Can. J. Math., Vol. XXIII, No. 2, 1971, pp. 374–380

A RELATIONSHIP BETWEEN LEFT EXACT AND REPRESENTABLE FUNCTORS

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1. Introduction. Our aim in this paper is to demonstrate a relationship between left exact and representable functors. More precisely, in the functor category $\mathfrak{Ab}^{\mathfrak{A}^{op}}$ whose objects are the additive functors from the dual of an abelian category $\mathfrak{A}$ to the category of abelian groups $\mathfrak{Ab}$ and whose morphisms are the natural transformations between them, the left exact functors can be characterized as those equivalent to a direct limit of representable functors taken over a directed class. The proof will proceed in the following manner. Lambek [**3**] and Ulmer [**7**] have shown that any functor T in $\mathfrak{Ab}^{\mathfrak{A}^{op}}$ can be expressed as a direct limit of representable functors taken over a comma category. When T is left exact, it is easily observed that this comma category is a filtered category. We shall show that for any filtered category $\mathfrak{D}_f$ there exists a directed class I and a cofinal functor $F: I \to \mathfrak{D}_f$. Our result then follows.

Acknowledgement. I would like to thank Saunders MacLane and R. G. Swan for many stimulating suggestions during the preparation of this paper.

2. Filtered categories and directed classes. Let us fix some terminology and prove a proposition which will be useful later on. Our aim will be to show that, given any filtered category (small filtered category) $\mathfrak{D}_f$, one can construct a directed class (directed set) I and a cofinal functor $F: I \to \mathfrak{D}_f$. But first, let us note the precise definitions involved.

A *filtered category* is a category $\mathfrak{D}_f$ satisfying the following two axioms:

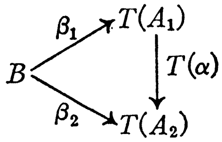
(i) given $D_1, D_2 \in |\mathfrak{D}_f|$, there exists $D_3 \in |\mathfrak{D}_f|$ and maps $\delta_1: D_1 \to D_3$ and $\delta_2: D_2 \to D_3$ in $\mathfrak{D}_f$, and,

(ii) given two maps $\delta_1, \delta_2: D_1 \to D_2$ in $\mathfrak{D}_f$, there exists a third map $\delta_3: D_2 \to D_3$ in $\mathfrak{D}_f$ such that $\delta_3 \circ \delta_1 = \delta_3 \circ \delta_2$.

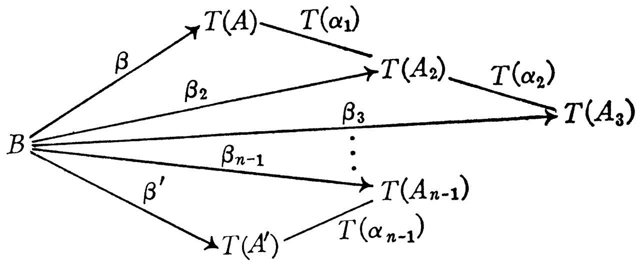
Given a directed set I, a subset $J \subseteq I$ is called *cofinal* if and only if for each $i \in I$ there is an element $j \in J$ such that $i \leqq j$ (cf. [**5**, pp. 47–48]). We wish to generalize this notion to include filtered categories. But first, we need to define the *comma category* (B, T), where $T: \mathfrak{A} \to \mathfrak{B}$ is a functor and $B \in |\mathfrak{B}|$ (cf. [**4**, pp. 13–14]). The objects of (B, T) are maps $\beta: B \to T(A)$ in $\mathfrak{B}$, $A \in |\mathfrak{A}|$. The morphisms of (B, T) from $\beta_1: B \to T(A_1)$ to $\beta_2: B \to T(A_2)$ are maps

Received September 11, 1970.

$\alpha: A_1 \to A_2$ in $\mathfrak{A}$ which yield the commutative diagram:



Given any category $\mathfrak{C}$, we say that two objects C and C' are *connected* if and only if there exists a finite sequence of objects $C_1 = C, C_2, C_3, \ldots, C_n = C'$ and maps $\gamma_i: C_i \to C_{i+1}$ or $C_{i+1} \to C_i$, $1 \leqq i \leqq n-1$, in $\mathfrak{C}$. A category $\mathfrak{C}$ is a *connected category* if and only if any two objects in $\mathfrak{C}$ are connected. Then objects $\beta: B \to T(A)$ and $\beta': B \to T(A')$ in the comma category (B, T) are connected if and only if there exist objects $A_1 = A, A_2, A_3, \ldots, A_n = A'$ and maps $\alpha_i: A_i \to A_{i+1}$ or $A_{i+1} \to A_i$, $1 \leqq i \leqq n-1$, in $\mathfrak{A}$ and maps $\beta_i: B \to T(A_i)$, $1 \leqq i \leqq n$, in $\mathfrak{B}$ with $\beta_1 = \beta$ and $\beta_n = \beta'$ yielding the following commutative diagram:



Now, a functor $F: \mathfrak{X} \to \mathfrak{D}_f$, where $\mathfrak{X}$ is any category and $\mathfrak{D}_f$ is a filtered category, is called *cofinal* if and only if for each $D \in |\mathfrak{D}_f|$ the comma category (D, F) is non-empty and connected.

One final definition. Let us say that a directed class I is *pointwise finitely preceded* (pfp) if and only if, for each $i_0 \in I$, the subclass $\{i | \, i \leqq i_0 \text{ in } I\}$ is finite.

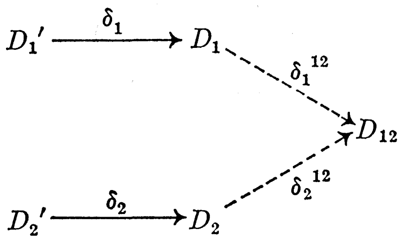
Now we are ready for the proposition.

PROPOSITION 2.1. *Let $\mathfrak{D}_f$ be a filtered category. Then there exist a directed class I and a functor $F: I \to \mathfrak{D}_f$ which is cofinal. Furthermore, I is pfp. As a function, $F: I \to |\mathfrak{D}_f|$ is onto. If $\mathfrak{D}_f$ is small, then I is a set.*

Proof. The objects of the category I are all finite non-empty sets of maps from $\mathfrak{D}_f$. The maps in I are the set inclusions. It is immediately clear that I is a pfp directed class. If $\mathfrak{D}_f$ is small, then I is a directed set. It remains to construct the cofinal functor $F: I \to \mathfrak{D}_f$.

For $n > 0$, let $\mathfrak{L}_n$ be the category whose objects are subsets $s \subseteq \{1, 2, \ldots, n\}$, $s \neq \emptyset$, and whose morphisms are the set inclusions. We proceed to define F inductively. For each object $\{\delta\}$ in I consisting of one map $\delta: D' \to D$ in $\mathfrak{D}_f$

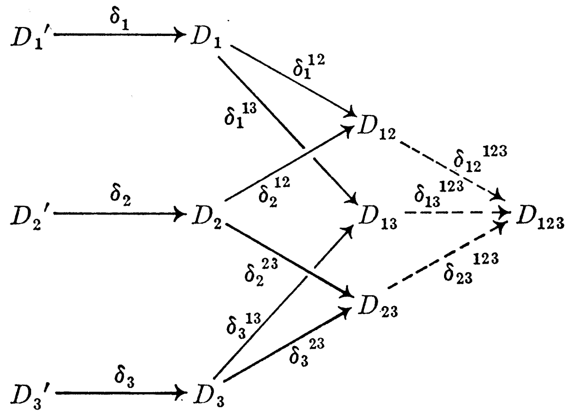
we define $F(\{\delta\}) = D$. For each object $\{\delta_1, \delta_2\}$ in I consisting of two maps, we use axiom (i) for a filtered category to fill in the diagram:



This yields a functor $D\colon \mathfrak{L}_2 \to \mathfrak{D}_f$ given by

$$s \not\subseteq s' \quad D{\leadsto}_s \xrightarrow{\delta_s^{s'}} D_{s'}.$$

Define $F(\{\delta_1, \delta_2\}) = D_{12}$ and $F(\{\delta_i\} \subset \{\delta_1, \delta_2\}) = \delta_i^{12}$, $i = 1, 2$. Next, for each $\{\delta_1, \delta_2, \delta_3\}$ in I, take the previously designated objects and maps and use the two axioms for a filtered category to fill in the following commutative diagram:



This yields a functor $D\colon \mathfrak{L}_3 \to \mathfrak{D}_f$. Define $F(\{\delta_1, \delta_2, \delta_3\}) = D_{123}$ and $F(\{\delta_i, \delta_j\} \subset \{\delta_1, \delta_2, \delta_3\}) = \delta_{ij}^{123}$, $1 \leqq i < j \leqq 3$. The other inclusions into $\{\delta_1, \delta_2, \delta_3\}$ can be handled by composing the obvious δ maps, a process which is well-defined since D is a functor.

Proceed inductively, defining F on each finite set of maps $\{\delta_1, \delta_2, \ldots, \delta_i\}$ and its subset inclusions, $i = 1, 2, \ldots, n - 1$. Now consider the object $\{\delta_1, \delta_2, \ldots, \delta_n\}$ in I. Let $t = \{1, 2, \ldots, n\}$ and $t_i = \{1, 2, \ldots, \hat{\imath}, \ldots, n\}$ in $\mathfrak{L}_n$. Using the two axioms for a filtered category, construct a diagram consisting of the previously designated objects

$$D_i', \quad 1 \leqq i \leqq n, \qquad D_s, \quad s \not\subseteq t,$$

and maps

$$\delta_i\colon D_i' \to D_i, \qquad 1 \leqq i \leqq n, \qquad \delta_s{}^{s'}, \qquad s \not\subseteq s' \not\subseteq t,$$

plus object D_t and maps $\delta_{ti}{}^t\colon D_{ti} \to D_t$, $1 \leqq i \leqq n$, such that:

(i) the diagram is commutative (i.e. $D\colon \mathfrak{L}_n \to \mathfrak{D}_f$ defines a functor), and

(ii) given $\delta_i\colon D_i' \to D_i$ and $\delta_j\colon D_j' \to D_j$ with $D_i' = D_j'$, $1 \leqq i, j \leqq n$, $\delta_s{}^t \circ \delta_i = \delta_{s'}{}^t \circ \delta_j$ whenever the compositions are defined (we want this condition to hold for $n \geqq 4$).

Define $F(\{\delta_1, \delta_2, \ldots, \delta_n\}) = D_t$ and

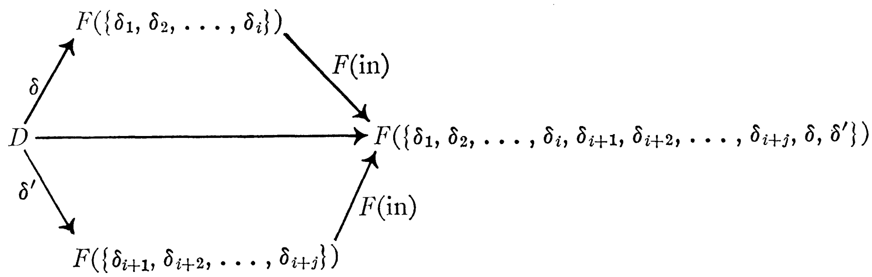
$$F(\{\delta_1, \delta_2, \ldots, \hat{\delta}_i, \ldots, \delta_n\} \subset \{\delta_1, \delta_2, \ldots, \delta_n\}) = \delta_{ti}{}^t.$$

Define F on the other inclusions into $\{\delta_1, \delta_2, \ldots, \delta_n\}$ by taking compositions, a well-defined process since (i) holds. (ii) will ensure the cofinality of F.

It is clear that F is a functor. It remains to show that $F\colon I \to \mathfrak{D}_f$ is cofinal. For each $D \in |\mathfrak{D}_f|$, the comma category (D, F) is certainly non-empty, since $F\colon I \to |\mathfrak{D}_f|$ considered as a function is onto; i.e. $F(\{\mathrm{id}_D\}) = D$ for every $D \in |\mathfrak{D}_f|$, where id_D is the identity map. We still need to show that (D, F) is connected for each $D \in |\mathfrak{D}_f|$. That is, we need to show that any two objects

$$\delta\colon D \to F(\{\delta_1, \delta_2, \ldots, \delta_i\}) \quad \text{and} \quad \delta'\colon D \to F(\{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+j}\})$$

in (D, F) are connected. But F was constructed to satisfy the following commutative diagram:



where "in" are the inclusions. Hence, (D, F) is connected for each $D \in |\mathfrak{D}_f|$, and we have completed the proof of the proposition.

3. Left exact and representable functors. Let $\mathfrak{A}$ be an abelian category. It is "well known" that every functor in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functors. Let us sketch this result, referring the reader to [**3**; **7**, pp. 79–82] for the details.

Let $J\colon \mathfrak{A} \to \mathfrak{A}'$ be a functor between abelian categories, and, for each $A' \in |\mathfrak{A}'|$, let (J, A') be the comma category. The objects of (J, A') are pairs (A, α') where $A \in |\mathfrak{A}|$ and $\alpha'\colon J(A) \to A'$ is a map in $\mathfrak{A}'$. A map in (J, A') from (A_1, α_1') to (A_2, α_2') is a map $\alpha\colon A_1 \to A_2$ in $\mathfrak{A}$ such that $\alpha_2' \circ J(\alpha) = \alpha_1'$. There exists a forgetful functor $F_J(A')\colon (J, A') \to \mathfrak{A}$ defined by $(A, \alpha') \rightsquigarrow A$.

The functor $J\colon \mathfrak{A} \to \mathfrak{A}'$ is called *dense* if and only if, for each $A' \in |\mathfrak{A}'|$, the natural transformation $\phi_J(A')\colon J \circ F_J(A') \to \mathrm{const}_{A'}$, the constant functor, defined by

$$\phi_J(A')[(A, \alpha')] = \alpha',$$

is universal. Then

$$\text{inj lim } J \circ F_J(A') = A'.$$

Let $Y_{\mathfrak{A}'}\colon \mathfrak{A}' \hookrightarrow \mathfrak{Ab}^{(\mathfrak{A}')^{\mathrm{op}}}$ denote the Yoneda embedding defined by

$$A' \rightsquigarrow \mathrm{Hom}_{\mathfrak{A}'}(\ \ , A').$$

Then we have the following result.

LEMMA 3.1. *Let $J\colon \mathfrak{A} \to \mathfrak{A}'$ be a functor between abelian categories. J is dense if and only if the composite functor*

$$\mathfrak{Ab}^J \circ Y_{\mathfrak{A}'}\colon \mathfrak{A}' \hookrightarrow \mathfrak{Ab}^{(\mathfrak{A}')^{\mathrm{op}}} \to \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$$

defined by

$$A' \to \mathrm{Hom}_{\mathfrak{A}'}(J(\ \ ), A')$$

is full and faithful.

Using this we can conclude the following result.

PROPOSITION 3.2. *The Yoneda embedding $Y_{\mathfrak{A}}\colon \mathfrak{A} \hookrightarrow \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is dense; i.e. each functor from $\mathfrak{A}^{\mathrm{op}}$ to $\mathfrak{Ab}$ is a direct limit of representable functors.*

For, the Yoneda lemma implies that the composite functor $M\colon \mathfrak{Ab}^{\mathfrak{A}_{\mathrm{do}}} \to \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ defined by

$$T \rightsquigarrow \mathrm{Hom}_{\mathfrak{Ab}}{}^{\mathfrak{A}^{\mathrm{op}}}(Y_{\mathfrak{A}}(\ \ ), T)$$

is full and faithful.

Now let us proceed to the main result of this paper. We have noted that each functor T in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functors:

$$T = \text{inj lim } Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T),$$

where $Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T)\colon (Y_{\mathfrak{A}}, T) \to \mathfrak{A} \hookrightarrow \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is the composite functor. Let us examine the comma category $(Y_{\mathfrak{A}}, T)$. Using the Yoneda lemma again, it is clear that the objects of $(Y_{\mathfrak{A}}, T)$ are the pairs (A, a) where $A \in |\mathfrak{A}|$ and $a \in T(A)$. A map in $(Y_{\mathfrak{A}}, T)$ from (A_1, a_1) to (A_2, a_2) is a map $\alpha\colon A_1 \to A_2$ such that $T(\alpha)[a_2] = a_1$.

LEMMA 3.3. *Let $\mathfrak{A}$ be abelian. Then any left exact functor L in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functor over a filtered category. If $\mathfrak{A}$ is small, then so is the filtered category.*

Proof. We have shown that

$$L = \text{inj lim } Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(L),$$

where the direct limit is taken over the comma category $(Y_{\mathfrak{A}}, L)$ described above. We now show that L left exact implies $(Y_{\mathfrak{A}}, L)$ filtered. A left exact functor preserves biproducts (cf. [**1**, pp. 64–65]). Thus, given

$$(A_1, a_1),\ (A_2, a_2) \in |(Y_{\mathfrak{A}}, L)|,$$

the object $(A_1 \oplus A_2, a_1 \oplus a_2)$ is well-defined, since

$$a_1 \oplus a_2 \in L(A_1) \oplus L(A_2) = L(A_1 \oplus A_2).$$

Furthermore, we have the two maps ι^1: $(A_1, a_1) \to (A_1 \oplus A_2, a_1 \oplus a_2)$ and ι^2: $(A_2, a_2) \to (A_1 \oplus A_2, a_1 \oplus a_2)$ in $(Y_{\mathfrak{A}}, L)$ induced from the biproduct injections, since $L(\iota^1)[a_1 \oplus a_2] = \pi^1[a_1 \oplus a_2] = a_1$ and $L(\iota^2)[a_1 \oplus a_2] = \pi^2[a_1 \oplus a_2] = a_2$, where π^1 and π^2 are the biproduct projections. Hence axiom (i) for a filtered category is satisfied. Given maps

$$\alpha_1, \alpha_2\colon\ (A_1, a_1) \to (A_2, a_2)$$

in $(Y_{\mathfrak{A}}, L)$, let α: $A_2 \to C$ be the coequalizer of α_1, α_2: $A_1 \to A_2$ in $\mathfrak{A}$. Then by the left exactness of L, the equalizer of $L(\alpha_1), L(\alpha_2)$: $L(A_2) \to L(A_1)$ is $L(\alpha)$: $L(C) \to L(A_2)$. Since $L(\alpha_1)[a_2] = L(\alpha_2)[a_2] = a_1$, (C, a_2) is well-defined as an object in $(Y_{\mathfrak{A}}, L)$. Furthermore, we have the commutative diagram

$$(A_1, a_1) \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} (A_2, a_2) \xrightarrow{\alpha} (C, a_2)$$

in $(Y_{\mathfrak{A}}, L)$. Thus axiom (ii) for a filtered category is satisfied. Note that if $\mathfrak{A}$ is small, then $(Y_{\mathfrak{A}}, L)$ is also small.

The following lemma is "well known" (cf. [**6**, p. 225]); in fact, it has motivated the definition of cofinal. Its proof can be left to the reader.

LEMMA 3.4. *Let F: $\mathfrak{X} \to \mathfrak{D}_f$ be a cofinal functor from any category to a filtered category. Then, for any functor G: $\mathfrak{D}_f \to \mathfrak{A}$, where $\mathfrak{A}$ is any category, we have*

$$\operatorname{inj\,lim} G = \operatorname{inj\,lim} G \circ F.$$

Now we are ready for the main result.

THEOREM 3.5. *Let $\mathfrak{A}$ be an abelian category (a small abelian category). Then a functor L in $\mathfrak{Ab}^{\mathfrak{A}^{op}}$ is left exact if and only if L is a direct limit of representable functors over a directed class (directed set) I; i.e.*

$$L = \operatorname*{inj\,lim}_{I} \operatorname{Hom}_{\mathfrak{A}}(\ \ , A.) = \operatorname*{inj\,lim}_{I} \operatorname{Hom}_{\mathfrak{A}}(\ \ , A_i).$$

Proof. The necessity follows from Lemma 3.3, Proposition 2.1, and Lemma 3.4. For let F: $I \to (Y_{\mathfrak{A}}, L)$ be a cofinal functor with I a directed class (directed set). Then $A. = F_{Y_{\mathfrak{A}}}(L) \circ F$: $I \to \mathfrak{A}$. The sufficiency follows, since a representable functor is left exact and the direct limit over a directed class (directed set) is exact in $\mathfrak{Ab}$.

Remark. Note that the directed class (directed set) I constructed in Proposition 2.1 is also a lattice class (lattice). Hence Theorem 3.5 could be restated in terms of lattice classes (lattices).

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