Cover product and Betti polynomial of graphs

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Abstract. For disjoint graphs $G$ and $H$, with fixed vertex covers $C(G)$ and $C(H)$, their cover product is the graph $G \boxtimes H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{i, j\} : i \in C(G), j \in C(H)\}$. We describe the graded Betti numbers of $G \boxtimes H$ in terms of those of $G$ and $H$. As applications we obtain: (i) For any positive integer $k$ there exists a connected bipartite graph $G$ such that $\text{reg} \ R/I(G) = \mu_S(G) + k$, where, $I(G)$ denotes the edge ideal of $G$, $\text{reg} \ R/I(G)$ is the Castelnuovo–Mumford regularity of $R/I(G)$ and $\mu_S(G)$ is the induced or strong matching number of $G$; (ii) The graded Betti numbers of the complement of a tree only depends upon its number of vertices; (iii) The $h$-vector of $R/I(G \boxtimes H)$ is described in terms of the $h$-vectors of $R/I(G)$ and $R/I(H)$. Furthermore, in a different direction, we give a recursive formula for the graded Betti numbers of chordal bipartite graphs.