On the Smallest and Largest Zeros of M\"untz–Legendre Polynomials

\textbf{\textit{Abstract.}} M\"untz–Legendre polynomials $L_n(\Lambda; x)$ associated with a sequence $\Lambda = \{\lambda_k\}$ are obtained by orthogonalizing the system $(x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \ldots)$ in $L_2[0,1]$ with respect to the Legendre weight. If the $\lambda_k$’s are distinct, it is well known that $L_n(\Lambda; x)$ has exactly $n$ zeros $l_{n,n} < l_{n-1,n} < \cdots < l_{2,n} < l_{1,n}$ on $(0,1)$.

First we prove the following global bound for the smallest zero,

$$\exp\left(-4 \sum_{j=0}^{n} \frac{1}{2\lambda_j + 1}\right) < l_{n,n}.$$ 

An important consequence is that if the associated M\"untz space is non-dense in $L_2[0,1]$, then

$$\inf_n x_{n,n} \geq \exp\left(-4 \sum_{j=0}^{\infty} \frac{1}{2\lambda_j + 1}\right) > 0,$$

so the elements $L_n(\Lambda; x)$ have no zeros close to 0.

Furthermore, we determine the asymptotic behavior of the largest zeros; for $k$ fixed,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} (2\lambda_j + 1) = \left(\frac{l_k}{2}\right)^2.$$ 

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where $j_k$ denotes the $k$-th zero of the Bessel function $J_0$.

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