Hilbert Transformation and Representation of the $ax + b$ Group
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Abstract. In this paper we study the Hilbert transformations over $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ from the viewpoint of symmetry. For a linear operator over $L^2(\mathbb{R})$ commutative with the $ax+b$ group we show that the operator is of the form $\lambda I + \eta H$, where $I$ and $H$ are the identity operator and Hilbert transformation respectively, and $\lambda$, $\eta$ are complex numbers. In the related literature this result was proved through first invoking the boundedness result of the operator, proved though a big machinery. In our setting the boundedness is a consequence of the boundedness of the Hilbert transformation. The methodology that we use is Gelfand-Naimark’s representation of the $ax+b$ group. Furthermore we prove a similar result on the unit circle. Although there does not exist a group like $ax+b$ on the unit circle, we construct a semigroup to play the same symmetry role for the Hilbert transformations over the circle $L^2(\mathbb{T})$. 