Nonconvexity of the Generalized Numerical Range Associated with the Principal Character

Chi-Kwong Li and Alexandru Zaharia

Abstract. Suppose $m$ and $n$ are integers such that $1 \leq m \leq n$. For a subgroup $H$ of the symmetric group $S_m$ of degree $m$, consider the generalized matrix function on $m \times m$ matrices $B = (b_{ij})$ defined by $d^H(B) = \sum_{\sigma \in H} \prod_{j=1}^m b_{\sigma(j)}$ and the generalized numerical range of an $n \times n$ complex matrix $A$ associated with $d^H$ defined by

$$W^H(A) = \{ \Re^H(X^*AX) : X \text{ is } n \times m \text{ such that } X^*X = I_m \}.$$ 

It is known that $W^H(A)$ is convex if $m = 1$ or if $m = n = 2$. We show that there exist normal matrices $A$ for which $W^H(A)$ is not convex if $3 \leq m \leq n$. Moreover, for $m = 2 < n$, we prove that a normal matrix $A$ with eigenvalues lying on a straight line has convex $W^H(A)$ if and only if $\nu A$ is Hermitian for some nonzero $\nu \in \mathbb{C}$. These results extend those of Hu, Hurley and Tam, who studied the special case when $2 \leq m \leq 3 \leq n$ and $H = S_m$. 

Received by the editors December 11, 1998.
AMS subject classification: 15A60.
Keywords: convexity, generalized numerical range, matrices.