A Singular Critical Potential for the Schrödinger Operator

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Abstract. Consider a real potential \( V \) on \( \mathbb{R}^d, d \geq 2 \), and the Schrödinger equation:

\[
    i \partial_t u + \Delta u - Vu = 0, \quad u|_{t=0} = u_0 \in L^2.
\]

In this paper, we investigate the minimal local regularity of \( V \) needed to get local in time dispersive estimates (such as local in time Strichartz estimates or local smoothing effect with gain of \( 1/2 \) derivative) on solutions of (LS). Prior works show some dispersive properties when \( V \) (small at infinity) is in \( L^{d/2} \) or in spaces just a little larger but with a smallness condition on \( V \) (or at least on its negative part).

In this work, we prove the critical character of these results by constructing a positive potential \( V \) which has compact support, bounded outside 0 and of the order \( (\log |x|)^2/|x|^2 \) near 0. The lack of dispersiveness comes from the existence of a sequence of quasimodes for the operator \( P := -\Delta + V \).

The elementary construction of \( V \) consists in sticking together concentrated, truncated potential wells near 0. This yields a potential oscillating with infinite speed and amplitude at 0, such that the operator \( P \) admits a sequence of quasi-modes of polynomial order whose support concentrates on the pole.