A Skolem–Mahler–Lech Theorem for Iterated Automorphisms of $K$-algebras

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Abstract. This paper proves a commutative algebraic extension of a generalized Skolem-Mahler-Lech theorem due to the first author. Let $A$ be a finitely generated commutative $K$-algebra over a field of characteristic 0, and let $\sigma$ be a $K$-algebra automorphism of $A$. Given ideals $I$ and $J$ of $A$, we show that the set $S$ of integers $m$ such that $\sigma^m(I) \supseteq J$ is a finite union of complete doubly infinite arithmetic progressions in $m$, up to the addition of a finite set. Alternatively, this result states that for an affine scheme $X$ of finite type over $K$, an automorphism $\sigma \in \text{Aut}_K(X)$, and $Y$ and $Z$ any two closed subschemes of $X$, the set of integers $m$ with $\sigma^m(Z) \subseteq Y$ is as above. The paper presents examples showing that this result may fail to hold if the affine scheme $X$ is not of finite type, or if $X$ is of finite type but the field $K$ has positive characteristic.