ON COUNTABLY PARACOMPACT SPACES

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LET X be a topological space, that is, a space with open sets such that the union of any collection of open sets is open and the intersection of any finite number of open sets is open. A covering of X is a collection of open sets whose union is X. The covering is called countable if it consists of a countable collection of open sets or finite if it consists of a finite collection of open sets; it is called locally finite if every point of X is contained in some open set which meets only a finite number of sets of the covering. A covering $\mathfrak{V}$ is called a refinement of a covering $\mathfrak{U}$ if every open set of $\mathfrak{V}$ is contained in some open set of $\mathfrak{U}$. The space X is called countably paracompact if every countable covering has a locally finite refinement.

The purpose of this paper is to study the properties of countably paracompact spaces. The justification of the new concept is contained in Theorem 4 below, where it is shown that, for normal spaces, countable paracompactness is equivalent to two other properties of known topological importance.

1. A space X is called compact if every covering has a finite refinement, paracompact if every covering has a locally finite refinement, and countably compact if every countable covering has a finite refinement. It is clear that every compact, paracompact or countably compact space is countably paracompact. Just as one shows[1] that every closed subset of a compact [paracompact, countably compact] space is compact [paracompact, countably compact], so one can show that every closed subset of a countably paracompact space is countably paracompact. It is known that the topological product of two compact spaces is compact and the topological product of a compact space and a paracompact space is paracompact [2, Theorem 5]. The following is an analogous theorem.

THEOREM 1. *The topological product $X \times Y$ of a countably paracompact space X and a compact space Y is countably paracompact.*

Proof. Let $\{U_i\}$ $(i = 1, 2, \ldots)$ be a countable covering of $X \times Y$. Let V_i be the set of all points x of X such that $x \times Y \subset \bigcup_{j \leqslant i} U_j$. If $x \in V_i$ every point (x, y) of $x \times Y$ has a neighbourhood $N \times M$, (N open in X, M open in Y), which is contained in the open set $\bigcup_{j \leqslant i} U_j$. A finite number of these open sets M cover Y; let N_x be the intersection of the corresponding finite number of sets N. Then $x \in N_x$, N_x is open and $N_x \times Y \subset \bigcup_{j \leqslant i} U_j$; and hence $N_x \subset V_i$. Therefore V_i is open. Also, for any $x \in X$, since $x \times Y$ is compact, $x \times Y$ is

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[1]See [1] page 86, Satz IV and [2] Theorem 2.

contained in some finite number of sets of the covering $\{U_i\}$; hence x is in some V_i. Therefore $\{V_i\}$ is a covering of X.

Since $\{V_i\}$ is countable and X is countably paracompact, $\{V_i\}$ has a locally finite refinement $\mathfrak{W}$. For each open set W of $\mathfrak{W}$ let $g(W)$ be the first V_i containing W and let G_i be the union of all W for which $g(W) = V_i$. Then G_i is open, $G_i \subset V_i$ and $\{G_i\}$ is a locally finite covering of X.

If $j \leqslant i$, let $G_{ij} = (G_i \times Y) \cap U_j$; then G_{ij} is an open set in $X \times Y$. If (x, y) is any point of (X, Y) then, for some i, $x \in G_i$ and hence $(x, y) \in G_i \times Y$. Also, since $x \in G_i \subset V_i$, $(x, y) \in x \times Y \subset \bigcup_{j \leqslant i} U_j$, and hence, for some $j \leqslant i$, $(x, y) \in U_j$. Hence $(x, y) \in G_{ij}$. Therefore $\{G_{ij}\}$ is a covering of $X \times Y$. Since $G_{ij} \subset U_j$, $\{G_{ij}\}$ is a refinement of $\{U_i\}$. Also, if $(x, y) \in X \times Y$, x is in an open set $H(x)$ which meets only a finite number of the sets of $\{G_i\}$. Then $H(x) \times Y$ is an open set containing (x, y) which can meet G_{ij} only if $H(x)$ meets G_i. But for each i there is only a finite number of sets G_{ij}. Hence $H(x) \times Y$ meets only a finite number of sets of $\{G_{ij}\}$; hence $\{G_{ij}\}$ is locally finite. Therefore $X \times Y$ is countably paracompact. This completes the proof.

It can similarly be shown that the topological product of a compact space and a countably compact space is countably compact.

2. A topological space X is called normal if for every pair of disjoint closed sets A and B of X there is a pair of disjoint open sets U and V with $A \subset U$ and $B \subset V$ (or, equivalently, there is an open set U with $A \subset U$, $\overline{U} \subset X - B$).

THEOREM 2. *The following properties of a normal space X are equivalent*:

(a) *The space X is countably paracompact.*

(b) *Every countable covering of X has a point-finite*[2] *refinement.*

(c) *Every countable covering $\{U_i\}$ has a refinement $\{V_i\}$ with $\overline{V}_i \subset U_i$.*

(d) *Given a decreasing sequence $\{F_i\}$ of closed sets with vacuous intersection, there is a sequence $\{G_i\}$ of open sets with vacuous intersection such that $F_i \subset G_i$.*

(e) *Given a decreasing sequence $\{F_i\}$ of closed sets with vacuous intersection, there is a sequence $\{A_i\}$ of closed G_δ-sets*[3] *with vacuous intersection such that $F_i \subset A_i$.*

Proof. (a) $\rightarrow$ (b). A locally finite covering is *a fortiori* point-finite.

(b) $\rightarrow$ (c). Let $\{U_i\}$ be any countable covering of X. Then, by (b), $\{U_i\}$ has a point-finite refinement $\mathfrak{W}$. For each open set W of $\mathfrak{W}$ let $g(W)$ be the first U_i containing W, and let G_i be the union of all W such that $g(W) = U_i$. Then $\{G_i\}$ is a point-finite covering of X and $G_i \subset U_i$. It is known [3, p. 26, (33-4); 2, Theorem 6] that every point-finite covering $\{G_i\}$ (whether countable or not) of a normal space X has a refinement $\{V_i\}$ with the closure of each V_i contained in the corresponding G_i. Then $\overline{V}_i \subset G_j \subset U_i$, hence $\overline{V}_i \subset U_i$.

[2] A covering of X is called point-finite if each point of X is in only a finite number of sets of the covering.

[3] A set A is called a G_δ-set if it is the intersection of some countable collection of open sets.

(c) → (d). Let $\{F_i\}$ be a sequence of closed sets with $F_{i+1} \subset F_i$ and $\bigcap_i F_i = 0$. Then, if $U_i = X - F_i$, $\{U_i\}$ is a covering of X. Then, by c, there is a covering $\{V_i\}$ with $\overline{V}_i \subset U_i$. Let G_i be the open set $X - \overline{V}_i$. Then, since $\overline{V}_i \subset U_i$, $F_i \subset G_i$ and, since $\bigcup \overline{V}_i = X$, $\bigcap G_i = 0$.

(d) → (e). Let $\{F_i\}$ be a sequence of closed sets with $F_{i+1} \subset F_i$ and $\bigcap F_i = 0$. Then, by d, there is a sequence $\{G_i\}$ of open sets with $F_i \subset G_i$ and $\bigcap G_i = 0$. Then, by Urysohn's lemma, there is a continuous function ϕ_i, $0 \leqslant \phi_i(x) \leqslant 1$, such that, if $x \in F_i$, $\phi_i(x) = 0$ and, if x non $\in G_i$, $\phi_i(x) = 1$. Let $G_{ij} = \{x \mid \phi_i(x) < 1/j\}$, and let $A_i = \bigcap_j G_{ij} = \{x \mid \phi_i(x) = 0\}$. Then G_{ij} is open, A_i is a closed G_δ-set, $F_i \subset A_i \subset G_i$ and $\bigcap A_i \subset \bigcap G_i = 0$.

(e) → (a). Let $\{U_i\}$ be a countable covering of X and let $F_i = X - \bigcup_{k \leqslant i} U_k$. Then F_i is closed, $F_{i+1} \subset F_i$ and, since $\bigcup U_i = X$, $\bigcap F_i = 0$. Then, by (e), there is a sequence $\{A_i\}$ of closed G_δ-sets with $F_j \subset A_j$ and $\bigcap A_j = 0$. Then $X - A_j$ is an F_σ-set; let $X - A_j = \bigcup_i B_{ji}$ where each B_{ji} is closed. Since X is normal we may assume that B_{ji} is contained in the interior of $B_{j,\,i+1}$. Let H_{ji} be the interior of B_{ji}; then $H_{ji} \subset B_{ji} \subset H_{j,\,i+1}$ and $X - A_j = \bigcup_i H_{ji}$. And $B_{ji} \subset X - A_j \subset X - F_j = \bigcup_{k \leqslant j} U_k$.

Let $V_i = U_i - \bigcup_{j<i} B_{ji}$; then V_i is open. If $j < i$, $B_{ji} \subset \bigcup_{k \leqslant j} U_k \subset \bigcup_{k<i} U_k$; hence $\bigcup_{j<i} B_{ji} \subset \bigcup_{k<i} U_k$. Hence $V_i \supset U_i - \bigcup_{k<i} U_k$. Thus, since each point x of X is in a first U_i, it is in the corresponding V_i. Therefore $\{V_i\}$ is a covering of X. Clearly $\{V_i\}$ is a refinement of $\{U_i\}$.

For each x of X there is some A_j such that x non ϵ A_j; hence, for some k, $x \in H_{jk}$. Then, if $i > j$ and $i > k$, $H_{jk} \subset B_{ji}$ and hence $H_{jk} \cap V_i = 0$. Thus the open set H_{jk} contains x and meets only a finite number of the sets V_i. Hence $\{V_i\}$ is locally finite. Therefore X is countably paracompact.

COROLLARY. *Every perfectly normal space is countably paracompact.*

Proof. A perfectly normal space is a normal space in which every closed set is a G_δ-set. Hence condition (e) is trivally satisfied with $A_i = F_i$.

Not every normal space is countably paracompact as the following example shows. Let X be a space whose points x are the real numbers. Let the open sets of X be the null set, the whole space X and the subsets $G_a = \{x \mid x < a\}$ for all real a. Then X is trivially normal since there are no non-empty disjoint closed sets. But the countable covering $\{G_i\}$ $(i = 1, 2, \ldots)$ where $G_i = \{x \mid x < i\}$, has no locally finite refinement. Hence X is not countably paracompact.[4]

3. We give here a sufficient condition for the normality of a product space.

LEMMA 3. *The topological product $X \times Y$ of a countably paracompact normal space X and a compact metric space Y is normal.*

Proof. Let A and B be two disjoint closed sets of $X \times Y$. Let $\{G_i\}$ be a

[4]This space is not a Hausdorff space. It would be interesting to have an example of a normal Hausdorff space which is not countably paracompact.

countable base for the open sets of Y and, if γ is any finite set of positive integers, let $H_\gamma = \bigcup_{i\epsilon\gamma} G_i$. For each $x \in X$ let A_x be the closed set of Y defined by $x \times A_x = (x \times Y) \cap A$; similarly let $x \times B_x = (x \times Y) \cap B$. Let

$$U_\gamma = \{x \mid A_x \subset H_\gamma \subset \overline{H}_\gamma \subset Y - B_x\}.$$

Let x_0 be a point of X for which $A_{x_0} \subset H_\gamma$. Then, for each $y \in Y - H_\gamma$, (x_0, y) non $\in A$ and, since A is closed, there is a neighbourhood $N \times M$ of (x_0, y) which does not meet A. A finite number of the open sets M cover the compact set $Y - H_\gamma$. If N_{x_0} is the intersection of the corresponding finite number of open sets N, $N_{x_0} \times (Y - H_\gamma)$ does not meet A. Hence, if $x \in N_{x_0}$, $A_x \subset H_\gamma$. Thus $\{x \mid A_x \subset H_\gamma\}$ is an open set. Similarly $\{x \mid \overline{H}_\gamma \subset Y - B_x\}$ is open and U_γ, which is the intersection of these two open sets, is also open.

Let $x \in X$; then for each point y of A_x there is an open set G_i of the base such that $y \in G_i$ and $\overline{G}_i \cap B_x = 0$. A finite number of these sets G_i cover A_x, i.e., for some finite set γ of positive integers, $A_x \subset \bigcup_{i\epsilon\gamma} G_i = H_\gamma$ and $\overline{H}_\gamma = \bigcup_{i\epsilon\gamma} \overline{G}_i \subset Y - B_x$. Hence $x \in U_\gamma$. Thus the open sets U_γ cover X. Since there are only a countable number of finite subsets γ of positive integers, the covering $\{U_\gamma\}$ of X is countable.

Since X is countably paracompact there is a locally finite covering $\{W_\gamma\}$ of X with $W_\gamma \subset U_\gamma$ and, by condition c of Theorem 2, $\{W_\gamma\}$ has a refinement $\{V_\gamma\}$ (still locally finite) such that $\overline{V}_\gamma \subset W_\gamma$. Let U be the open set $\bigcup_\gamma (V_\gamma \times H_\gamma)$. For any point (x, y) of A and for some V_γ, $x \in V_\gamma \subset U_\gamma$. Then $y \in A_x \subset H_\gamma$ and hence $(x, y) \in V_\gamma \times H_\gamma$; therefore $A \subset U$. Since $\{V_\gamma\}$ is locally finite, each point x of X is contained in an open set $G(x)$ which meets only a finite number of sets V_γ; and hence the neighbourhood $G(x) \times Y$ of (x, y) meets only a finite number of the sets $V_\gamma \times H_\gamma$. It follows that (x, y) is in the closure of U if and only if it is in the closure of some $V_\gamma \times H_\gamma$, i.e., $\overline{U} = \bigcup(\overline{V_\gamma \times H_\gamma})$. But $\overline{V_\gamma \times H_\gamma} = \overline{V}_\gamma \times \overline{H}_\gamma$. Hence $\overline{U} = \bigcup(\overline{V}_\gamma \times \overline{H}_\gamma) \subset \bigcup(U_\gamma \times \overline{H}_\gamma)$. But $(U_\gamma \times \overline{H}_\gamma) \cap B = 0$; hence $\overline{U} \cap B = 0$. Thus the open set U contains A and its closure does not meet B. Hence $X \times Y$ is normal.

4. In Theorem 4 below we extend some results of J. Dieudonné [2]. He showed[5] that paracompactness of a Hausdorff space X implies condition β (see below) on semicontinuous functions on X and our proof that $\alpha \rightarrow \beta$ is a trivial modification of his proof. It also follows immediately from Dieudonné's results that if X is a paracompact Hausdorff space, $X \times I$ is a paracompact Hausdorff space and hence is normal. However, in terms of countable paracompactness we are able to give a necessary and sufficient condition for β and γ to hold. The equivalence of conditions β and γ was conjectured by S. Eilenberg.

THEOREM 4. *The following three properties of a topological space X are equivalent.*

(α). *The space X is countably paracompact and normal.*

[5] See [2], Theorem 9.

(β). *If g is a lower semicontinuous real function on X and h is an upper semicontinuous real function on X and if $h(x) < g(x)$ for all $x \in X$, then there exists a continuous real function f such that $h(x) < f(x) < g(x)$ for all $x \in X$.*

(γ). *The topological product $X \times I$ of X with the closed line interval $I = [0, 1]$ is normal.*

Proof. (α) $\to$ (β). Let X be a countably paracompact normal space and let g and h be lower and upper semicontinuous functions respectively with $h(x) < g(x)$. If r is a rational number let $G_r = \{x \mid h(x) < r < g(x)\}$. Since g is lower semicontinuous, $\{x \mid g(x) > r\}$ is open, and, since h is upper semicontinuous, $\{x \mid h(x) < r\}$ is open. Hence G_r is open. Since, for every x, $h(x) < g(x)$ there is some rational number $r(x)$ with $h(x) < r(x) < g(x)$; hence $x \in G_{r(x)}$. Thus $\{G_r\}$ is a covering of X. And, since the rational numbers are countable, $\{G_r\}$ is a countable covering. Hence, since X is countably paracompact and normal, there is a locally finite covering $\{U_r\}$ of X with $U_r \subset G_r$ and there is a (locally finite) covering $\{V_r\}$ with $\overline{V}_r \subset U_r$.

There is a continuous function f_r with $-\infty \leqslant f_r(x) \leqslant r$ such that $f_r(x) = -\infty$ if x non$\in U_r$ and $f_r(x) = r$ if $x \in \overline{V}_r$. Let $f(x)$ be the least upper bound of $f_r(x)$ for all r. Each point x_0 of X is contained in an open set $N(x_0)$ which meets only a finite number of the sets U_r. Hence, in $N(x_0)$, for all but a finite number of values of r, $f_r(x) = -\infty$. Thus, in each neighbourhood $N(x_0)$, $f(x)$ is the least upper bound of a finite number of continuous functions, hence f is continuous. In U_r, $f_r(x) \leqslant r < g(x)$ and, in $X - U_r$, $f_r(x) = -\infty < g(x)$. Thus $f_r(x) < g(x)$ and, for each x, $f(x)$ is the least upper bound of a finite number of $f_r(x)$ each less than $g(x)$. Therefore $f(x) < g(x)$. Each x is in some V_r and, for this r, $f_r(x) = r$; hence $f(x) \geqslant f_r(x) = r > h(x)$. Hence $f(x) > h(x)$. Therefore $h(x) < f(x) < g(x)$.

(β) $\to$ (α). Let X be a space satisfying condition (β) and let A and B be two disjoint closed sets in X. Let h be the characteristic function of A, i.e., $h(x) = 1$ if $x \in A$ and $h(x) = 0$ if x non $\in A$. Let g be defined by $g(x) = 1$ if $x \in B$ and $g(x) = 2$ if x non $\in B$. Then g is lower semicontinuous, h is upper semicontinuous and $h(x) < g(x)$ for all $x \in X$. Hence there is a continuous function f with $h(x) < f(x) < g(x)$. Let $U = \{x \mid f(x) > 1\}$ and $V = \{x \mid f(x) < 1\}$. Then U and V are disjoint open sets and $A \subset U$ and $B \subset V$. Hence X is normal.

Let $\{F_i\}$ ($i = 1, 2, \ldots$) be a decreasing sequence of closed sets with $\cap F_i = 0$. Let g be defined by $g(x) = 1/(i + 1)$ for $x \in F_i - F_{i+1}$ ($i = 0, 1, \ldots$), where F_0 means the whole space X. Let $h(x) = 0$ for all $x \in X$. Then g is lower semicontinuous, h is upper semicontinuous and $h(x) < g(x)$ for all x. Hence there is a continuous function f with $0 < f(x) < g(x)$. Let $G_i = \{x \mid f(x) < 1/(i + 1)\}$. Then G_i is open, $F_i \subset G_i$ and, since $f(x) > 0$ for all x, $\bigcap G_i = 0$. Thus condition *d* of Theorem 2 is satisfied and therefore X is countably paracompact.

(α) $\to$ (γ). This follows immediately from Lemma 3 and the fact that the interval I is a compact metric space.

$(\gamma) \rightarrow (\alpha)$. Let X be a space for which $X \times I$ is normal. Then X is homeomorphic to the closed subset $X \times 0$ of the normal space $X \times I$; therefore X is normal.

Let $\{F_i\}(i = 1, 2, \ldots)$, be a decreasing sequence of closed sets with $\bigcap F_i = 0$. Then, since the half open interval $[0, 1/i[$ is open in $I = [0, 1]$, $W_i = (X - F_i) \times [0, 1/i[$ is open in $X \times I$. Let A be the closed set $X \times I - \bigcup_i W_i$. If $x \in X$, then, for some i, $x \in X - F_i$ and $(x, 0) \in W_i$ and hence $(x, 0)$ non$\in A$. Hence, if $B = X \times 0$, A and B are disjoint closed sets of the normal space $X \times I$. Therefore there are disjoint open sets U and V with $A \subset U$ and $B \subset V$. Let $G_i = \{x \mid (x, 1/i) \in U\}$; then G_i is open. For each $x \in X$, $(x, 0) \in B$ and hence, for sufficiently large i, $(x, 1/i) \in V$ and hence non$\in U$. Therefore $\bigcap G_i = 0$. Let $x \in F_i$. Then, if $j \leqslant i$, $F_i \subset F_j$ and x non$\in X - F_j$, and, if $j \geqslant i$, $1/i$ non ϵ $[0, 1/j[$. Hence $(x, 1/i)$ non$\in \bigcup_j W_j$; hence $(x, 1/i) \in A \subset U$ and hence $x \in G_i$. Therefore $F_i \subset G_i$. Thus condition (d) of Theorem 2 is satisfied and therefore X is countably paracompact. This completes the proof of the theorem.

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