THE TRIGONOMETRY OF HYPERBOLIC TESSELLATIONS

H. S. M. COXETER

ABSTRACT. For positive integers $p$ and $q$ with $(p - 2)(q - 2) > 4$ there is, in the hyperbolic plane, a group $[p, q]$ generated by reflections in the three sides of a triangle $ABC$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$. Hyperbolic trigonometry shows that the side $AC$ has length $\hat{t}$, where $\cosh \hat{t} \geq c/s$, $c = \cos \frac{\pi}{q}$, $s = \sin \frac{\pi}{p}$. For a conformal drawing inside the unit circle with centre $A$, we may take the sides $AB$ and $AC$ to run straight along radii while $BC$ appears as an arc of a circle orthogonal to the unit circle. The circle containing this arc is found to have radius $1/ \sinh \hat{t} \geq s/z$, where $z = \sqrt{c^2 - s^2}$, while its centre is at distance $1/ \tanh \hat{t} \geq c/z$ from $A$. In the hyperbolic triangle $ABC$, the altitude from $AB$ to the right-angled vertex $C$ is $\zeta$, where $\sinh \zeta = z$. 

Received by the editors November 22, 1995.
AMS subject classification: 51F15, 51N30, 52A55.