Monotonicity Properties of the Hurwitz Zeta Function

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Abstract. Let
\[ \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad (s > 1, x > 0) \]
be the Hurwitz zeta function and let
\[ Q(x) = Q(x; \alpha, \beta; a, b) = \frac{(\zeta(\alpha, x))^a}{(\zeta(\beta, x))^b}, \]
where \( \alpha, \beta > 1 \) and \( a, b > 0 \) are real numbers. We prove: (i) The function \( Q \) is decreasing on \((0, \infty)\) if and only if \( \alpha a - \beta b \geq \max(a - b, 0) \). (ii) \( Q \) is increasing on \((0, \infty)\) if and only if \( \alpha a - \beta b \leq \min(a - b, 0) \). An application of part (i) reveals that for all \( x > 0 \) the function \( s \mapsto [(s - 1)\zeta(s, x)]^{1/(s-1)} \) is decreasing on \((1, \infty)\). This settles a conjecture of Bastien and Rogalski.