Local Complexity of Delone Sets and Crystallinity

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Abstract. This paper characterizes when a Delone set $X$ in $\mathbb{R}^n$ is an ideal crystal in terms of restrictions on the number of its local patches of a given size or on the heterogeneity of their distribution. For a Delone set $X$, let $N_X(T)$ count the number of translation-incomparable patches of radius $T$ in $X$ and let $M_X(T)$ be the minimum radius such that every closed ball of radius $M_X(T)$ contains the center of a patch of every one of these kinds. We show that for each of these functions there is a “gap in the spectrum” of possible growth rates between being bounded and having linear growth, and that having sufficiently slow linear growth is equivalent to $X$ being an ideal crystal.

Explicitly, for $N_X(T)$, if $R$ is the covering radius of $X$ then either $N_X(T)$ is bounded or $N_X(T) \geq T/2R$ for all $T > 0$. The constant $1/2R$ in this bound is best possible in all dimensions.

For $M_X(T)$, either $M_X(T)$ is bounded or $M_X(T) \geq T/3$ for all $T > 0$. Examples show that the constant $1/3$ in this bound cannot be replaced by any number exceeding $1/2$. We also show that every aperiodic Delone set $X$ has $M_X(T) \geq c(n)T$ for all $T > 0$, for a certain constant $c(n)$ which depends on the dimension $n$ of $X$ and is $>1/3$ when $n > 1$. 

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