Characterizing Two-Dimensional Maps Whose Jacobians Have Constant Eigenvalues

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Abstract. Recent papers have shown that $C^1$ maps $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose Jacobians have constant eigenvalues can be completely characterized if either the eigenvalues are equal or $F$ is a polynomial. Specifically, $F = (u, v)$ must take the form

\begin{align*}
u &= ax + by + \beta \phi(\alpha x + \beta y) + e \\
v &= cx + dy - \alpha \phi(\alpha x + \beta y) + f
\end{align*}

for some constants $a, b, c, d, e, f, \alpha, \beta$ and a $C^1$ function $\phi$ in one variable. If, in addition, the function $\phi$ is not affine, then

\begin{equation}
\alpha \beta (d - a) + b \alpha^2 - c \beta^2 = 0.
\end{equation}

This paper shows how these theorems cannot be extended by constructing a real-analytic map whose Jacobian eigenvalues are $\pm 1/2$ and does not fit the previous form. This example is also used to construct non-obvious solutions to nonlinear PDEs, including the Monge–Ampère equation.