i=1}^{n} \omega_i^i, \\
\beta &= b \prod_{i=1}^{n} \omega_i^i, \\
\gamma &= (\prod_{i=1}^{n} \omega_i^i)^{n+1}.
\end{align*} \tag{5}
\]
We now show that solution (5) is primitive. Suppose it is not. Then there is some prime $p$ such that

$$p^{n+1} | x, \quad p^{n+1} | y, \quad \text{and} \quad p^n | z.$$  

This prime $p$ must be one of $p_1, p_2, \ldots, p_m$, since these are the prime divisors of our $z$, and therefore $p | c$. Since $a$ and $b$ are relatively prime, we must have $p | a$. Now $p$ divides at most one $\omega^s$ of our $\omega_1^s \omega_2^s \ldots \omega_n^s$; hence $p^{n+1} | x$. The contradiction shows that (5) is a primitive solution.

Finally, we show that all solutions of (1) can be generated from the primitive solutions (5). Let $(r, s, t)$ be a solution of (1), so that

$$r^n + s^n = t^{n+1},$$

and let $d = (r, s)$. Then each side of (6) can be divided by $d^n$, resulting in

$$a^n + b^n = c,$$

where $a = r/d$ and $b = s/d$ are relatively prime positive integers, and $c = t^{n+1}/d^n$. The primitive solution (5) corresponding to this pair $(a, b)$ can then be "augmented" to $(r, s, t)$. To see how this can be done, see the author's article [1], where the procedure is applied to the more general equation

$$x^n + y^n = z^k.$$


We first state and prove a lemma that generalizes our problem. The lemma will then lead to a theorem that is an even more sweeping generalization of our problem.

**Lemma.** Let $n$ and $k$ be given positive integers, and let $a_0, a_1, \ldots, a_k$ be arbitrary nonzero integers. Then the Diophantine equation

$$a_1y_1^n + a_2y_2^n + \ldots + a_ky_k^n = a_0y_0^{n+1}$$

has infinitely many integral solutions $(y_0, y_1, \ldots, y_k)$.

**Proof.** Infinitely many solutions are given by

$$y_i = \frac{a_i}{a_0} (a_1c_1^n + a_2c_2^n + \ldots + a_kc_k^n), \quad i = 0, 1, \ldots, k,$$

where the $c_j$ are arbitrary integral multiples of $a_0$ for $j = 1, 2, \ldots, k$ and $c_0 = 1$. This is easily verified by substituting into (1) and verifying that the result is an identity. The requirement that the $c_j$ (other than $c_0$) be multiples of $a_0$ ensures that all the $y_i$ are integral. □
THEOREM. Let \( k \) be a given positive integer; let \( a_0, a_1, \ldots, a_k \) be arbitrary non-zero integers; and let \( r_0, r_1, \ldots, r_k \) be positive integers such that \((r_i, r_0) = 1\) for \( i = 1, 2, \ldots, k \). Then the Diophantine equation

\[
a_1(x_1)^{r_1} + a_2(x_2)^{r_2} + \ldots + a_k(x_k)^{r_k} = a_0(x_0)^{r_0} \tag{2}
\]

has infinitely many integral solutions \((x_0, x_1, \ldots, x_k)\).

Proof. Let \( r \) be the least common multiple of \( r_1, r_2, \ldots, r_k \). It follows from \((r_i, r_0) = 1\) for \( i = 1, 2, \ldots, k \) that \((r, r_0) = 1\). Thus there are positive integers \( p \) and \( q \) such that

\[
qr_0 - pr = 1.
\]

Then any solution \((y_0, y_1, \ldots, y_k)\) of

\[
a_1(y_1)^{pr} + a_2(y_2)^{pr} + \ldots + a_k(y_k)^{pr} = a_0(y_0)^{qr_0} \tag{3}
\]

provides a solution \((x_0, x_1, \ldots, x_k)\) to (2). But (3) has infinitely many solutions by the lemma, since \( qr_0 = pr + 1 \). Therefore (2) has infinitely many solutions.

III. Solution by Paul R. Beesack, Carleton University, Ottawa.

We collect in the form of two theorems information about equations that generalize the one in the proposal. Some of the information in Theorem 1, at least, is already known (see [2] and [3]).

THEOREM 1. Let \( n \) and \( a \) be given positive integers, and let \( d = (n, a) \).

(i) If \( d = 1 \), then the Diophantine equation

\[
x^n + y^n = z^d \tag{1}
\]

has infinitely many positive integral solutions \((x, y, z)\).

(ii) If \( d \geq 3 \), then (1) has no positive integral solutions provided \( x^d + y^d = z^d \) has no such solutions (hence never, if Fermat's last theorem is valid).

(iii) If \( d = 2 \) and \( n = 2n_1 \), then (1) has no positive integral solutions provided \( x^{2n_1} + y^{2n_1} = z^2 \) has no such solutions. (This is the case if \( n_1 = 2 \), or if \( n_1 \geq 3 \) and \( x^{n_1} + y^{n_1} = z^{n_1} \) has no such solutions.)

THEOREM 2. Let \( a_1, a_2, \ldots, a_n, b \) be positive integers and let \( A_1, A_2, \ldots, A_n, B \) be nonzero integers such that \( A_i \) and \( B \) have the same sign for at least one \( i \). If \( s.c.d. \{a_1, a_2, \ldots, a_n, b\} = 1 \), then the Diophantine equation

\[
\sum_{i=1}^{n} A_i (x_i) ^{a_i} = Bb^b
\]

has infinitely many positive integral solutions \((x_1, x_2, \ldots, x_n, z)\).

[The proofs are omitted. (Editor)]
Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; PAUL R. BEESACK, Carleton University, Ottawa (second solution); CURTIS COOPER, Central Missouri State University at Warrensburg; the COPS of Ottawa; JORDI DOU, Barcelona, Spain; UNDERWOOD DUDLEY, DePauw University; MEIR FEDER, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin-Whitewater; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire (second solution); DAVID R. STONE, Georgia Southern College, Statesboro; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

Beesack submitted full proofs for his Theorems 1 and 2. We decided not to publish them because of their length, because parts of Theorem 1 are already in the literature (see also references [4]-[6], all submitted by readers), and because of the obvious similarity of Theorem 2 to Rabinowitz's theorem which is proved in solution II.

REFERENCES


Find the remainder when 72! is divided by 79.

I. Solution by W.J. Blundon, Memorial University of Newfoundland.

We seek an integer $r$ such that $0 \leq r < 79$ and

$$72! \equiv r \quad (1)$$

(congruences throughout are modulo 79).
From

\[ 73 \cdot 74 \cdot \ldots \cdot 78 \equiv (-6) \cdot (-5) \cdot \ldots \cdot (-1) = 720 \equiv 9 \]

and (1) we obtain (using Wilson's Theorem to get \( 78! \equiv -1 \))

\[ 9r \equiv 78! \equiv -1 \equiv 315, \quad \text{so} \quad r \equiv 35, \]

and the required remainder is 35.


Dividing 72! by 79 gives a quotient of

\[
775 119 726 289 697 302 044 608 485 889 552 819 503 520 149 173 383 911 187 633 100 809
\]

\[
852 974 521 793 535 116 508 354 430 379 746 835
\]

and a remainder of 35.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; CURTIS COOPER, Central Missouri State University at Warrensburg (two solutions); the COPS of Ottawa; JORDI DOU, Barcelona, Spain; UNDERWOOD DUDLEY, DePauw University, Greencastle, Indiana; MEIR FEDER, Haifa, Israel; A.P. GUINAND, Trent University, Peterborough, Ontario; WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN and M.V. SUBBARAO, University of Alberta (jointly); EDWIN M. KLEIN, University of Wisconsin-Whitewater; GLEN E. MILLS, Pensacola Junior College, Florida; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; LAWRENCE SOMER, Washington, D.C.; DAVID STONE, Georgia Southern College, Statesboro; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer. One incorrect solution was received.


The adjoined alphametic was inspired by Gertrude Stein's put-down of Oakland, California [1983: 55]. It is in the lowest possible base, and the punctuation is of literary, rather than arithmetic, significance.

Solutions were received from MEIR FEDER, Haifa, Israel; ALLAN WM. JOHNSON JR., Washington, D.C.; EDWIN M. KLEIN, University of Wisconsin-Whitewater; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The lowest possible base is eleven, and the unique solution in that base, found by all solvers, is

\[
188
964t42
51
964t4
964t4,
\]

where \( t = \text{ten} \).

\[ 1037058 \]
All but two solvers submitted only the answer. Only Mills and Wilke submitted full-blown solutions, and these involved a lot of brute force which computers can apply much more efficiently. We are grateful to Mills and Wilke for keeping the game honest, but we do not feel that any useful purpose would be served by publishing in this journal, where space is at a premium, a blow-by-blow description of a brute force solution to this, or any other, alphametic. Solvers should look for finesses that make most of the brute force unnecessary. If they do not find them, then they may as well keep all the gory details to themselves and send in only the answer, or else go to a computer and get, at least, some programming experience out of the problem.

842, [1953: 142] Proposed by W.J. Blundon, Memorial University of Newfoundland.

Find (a) necessary and (b) sufficient conditions on $a, b, c$ for the system

\[ x + \frac{1}{x} = a, \quad y + \frac{1}{y} = b, \quad xy + \frac{1}{xy} = c \]

to be consistent, that is, to have at least one common solution $(x, y)$. (These necessary and sufficient conditions constitute the eliminant of the system.)

Solution by Gali Salvatore, Perkin, Québec.

We assume that the equations are over the field of complex numbers. Suppose the given system is consistent, and let $(x, y)$ be a common solution. If we multiply together corresponding members of the three equations, we obtain

\[ (a^2 - 2) + (b^2 - 2) + (c^2 - 2) + 2 = abc, \]

and thus a necessary condition for consistency is

\[ a^2 + b^2 + c^2 - 4 = abc. \] (1)

Conversely, suppose (1) holds, and let $(x, y)$ be any solution of the first two equations of the system. Then $(x, 1/y)$ is also a solution of the same first two equations. Now, from (1),

\[ a^2 - abc + (a^2 + b^2 - 4) = 0, \]

and so

\[ c = \frac{ab \pm \sqrt{(a^2 - 4)(b^2 - 4)}}{2} \]

\[ = \frac{(x + \frac{1}{x})(y + \frac{1}{y}) \pm (x - \frac{1}{x})(y - \frac{1}{y})}{2} \]

\[ = xy + \frac{1}{xy} \quad \text{or} \quad \frac{x}{y} + \frac{y}{x}. \]
Now $a = xy + 1/xy$ implies that $(x, y)$ is a common solution of the system, and $a = x/y + y/x$ implies that $(x, 1/y)$ is a common solution. In either case the system is consistent. We conclude that (1) is the required eliminant of the system.

It is clear from the above solution that (1) is also the eliminant of the system

$$x + \frac{1}{x} = a, \quad y + \frac{1}{y} = b, \quad \frac{x}{y} + \frac{y}{x} = c.$$ 

If we are restricted to the real field, it is easy to see that the eliminant consists of (1) and the additional two conditions $|a| \geq 2$ and $|b| \geq 2$. (The condition $|c| \geq 2$ is then automatically satisfied, so it need not appear in the eliminant.)

Also solved by the COPS of Ottawa; G.P. HENDERSON, Campbellcroft, Ontario; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) only); FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio (part (a) only); M.S. KLAMKIN, University of Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; DAVID R. STONE, Georgia Southern College, Statesboro; and the proposer.

Also solved by the COPS of Ottawa; G.P. HENDERSON, Campbellcroft, Ontario; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) only); FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio (part (a) only); M.S. KLAMKIN, University of Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; DAVID R. STONE, Georgia Southern College, Statesboro; and the proposer.


For integers $m > 1$ and $n > 2$, and real numbers $p, q > 0$ such that $p + q = 1$, prove that

$$p^m n^+ + np^m (1-p)^n + (1-q-npq)^n > 1.$$ 

I. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We rewrite the inequality in the equivalent form

$$(1 - q^n - npq^{n-1}) > (1 - (1-p)^n) - np^m(1-p)^n.$$ (1)

and use the argument and notation in the proposer's article [1].

As noted in [1], $[1 - (1-p)^n]$ is the probability that at least one of the $n$ lines from $D$ is solid, and it is easy to see that $np^m (1-p)^n$ is the probability that precisely one of the $n$ lines from $D$ is solid. Hence the right side of (1) is the probability of the event

$\alpha$: at least two of the $n$ lines from $D$ are solid.

On the other hand, the left side of (1) is the probability of the event

$\beta$: each of the $m$ lines from $E$ is broken at least twice.

It is clear that Pr ($\beta$) $\geq$ Pr ($\alpha$), and the inequality is strict when $n > 2$.

II. Comment by the proposer.

This inequality was first stated and proved in [2]. A much simpler proof can be obtained from a triangle as in [1] or from a rectangular lattice as in [3]. We
noted in [3] that the inequalities in [1] and [3] and several others (including the present one) remain valid when the parameters are real, and that this extension would be carried out in a more technical article, to appear in 1985 in J. Math. Anal. & Appl.

Also solved by the proposer.

Editor's comment.

Solution I was obtained from a triangle as in [1], and the proposer submitted a solution from a rectangular lattice as in [3]. The editor has seen the proposer's proof that the present inequality remains valid when the parameters \( m > 1 \) and \( n > 2 \) are real, but J. Math. Anal. & Appl. has the priority of publication and common mortals will have to wait until 1985 to see it.

REFERENCES


(continued from p. 215)

side CD. Define \( u_{BC} \), \( u_{CD} \), \( u_{DA} \) likewise, using in each case perpendiculars to the opposite side. Show that

\[
\begin{align*}
\mu_{AB} &= \mu_{BC} = \mu_{CD} = \mu_{DA}.
\end{align*}
\]

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

THE PUZZLE CORNER

Answer to Puzzle No. 55 [1984: 204]: Reflections (Reflect I on S).
Answer to Puzzle No. 56 [1984: 204]: Parabola, parabole; Hyperbola, hyperbole.
Answer to Puzzle No. 57 [1984: 204]: Calculus, calculus.